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A Bayesian Approach to Survival Analysis of COVID-19 Data Using the Additive Flexible Weibull Extension-Lomax Distribution

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Abstract: This study investigates the Bayesian estimation for the additive flexible Weibull extension-Lomax (AFWE-L) distribution, a versatile model designed to capture complex survival data patterns. Using Type II censored samples and a joint bivariate informative prior, Bayes estimators are derived for the distribution's unknown model parameters, reliability metrics such as the hazard and reversed hazard rate functions. Bayesian inference is conducted with the squared error (symmetric) and linear-exponential (asymmetric) loss functions, allowing flexibility in handling estimation errors. Credible intervals are constructed to quantify parameter uncertainty, providing a measure of precision for the Bayes estimates. The effectiveness of the Bayes estimators under various censoring levels is evaluated through a comprehensive simulation study, specifically 0% and 30%, revealing increased estimation efficiency with larger sample sizes and reduced censoring. The simulation study is performed through the implementation of the adaptive Metropolis algorithm. The simulation results demonstrate the efficacy of the proposed methodology in capturing diverse survival patterns and highlight the impact of censoring on estimation accuracy. Furthermore, The AFWE-L distribution model is utilized to analyze three real-world datasets concerning COVID-19, showcasing its practical utility in analyzing complex survival data characterized by competing risks. This research advances Bayesian survival analysis techniques, providing a robust framework for modeling lifetime data with competing risks in medical and reliability studies.

Keywords: Competing risks; additive flexible Weibull extension-Lomax distribution; linear exponential and squared error loss functions; Bayesian estimation; adaptive Metropolis algorithm.
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1. Introduction

Engineering applications, reliability studies, demography, and the medical and biological sciences sometimes face competing risks due to several causes or modes of failure. These failure types contend to induce an item's failure, an idea commonly described as competing risks in the statistical field. Moreover, conflicting hazards are observed in series systems, where elements are organized sequentially. The system's lifespan is determined by the minimum of its components' lifetimes, as component lifetimes are statistically independent. Furthermore, a model involving competing risks is referred to as the series model, additive model, and multiple risk.

Statistical literature has many lifespan distributions developed using the concept of competing hazards. Numerous citations in this domain encompass: the additive Weibull distribution established by Xie and Lai [1], the additive Burr XII distribution suggested by Wang [2], and the newly modified Weibull distribution provided by Almalki and Yuan [3]. The newly modified Weibull distribution was created by integrating Weibull and modified Weibull distributions into a series structure [See Lai et al. [4]]. Additional contributions include He et al. [5], who developed the additive modified Weibull distribution, and Oluyede et al. [6], who introduced the log-logistic Weibull distribution. Singh [7] discovered the additive Perks-Weibull distribution; the Burr XII modified Weibull distribution was first introduced by Mdlongwa et al. [8], and Tarvirdizade and Ahmedpour [9] got the Weibull-Chen distribution. Additionally, the log-normal modified Weibull distribution was developed by Shakhatreh et al. [10], the Lomax-Weibull distribution by Osagie and Osemwenkhae [11], and the flexible Weibull extension-Burr XII distribution by Kamal and Ismail [12], which was created by integrating the Flexible Weibull extension (FWE) [Refer to Bebbington et al. [13]] and Burr XII distributions in a sequential framework. Recent advancements in competing risk lifetime distributions include the additive Chen-Weibull distribution proposed by Thach and Bris [14], the Lindley-Burr XII distribution examined by Makubate et al. [15], and the flexible additive Chen-Gompertz distribution introduced by Abba et al. [16], which was developed by integrating the Chen distribution with a particular instance of the Gompertz distribution within a series system framework. In addition to the additive power-transformed half-logistic model by Xavier et al. [17], the three-component additive Weibull distribution by Thach [18], and the additive Perks distribution by Méndez-González et al. [19]. Méndez-González et al. [20] produced the additive Chen distribution; they also [21] provided the additive Chen-Perks distribution, whereas Mohammad et al. [22] developed the additive xgamma-Burr XII distribution. Kalantan et al. [23] developed a competing risks model known as the Chen-Burr XII distribution. Amiru et al. [24] introduced the Additive Dhillon-Chen distribution.

The additive flexible Weibull extension-Lomax (AFWE-L) distribution was first studied by Salem et al. [25] via the modeling of a series system with two independent components. The FWE distribution with parameters α and β is used to describe the lifetime of the first element, X_1. Conversely, the Lomax (L) distribution with parameters λ and θ is used to describe the lifetime of the second component, X_2. Consequently, the system's lifetime, $X = min\{X_1, X_2\}$, follows the AFWE-L distribution characterized by the parameter vector $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$. The following formulas are for the probability density function (pdf), cumulative distribution function (cdf), and the reliability measures such as hazard rate function (hrf), and reversed hazard rate function (rhrf) of the AFWE-L distribution:

$$f(x;\underline{\psi}) = \left[\left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x}} + \frac{\theta}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-1}\right]e^{-e^{\alpha x - \frac{\beta}{x}}}\left(1 + \frac{x}{\lambda}\right)^{-\theta} \quad , for \quad x > 0; and \quad \underline{\psi} > \underline{0}, \tag{1.1}$$

$$F(x;\underline{\psi}) = 1 - e^{-e^{\alpha x-\frac{\beta}{x}}}\left(1+\frac{x}{\lambda}\right)^{-\theta} \quad ,for \quad x>0; and \quad \underline{\psi} > \underline{0}, \tag{1.2}$$

$$R(x;\underline{\psi}) = \prod_{i-1}^{2} R_i(x) = e^{-e^{\alpha x-\frac{\beta}{x}}}\left(1+\frac{x}{\lambda}\right)^{-\theta} \quad ,for \quad x>0; and \quad \underline{\psi} > \underline{0}, \tag{1.3}$$

$$h(x;\underline{\psi}) = \left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}} + \frac{\theta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}; \quad ,for \quad x>0; and \quad \underline{\psi} > \underline{0}, \tag{1.4}$$

and

$$r(x;\underline{\psi}) = \frac{\left[\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}} + \frac{\theta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}\right]e^{-e^{\alpha x-\frac{\beta}{x}}}\left(1+\frac{x}{\lambda}\right)^{-\theta}}{1-e^{-e^{\alpha x-\frac{\beta}{x}}}\left(1+\frac{x}{\lambda}\right)^{-\theta}} \quad ,for \quad x>0; and \quad \underline{\psi} > \underline{0}, \tag{1.5}$$

where

$$\underline{\psi} = (\alpha, \beta, \lambda, \theta). \tag{1.6}$$

The AFWE-L distribution exhibits high adaptability and versatility in the shapes of its pdf and hrf, enabling it to fit various types of lifetime data. The pdf of the distribution can exhibit various shapes, including decreasing, unimodal, decreasing followed by a unimodal form, and vice versa. Correspondingly, its hrf may take on increasing, decreasing, or bathtub, bi-bathtub (also known as w shape) and modified bathtub (The hrf initially increases and subsequently exhibits a bathtub shape). Salem et al. [25] derived key characteristics of the AFWE-L distribution and obtained estimates of its parameters, reliability function (rf) and hrf, through maximum likelihood estimation applied with Type II censoring. They also obtained asymptotic confidence intervals for these parameters, rf and hrf. Furthermore, they demonstrated the AFWE-L distribution's applicability and versatility over existing distributions through three applications to various countries COVID-19 data. EL-Helbawy et al. [26] developed methodologies for both one-sample prediction and two-sample prediction of future observations from the AFWE-L distribution, employing non-Bayesian and Bayesian approaches employing Type II censored samples.

Estimation within a Bayesian framework has gained importance due to its advantages. Notably, it incorporates prior knowledge regarding the unknown parameters through a joint prior distribution and combines this with data-derived evidence from the likelihood function (LF) to form the posterior distribution. Bayesian estimation is also particularly suitable for small datasets and censored observations. The symmetric squared error (SE) loss function and the asymmetric linear-exponential (LINEX) loss function are widely used in Bayesian estimation. Varian [27] introduced these function, which were later popularized by Zellner [28]. In terms of mathematical formulation, the SE and LINEX loss functions are as follows:

$$L_1(\psi, \tilde{\psi}) = c(\psi - \tilde{\psi})^2, \tag{1.7}$$

and

$$L_2(\psi, \tilde{\psi}) \propto e^{v\Delta} - v\Delta - 1 \quad ,v \neq 0, \tag{1.8}$$

in which ψ is an unknown parameter, $\tilde{\psi}$ is its Bayes estimator, c is a constant, v is a shape-controlling constant of the loss function, and $\Delta = \tilde{\psi} - \psi$ is the estimation error.

The Bayesian framework has been extensively employed by numerous researchers for parameter estimation in a variety of lifetime distributions, often under different loss functions, including Amein [29], Ateya and Mohammed [30], Nassar et al. [31], Almetwally et al. [32], Hasaballah et al. [33], Najmaldin et al. [34], Xu [35], Muhammed and Almetwally [36], Jeslin et al. [37] and Habtewold et al. [38].

The motivation for this study arises from the increasing need for flexible statistical models capable of effectively capturing complex and diverse survival data patterns encountered across various fields, including reliability engineering, biomedical research, and actuarial science. The AFWE-L distribution holds potential as a widely applicable model to address this need due to its ability to exhibit various hazard rate shapes. Furthermore, the motivation for this study lies in the sound rationale for applying Bayesian estimation techniques, particularly with informative priors and different loss functions, to provide robust and flexible inference for the parameters and reliability properties of the AFWE-L distribution, especially when dealing with censored observations. Finally, the practical relevance of this research is underscored by the application of the proposed Bayesian methodology to real-world COVID-19 survival datasets characterized by competing risks.

Our work advances current Bayesian survival analysis methodologies by introducing a flexible and adaptable model AFWE-L distribution, employing informative priors, conducting inference under multiple loss functions, accurately evaluating performance under censoring, and demonstrating practical applicability to complex real-world data. This provides researchers with a more robust and flexible framework for analyzing a wider range of survival phenomena.

While the AFWE-L distribution can be utilized to employ reliability analysis, our paper focuses fundamentally on the application of survival analysis to model time-to-event data in the context of COVID-19. The ”event” is a health-related outcome, and our analysis aims to understand the time until this outcome occurs and the factors influencing it, which are the essential principles of survival analysis.

The paper is organized as follows: Section 2 delineates the derivation of the Bayes estimators for the unknown parameters, including the rf, hrf, and rhrf of the AFWE-L distribution, using both the SE loss function and the LINEX loss function under Type II censoring. It also acquires credible intervals (CIs). Section 3 provides a numerical example to showcase the theoretical findings derived from Bayesian estimation. Section 4 presents three uses of COVID-19 data in some countries. Ultimately, a thorough conclusive summary is presented in Section 5.

2. Bayesian Parameters and Functions Estimation

The present paper focuses on Bayesian estimation for the AFWE-L distribution due to its several key advantages within the context of time-to-event analysis and reliability assessment. The Bayesian framework provides a convenient method for incorporating existing knowledge or historical data about the distribution's parameters, which can be particularly valuable when dealing with limited datasets or established understanding of related phenomena. This allows for more informed and potentially more stable estimates. Also, Bayesian inference quantifies uncertainty through the posterior distribution and credible intervals, offering a more thorough assessment of estimation precision relative to point estimates. Furthermore, for complex and flexible distributions like the AFWE-L, Bayesian computational techniques facilitate parameter estimation and inference, which can be challenging with traditional fre-

quentist methods. The ability to incorporate different loss functions within the Bayesian framework also allows for a more detailed approach to estimation, reflecting varying costs associated with different types of errors. Finally, in the context of competing risks analysis, which is relevant to the utilization of the AFWE-L distribution to real-world data, Bayesian methods provide a coherent framework for modeling the probabilities of different failure events, potentially integrating prior information about these risks. For these reasons, Bayesian methods offer a reliable and flexible approach for analyzing the AFWE-L distribution and drawing meaningful conclusions from survival data.
In accordance with Type II censored samples, this section develops the Bayes estimators for the unknown parameters and the rf, hrf, and rhrf of the AFWE-L distribution using the SE loss function and the LINEX loss function. The CIs are also determined. The LF of the AFWE-L distribution is given below

$$L(\underline{\psi}|\underline{x}) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(x_{(i)};\underline{\psi})\right]\left[R(x_{(r)};\underline{\psi})\right]^{n-r}$$
$$= \frac{n!}{(n-r)!}\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^{2}}\right)w_{(i)}+\frac{\theta}{\lambda z_{(i)}}\right]\right\}\left[\prod_{i=1}^{r} z_{(i)}^{-\theta}\right] z_{(r)}^{-\theta(n-r)} exp\left\{-(n-r)w_{(r)}-\sum_{i=1}^{r} w_{(i)}\right\} \tag{2.1}$$

where

$$w_{(i)} = e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}, \qquad w_{(r)} = e^{\alpha x_{(r)}-\frac{\beta}{x_{(r)}}}, \qquad z_{(i)} = \left(1+\frac{x_{(i)}}{\lambda}\right), and \qquad z_{(r)} = \left(1+\frac{x_{(r)}}{\lambda}\right). \tag{2.2}$$

Assuming that the parameters $\underline{\psi} = (\alpha, \beta, \lambda, \theta) = (\psi_1, \psi_2, \psi_3, \psi_4)$ of the AFWE-L distribution are unknown random variables and the prior joint distribution assigned to the parameters α and β is independent of the prior joint distribution of parameters λ and θ. Consequently, the joint prior distribution of $\underline{\psi}$ can be obtained as follows:

$$\pi(\underline{\psi}) = \pi(\alpha, \beta)\pi(\lambda, \theta). \tag{2.3}$$

Assuming dependence between the parameters α and β we utilize a joint bivariate prior distribution previously employed by AL-Hussaini and Jaheen [39,40], Mousa and Jaheen [41], and Sharaf El-Deen et al. [42], expressed as

$$\pi(\alpha, \beta) = \pi(\alpha|\beta)\pi(\beta), \tag{2.4}$$

where

$$\pi(\alpha|\beta) = \frac{\alpha^{a_1}\beta^{a_1+1}}{\Gamma(a_1+1)b_1^{a_1+1}} e^{-\frac{\alpha\beta}{b_1}} \quad , \alpha, \beta > 0; a_1 > -1, b_1 > 0. \tag{2.5}$$

and β is assigned a marginal prior that follows a gamma distribution characterized by parameters a_1 and b_2 and the following pdf:

$$\pi(\beta) = \frac{\beta^{a_2-1}}{\Gamma(a_2)b_2^{a_2}} e^{-\frac{\beta}{b_2}}, \quad \beta > 0; a_2, b_2 > 0. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), the prior joint distribution of α and β can be obtained as follows:

$$\pi(\alpha,\beta) \propto \alpha^{a_1}\beta^{a_1+a_2}e^{-\beta(\frac{\alpha}{b_1}+\frac{1}{b_2})}, \quad \alpha,\beta > 0; a_1 > -1, a_2, b_1, b_2 > 0. \tag{2.7}$$

Similarly, by assuming that λ and θ are dependent with the following joint bivariate prior distribution:

$$\pi(\lambda,\theta) = \pi(\lambda|\theta)\pi(\theta), \tag{2.8}$$

where

$$\pi(\lambda|\theta) = \frac{\lambda^{a_3}\theta^{a_3+1}}{\Gamma(a_3+1)b_3^{a_3+1}}e^{-\frac{\lambda\theta}{b_3}}, \quad \lambda,\theta > 0; a_3 > -1, b_3 > 0, \tag{2.9}$$

and the marginal prior distribution of θ is a gamma prior distribution as:

$$\pi(\theta) = \frac{\theta^{a_4-1}}{\Gamma(a_4)b_4^{a_4}}e^{-\frac{\theta}{b_4}}, \quad \theta > 0; a_4, b_4 > 0. \tag{2.10}$$

Replacing (2.9) and (2.10) into (2.8) ,then, the prior joint distribution of λ and θ can be given by

$$\pi(\lambda,\theta) \propto \lambda^{a_3}\theta^{a_3+a_4}e^{-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)}, \quad \lambda,\theta > 0; a_3 > -1, a_4, b_3, b_4 > 0. \tag{2.11}$$

Substituting (2.7) and (2.11) into (2.3) the prior joint distribution of $\underline{\psi} = (\alpha,\beta,\lambda,\theta)$ can be obtained as given below

$$\pi(\underline{\psi}) \propto \alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}e^{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)}, \quad \underline{\psi} > 0; a_1, a_3 > -1, a_2, a_4, b_i > 0, \tag{2.12}$$

where a_j and b_j, $j = 1, 2, 3, 4$, are the hyperparameters of the prior joint distribution.
Therefore, the joint posterior distribution of $\underline{\psi} = (\alpha,\beta,\lambda,\theta)$ may be expressed as follows using (2.1) and (2.12):

$$\pi\left(\underline{\psi}|\underline{x}\right) = L\left(\underline{\psi}|\underline{x}\right)\pi\left(\underline{\psi}\right).$$

Then,

$$\begin{aligned}\pi\left(\underline{\psi}|\underline{x}\right) = A\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}&\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)w_{(i)}+\frac{\theta}{\lambda z_{(i)}}\right]\right\}\left[\prod_{i=1}^{r}z_{(i)}^{-\theta}\right]z_{(r)}^{-\theta(n-r)}\\ &\times\exp\left\{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)w_{(r)}-\sum_{i=1}^{r}w_{(i)}\right\},\end{aligned} \tag{2.13}$$

where A represents the normalization constant, defined as:

$$\begin{aligned}A^{-1} = \int_{\underline{\psi}}\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}&\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)w_{(i)}+\frac{\theta}{\lambda z_{(i)}}\right]\right\}\left[\prod_{i=1}^{r}z_{(i)}^{-\theta}\right]z_{(r)}^{-\theta(n-r)}\\ &\times\exp\left\{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_1}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)w_{(r)}-\sum_{i=1}^{r}w_{(i)}\right\}d\underline{\psi},\end{aligned} \tag{2.14}$$

where

$$\int_{\underline{\psi}} = \int_{\alpha=0}^{\infty}\int_{\beta=0}^{\infty}\int_{\lambda=0}^{\infty}\int_{\theta=0}^{\infty} \quad \textit{and} \quad d\underline{\psi} = d\alpha d\beta d\lambda d\theta. \tag{2.15}$$

The parameter ψ_j has the following marginal posterior distribution

$$\pi^*\left(\psi_j|\underline{x}\right) = \int_{\underline{\psi_l}}\pi\left(\underline{\psi}|\underline{x}\right)d\underline{\psi_l}, \quad l \neq j, \quad l, j = 1, 2, 3, 4. \tag{2.16}$$

2.1. Bayesian Estimation of Parameters and Functions

The present subsection provides the Bayes estimators for the unknown parameters and the rf, hrf, and rhrf of the AFWE-L distribution, through Type II censored samples along with the SE loss function and LINEX loss function.

I The squared error loss function
A loss function based on squared error has attractive properties, such as its symmetric nature, where equal weight is assigned to overestimation and underestimation. This makes it a suitable choice when there is no specific reason to penalize errors in one direction more heavily than the other. Additionally, its mathematical tractability often simplifies derivations.
With respect to the loss criterion using squared error, the Bayes estimator of a parameter ψ, represented by $\tilde{\psi}_{SE}$, is the mean of the posterior distribution which minimizes the posterior risk (PR), and is specified as

$$PR\left(\tilde{\psi}_{SE}\right) = E\left[L_1\left(\psi, \tilde{\psi}\right)\right] = \int_{\psi} L_1\left(\psi, \tilde{\psi}\right)\pi^*\left(\psi|\underline{x}\right)d\psi = E\left(\psi^2|\underline{x}\right) - \left[E\left(\psi|\underline{x}\right)\right]^2 = V\left(\psi|\underline{x}\right). \quad (2.17)$$

[For further information, refer to Berger [43] and Press [44]].
The Bayes estimator $\tilde{\psi}_{SE}$ is

$$\tilde{\psi}_{SE} = E\left(\psi|\underline{x}\right). \quad (2.18)$$

Accordingly, the Bayes estimators for the parameters $\underline{\psi} = (\psi_1, \psi_2, \psi_3, \psi_4)$ are obtained as the expected values of their respective marginal posterior distributions. Therefore, the Bayes estimators can be derived using the marginal posterior distribution given in (2.16) as follows:

$$\tilde{\psi}_{jSE} = E\left(\psi_j|\underline{x}\right) = \int_{\psi_j} \psi_j \pi^*\left(\psi_j|\underline{x}\right)d\psi_j = \int_{\underline{\psi}} \psi_j \pi\left(\underline{\psi}|\underline{x}\right)d\underline{\psi} = \int_{\underline{\psi}} \psi_j A\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\left\{\prod_{i=1}^{r}\left[\left(\alpha + \frac{\beta}{x_{(i)}^2}\right)w_{(i)} + \frac{\theta}{\lambda z_{(i)}}\right]\right\}$$
$$\times\left[\prod_{i=1}^{r} z_{(i)}^{-\theta}\right] z_{(r)}^{-\theta(n-r)} exp\left\{-\beta\left(\frac{\alpha}{b_1} + \frac{1}{b_2}\right) - \theta\left(\frac{\lambda}{b_3} + \frac{1}{b_4}\right) - (n-r)w_{(r)} - \sum_{i=1}^{r} w_{(i)}\right\} d\underline{\psi}, \quad (2.19)$$

Here, $j = 1, 2, 3, 4$, $\underline{\psi}$ is defined in Equation (1.6), A denotes the normalization constant as specified in Equation (2.14), while $w_{(i)}, w_{(r)}, z_{(i)}$, and $z_{(r)}$ are outlined in Equations (2.2) and (2.15), which also present the formulas for $\int_{\underline{\psi}}$ and $d\underline{\psi}$.
For Type II censored samples, the Bayes estimators of the rf, hrf, and rhrf may be obtained from Equations (1.3), (1.4), (1.5), and (2.13) as shown below:

$$\tilde{R}_{SE}(x_0) = E\left[R(x_0)|\underline{x}\right] = \int_{\underline{\psi}} R(x_0)\pi\left(\underline{\psi}|\underline{x}\right)d_{\underline{\psi}} = \int_{\underline{\psi}} A\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\left(1 + \frac{x_0}{\lambda}\right)^{-\theta}\left\{\prod_{i=1}^{r}\left[\left(\alpha + \frac{\beta}{x_{(i)}^2}\right)w_{(i)} + \frac{\theta}{\lambda z_{(i)}}\right]\right\}$$
$$\times\left[\prod_{i=1}^{r} z_{(i)}^{-\theta}\right] z_{(r)}^{-\theta(n-r)} exp\left\{-\beta\left(\frac{\alpha}{b_1} + \frac{1}{b_2}\right) - \theta\left(\frac{\lambda}{b_3} + \frac{1}{b_4}\right) - e^{\alpha x_0 - \frac{\beta}{x_0}} - (n-r)w_{(r)} - \sum_{i=1}^{r} w_{(i)}\right\} d\underline{\psi}, \quad (2.20)$$

$$\tilde{h}_{SE}(x_0) = E\left[h(x_0)|\underline{x}\right] = \int_{\underline{\psi}} h(x_0)\,\pi\left(\underline{\psi}|\underline{x}\right) d_{\underline{\psi}} = \int_{\underline{\psi}} A\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\left[\left(\alpha+\frac{\beta}{x_0^2}\right)e^{\alpha x_0-\frac{\beta}{x_0}}+\frac{\theta}{\lambda}\left(1+\frac{x_0}{\lambda}\right)^{-1}\right]\left[\prod_{i=1}^{r} z_{(i)}^{-\theta}\right]$$
$$\times z_{(r)}^{-\theta(n-r)}\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)w_{(i)}+\frac{\theta}{\lambda z_{(i)}}\right]\right\} exp\left\{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)\,w_{(r)}-\sum_{i=1}^{r} w_{(i)}\right\} d\underline{\psi}, \tag{2.21}$$

and

$$\tilde{r}_{SE}(x_0) = E\left[r(x_0)|\underline{x}\right] = \int_{\underline{\psi}} \frac{A\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\left[\left(\alpha+\frac{\beta}{x_0^2}\right)e^{\alpha x_0-\frac{\beta}{x_0}}+\frac{\theta}{\lambda}\left(1+\frac{x_0}{\lambda}\right)^{-1}\right]e^{-e^{\alpha x_0-\frac{\beta}{x_0}}}\left(1+\frac{x_0}{\lambda}\right)^{-\theta}}{1-e^{-e^{\alpha x_0-\frac{\beta}{x_0}}}\left(1+\frac{x_0}{\lambda}\right)^{-\theta}}\left[\prod_{i=1}^{r} z_{(i)}^{-\theta}\right]$$
$$\times z_{(r)}^{-\theta(n-r)}\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)w_{(i)}+\frac{\theta}{\lambda z_{(i)}}\right]\right\} exp\left\{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)\,w_{(r)}-\sum_{i=1}^{r} w_{(i)}\right\} d\underline{\psi}, \tag{2.22}$$

where $\underline{\psi}$ is given in Equation (1.6), A represents the normalization constant defined in Equation (2.14), $w_{(i)}, w_{(r)}, z_{(i)}$ and $z_{(r)}$ are defined in Equations (2.2) and (2.15) provides the expressions for $\int_{\underline{\psi}}$ and $d\underline{\psi}$.

Numerical computation of the Bayes estimates of the parameters, as well as the rf, hrf, and rhrf, can be achieved through the SE loss function using the Metropolis-Hastings algorithm, an MCMC (Markov Chain Monte Carlo) methods simulation technique.

II The linear-exponential loss function

The importance of the LINEX loss originates from its asymmetric nature, characterized by the shape parameter ' v '. This parameter allows for differential weighting of overestimation and underestimation. Specifically, a positive ' v ' penalizes overestimation, i.e., assigns more weight (and thus a higher penalty) to overestimation. While a negative ' v ' penalizes underestimation, i.e., assigns more weight (and thus a higher penalty) to underestimation. As the parameter 'v ' approaches zero, the LINEX loss function converges to approximately symmetric and behaves like the loss function based on squared error. This flexibility is crucial in situations where one type of estimation error has more serious consequences than the other.

The LINEX-based loss criterion specifies that the posterior risk (PR) associated with the Bayes estimator of ψ, denoted as $\tilde{\psi}_{LIN}$, is as follows:

$$PR\left(\tilde{\psi}_{LIN}\right) = e^{v\tilde{\psi}_{LIN}}E\left(e^{-v\tilde{\psi}_{LIN}}|\underline{x}\right) - v\left(\tilde{\psi}_{LIN} - \tilde{\psi}_{SE}\right) - 1. \tag{2.23}$$

For more details, see Varian [27], Zellner [28] and Jaheen [45].

Considering the LINEX-based loss criterion, the Bayes estimator that achieves the minimum of (2.23) is given by:

$$\tilde{\psi}_{LIN} = \frac{-1}{v} ln\left[E\left(e^{-v\psi}|\underline{x}\right)\right]. \tag{2.24}$$

Hence, The parameters' Bayes estimators $\underline{\psi} = (\psi_1, \psi_2, \psi_3, \psi_4)$ can be obtained using (2.24) as follows:

$$\tilde{\psi}_{jLIN} = \frac{-1}{v} ln\left[E\left(e^{-v\psi_j}|\underline{x}\right)\right], \qquad j = 1, 2, 3, 4, \tag{2.25}$$

where

$$E\left(e^{-v\psi_j}|\underline{x}\right) = \int_{\psi_j} e^{-v\psi_j}\pi^*\left(\psi_j|\underline{x}\right)d\psi_j = \int_{\underline{\psi}} e^{-v\psi_j}\pi\left(\underline{\psi}|\underline{x}\right)d\underline{\psi} = \int_{\underline{\psi}} \psi_j A\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)w_{(i)}+\frac{\theta}{\lambda z_{(i)}}\right]\right\}$$
$$\times\left[\prod_{i+1}^{r} z_{(i)}^{-\theta}\right]z_{(r)}^{-\theta(n-r)}exp\left\{-v\psi_j-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)w_{(r)}-\sum_{i+1}^{r}w_{(i)}\right\}d\underline{\psi}, \tag{2.26}$$

where $\underline{\psi}$ is given in (1.6), A represents the normalization constant determined in (2.14), $w_{(i)}, w_{(r)}, z_{(i)}$ and $z_{(r)}$ are defined in (2.2) and (2.15) provides the expressions for $\int_{\underline{\psi}}$ and $d\underline{\psi}$.
Using Type II censored samples, the estimators under the Bayesian framework for the rf, hrf, and rhrf can be derived from Equations (1.3), (1.4), (1.5), (2.13), and (2.24) as outlined below:

$$\tilde{R}_{LIN}(x_0) = \frac{-1}{v}ln\left[E\left(e^{vR(x_0)}|\underline{x}\right)\right], \tag{2.27}$$

where

$$E\left(e^{-vR(x_0)}|\underline{x}\right) = \int_{\underline{\psi}} A\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)w_{(i)}+\frac{\theta}{\lambda z_{(i)}}\right]\right\}\left[\prod_{i=1}^{r} z_{(i)}^{-\theta}\right]z_{(r)}^{-\theta(n-r)}$$
$$\times\exp\left\{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)w_{(i)}-\sum_{i=1}^{r}w_{(r)}-ve^{e^{\alpha x_0-\frac{\beta}{x_0}}}\left(1+\frac{x_0}{\lambda}\right)^{-\theta}\right\}d\underline{\psi}, \tag{2.28}$$

$$\tilde{h}_{LIN}(x_0) = \frac{-1}{v}ln\left[E\left(e^{-vh(x_0)}|\underline{x}\right)\right], \tag{2.29}$$

where

$$E\left(e^{-vh(x_0)}|\underline{x}\right) = \int_{\underline{\psi}} A\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)w_{(i)}+\frac{\theta}{\lambda z_{(i)}}\right]\right\}\left[\prod_{i=1}^{r} z_{(i)}^{-\theta}\right]z_{(r)}^{-\theta(n-r)}$$
$$\times\exp\left\{-v\left[\left(\alpha+\frac{\beta}{x_0^2}\right)e^{\alpha x_0-\frac{\beta}{x_0}}+\frac{\theta}{\lambda}\left(1+\frac{x_0}{\lambda}\right)^{-1}\right]-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)w_{(r)}-\sum_{i=1}^{r}w_{(i)}\right\}d\underline{\psi}, \tag{2.30}$$

and

$$\tilde{r}_{LIN}(x_0) = \frac{-1}{v}ln\left[E\left(e^{-vr(x_0)}|\underline{x}\right)\right], \tag{2.31}$$

where

$$E\left(e^{-vr(x_0)}|\underline{x}\right) = \int_{\underline{\psi}} A\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)w_{(i)}+\frac{\theta}{\lambda z_{(i)}}\right]\right\}\left[\prod_{i=1}^{r} z_{(i)}^{-\theta}\right]z_{(r)}^{-\theta(n-r)}$$
$$\times\exp\left\{-v\left(\frac{\left[\left(\alpha+\frac{\beta}{x_0^2}\right)e^{\alpha x_0-\frac{\beta}{x_0}}+\frac{\theta}{\lambda}\left(1+\frac{x_0}{\lambda}\right)^{-1}\right]e^{-e^{\alpha x_0-\frac{\beta}{x_0}}}\left(1+\frac{x_0}{\lambda}\right)^{-\theta}}{1-e^{-e^{\alpha x_0-\frac{\beta}{x_0}}}\left(1+\frac{x_0}{\lambda}\right)^{-\theta}}\right)\right\}$$
$$\times\exp\left\{-\beta\left(\frac{\lambda}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)\,w_{(r)}-\sum_{i=1}^{r}w_{(i)}\right\}d\underline{\psi}, \tag{2.32}$$

where $\underline{\psi}$ is determined in (1.6), A represents the normalization constant given in (2.14), $w_{(i)}, w_{(r)}, z_{(i)}$ and $z_{(r)}$ are defined in (2.2) and (2.15) provides the expressions $\int_{\underline{\psi}}$ and $d\underline{\psi}$.

The Metropolis-Hastings algorithm, an MCMC simulation technique implemented in the R programming language, can be used to numerically solve (2.25) - (2.32) to derive the Bayes estimates under Type II censoring, using the LINEX loss function.

2.2. *Credible intervals*

A two-sided, $\left(L_j\left(\underline{x}\right), U_j\left(\underline{x}\right)\right)$, $(1-\omega)$ 100% CIs of $\underline{\psi} = (\psi_1, \psi_2, \psi_3, \psi_4)$ are expressed as:

$$P\left[L_j(\underline{x}) < \psi_j < U_j(\underline{x})|\underline{x}\right] = \int_{L_j(\underline{x})}^{U_j(\underline{x})} \pi^*\left(\psi_j|\underline{x}\right) d\psi_j = 1-\omega, \qquad j = 1,2,3,4, \tag{2.33}$$

where the upper limit (UL) and lower limit (LL); $U_j(x)$ and $L_j(x)$, and $(1-\omega)$ is the credible coefficient. A symmetric CI for $\underline{\psi} = (\psi_1, \psi_2, \psi_3, \psi_4)$ with a confidence level of $(1-\omega)100\%$ can be obtained utilizing the marginal posterior distributions associated with the model parameters $\underline{\psi} = (\psi_1, \psi_2, \psi_3, \psi_4)$ given by (2.16) as follows:

$$P\left[\psi_j < L_j(\underline{x})|\underline{x}\right] = \int_{L_j(\underline{x})}^{\infty} \pi^*\left(\psi_j|\underline{x}\right) d\psi_j = 1-\frac{\omega}{2}, \qquad j = 1,2,3,4, \tag{2.34}$$

and

$$P\left[\psi_j < U_j(\underline{x})|\underline{x}\right] = \int_{U_j(\underline{x})}^{\infty} \pi^*\left(\psi_j|\underline{x}\right) d\psi_j = \frac{\omega}{2}, \qquad U = 1,2,3,4, \tag{2.35}$$

Equations (2.34) and (2.35) can be evaluated numerically to find the symmetric CI for $\underline{\psi} = (\psi_1, \psi_2, \psi_3, \psi_4)$ with a confidence level of $(1-\omega)100\%$ utilizing R programming language.

3. Numerical Illustration

This section presents a simulation study that evaluates the efficacy of the Bayes estimators for the unknown parameters rf, hrf, and rhrf of the AFWE-L distribution under Type II censoring. The adaptive Metropolis (AM) algorithm, which was introduced by Haario et al. [46], was employed in the R programming language to implement the study. The following procedure was employed:

The steps of the AM algorithm are outlined below:

Step 1. Choose an initial $(l \times 1)$ vector of values for $\underline{\psi}^{(0)}$.

Step 2. At each iteration ξ, where $\xi = 1, 2, ..., h$, a proposed value $\underline{\psi}^*$ is sampled from the candidate distribution $j_\xi(\underline{\psi}^*|\underline{\psi}^{(0)}, \underline{\psi}^{(1)}, ..., \underline{\psi}^{(\xi-1)})$. The algorithm utilizes a Gaussian candidate distribution with the current point as its mean $\underline{\psi}^{(\xi-1)}$, with a covariance matrix C_ξ that depends on the sequence of previous points $C_\xi(\underline{\psi}^{(0)}, \underline{\psi}^{(1)}, ..., \underline{\psi}^{(\xi-1)})$.

Step 3. Evaluate the acceptance rate:

$$AR = \frac{\pi\left(\underline{\psi}^*|\underline{x}\right)}{\pi\left(\underline{\psi}^{(\xi-1)}|\underline{x}\right)},$$

the posterior distribution $\pi\left(\underline{\psi}|\underline{x}\right)$ is considered without including the normalization constant.

Step 4. Retain $\underline{\psi}^{*}$ as $\underline{\psi}^{(\xi)}$ with probability $\min(AR, 1)$. If $\underline{\psi}^{*}$ is rejected, then set $\underline{\psi}^{(\xi)} = \underline{\psi}^{(\xi-1)}$. This acceptance decision can be implemented through simulating a random variable U from a uniform distribution. If the value U is less than or equal to AR, then $\underline{\psi}^{(\xi)}$ at iteration ξ is updated to $\underline{\psi}^{*}$; otherwise, it retains the value from the previous iteration, $\underline{\psi}^{(\xi)} = \underline{\psi}^{(\xi-1)}$.

Step 5. Execute Steps 2–4 repeatedly for h iterations, where h should be substantial enough to ensure stability of the results.

Step 6. A warm-up phase is applied to mitigate the influence of initial values, during which the first M simulated $\underline{\psi}$ values are rejected. Using the AM algorithm, the Bayes estimates of ψ_j, $j = 1, 2, 3, 4$, along with their corresponding posterior risks (PRs), can be derived within terms of both the SE loss and LINEX function as follows:

$$\tilde{\psi}_{jSE} = \frac{1}{h-M}\sum_{l=M+1}^{h}\psi_{jl},$$

$$PR\left(\tilde{\psi}_{jSE}\right) = \frac{1}{h-M}\sum_{l=M+1}^{h}\left(\psi_{jl} - \tilde{\psi}_{jSE}\right)^2,$$

$$\tilde{\psi}_{jLIN} = \frac{-1}{v}ln\left[\frac{1}{h-M}\sum_{l=M+1}^{h}exp\left(-v\psi_{jl}\right)\right],$$

end

$$PR\left(\tilde{\psi}_{jLIN}\right) = e^{v\tilde{\psi}_{jLIN}}\left[\frac{1}{h-M}\sum_{l=M+1}^{h}exp\left(-v\psi_{jl}\right)\right] - v\left(\tilde{\psi}_{jLIN} - \tilde{\psi}_{jSE}\right) - 1,$$

where $(\psi_{1l}, \psi_{2l}, \psi_{3l}, \psi_{4l}) = (\alpha_l, \beta_l, \lambda_l, \theta_l) = 1, 2, ..., h$, are drawn from the posterior distribution, with M denoting the warm-up phase.

The following steps are used to conduct the simulation study:

- The following procedure is employed to draw a simulated sample from the AFWE-L distribution:
    1. Setting the sample sizes as n=(30,60,100) along with the corresponding parameter values I. $(\alpha = 0.8, \beta = 1.15, \lambda = 1.5, \theta = 2)$, II. $(\alpha = 2, \beta = 4, \lambda = 0.3, \theta = 1.5)$
    2. The uniroot function in R programming language was utilized to numerically compute the quantile function of the AFWE-L distribution, as defined by:

    $$e^{\alpha x_u - \frac{\beta}{x_u}} + \theta ln\left(1 + \frac{x_u}{\lambda}\right) + ln\left(1 - u\right) = 0.$$

    hence, a simulated sample consisting of n observations are drawn.
- Applying the conditional gamma prior distributions specified in (2.3) and (2.7), the Bayes estimates are derived for the parameters α and λ and gamma prior distribution obtained in (2.4) and (2.8) for the parameters β and θ. Table 1 has specified the prior means and variances. Using the same approach Priors I and II are chosen with the same mean but differ in their variances. Similarly, Priors III and IV are designed with identical means yet distinct variances.
- Using $h = 22000$, and $M = 2000$. At each time, $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$ are sampled from the joint posterior distribution specified in (2.13).

Table 1. Hyperparameter Prior Means and Corresponding Variances

$\underline{\psi}$	Prior I		Prior II		Prior III		Prior IV	
	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
α	0.8	0.05	0.8	0.03	2	0.05	2	0.08
β	1.15	0.01	1.15	0.05	4	0.03	4	0.08
λ	1.5	0.08	1.5	0.1	0.3	0.05	0.3	0.08
θ	2	0.02	2	0.05	1.5	0.03	1.5	0.08

- Two distinct levels of Type II censoring (30%,0%) are considered in the simulation study.
- Estimate the model parameters using Bayesian methods with the SE loss function and LINEX loss function for ($v = 1.5, 0.5$) through the previously described AM algorithm and determine their corresponding PRs.
- Determine the Bayes estimate of the rf, hrf, and rhrf using the AM method and their PRs, using the SE and LINEX loss functions for ($v = 1.5, 0.5$) and time ($x_0 = 0.5, 1.5, 2$).
- Considering both the SE loss function and LINEX loss functions, the 95% CIs and their respective lengths are computed for $\underline{\psi} = (\alpha, \beta, c, k)$, rf, hrf, and rhrf.

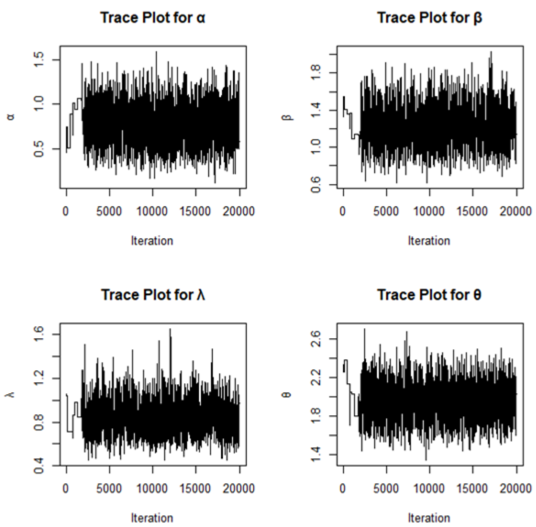

Figure 1. Trace plots of α, β, λ and θ for $n = 30, r = 30$

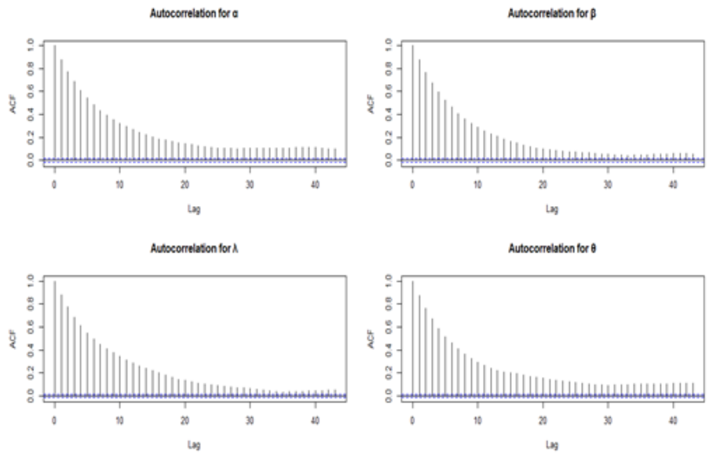

Figure 2. Autocorrelation plots of α, β, λ and θ for $n = 30, r = 30$

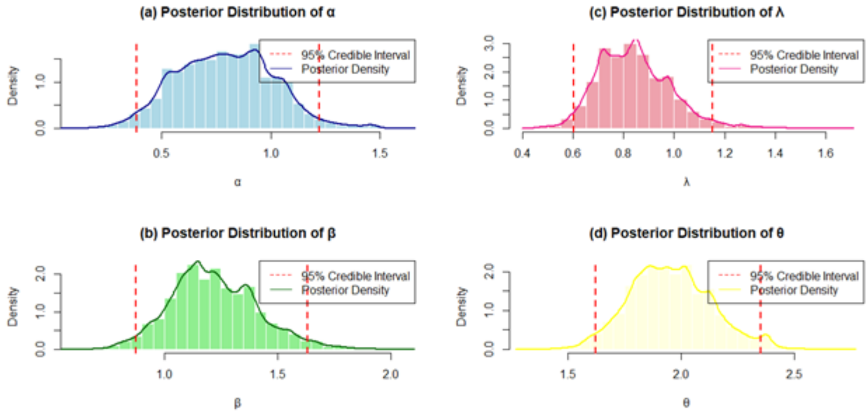

Figure 3. Posterior Density and histogram with 95% credible interval bounds for α, β, λ and θ for $(n = 30, r = 30)$.

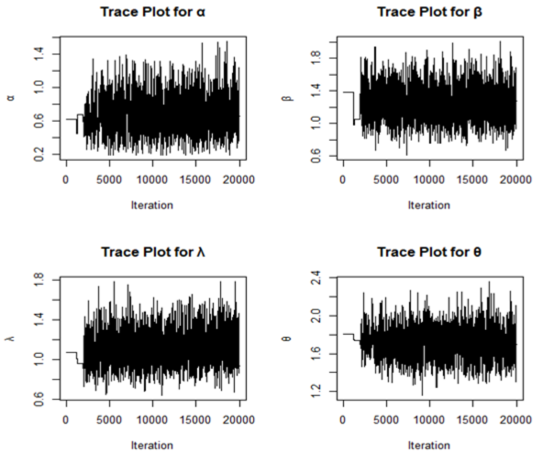

Figure 4. Trace plots of α, β, λ and θ for $n = 60, r = 42$

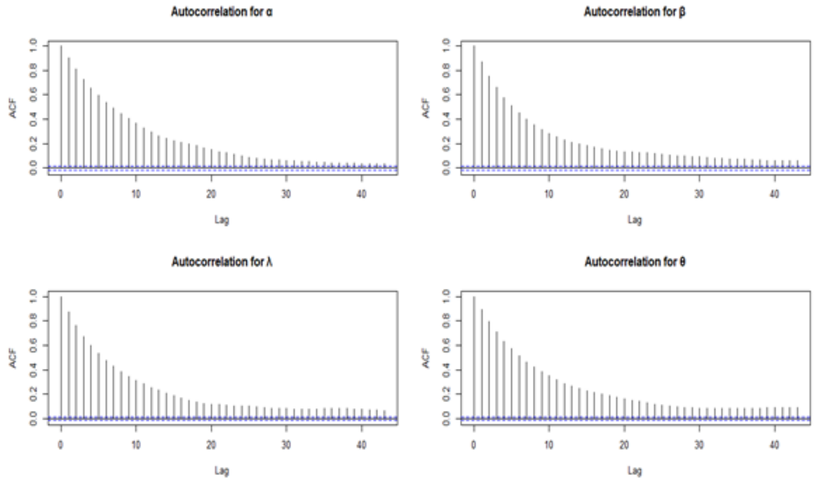

Figure 5. Autocorrelation plots of α, β, λ and θ for $n = 60, r = 42$

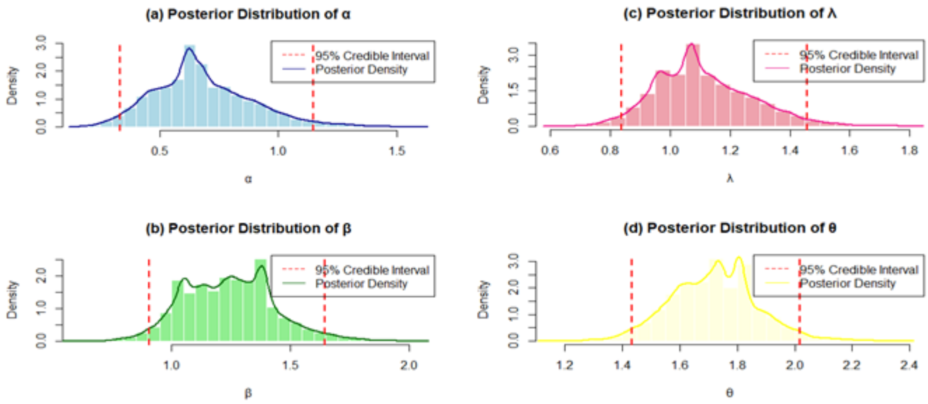

Figure 6. Posterior Density and histogram with 95% credible interval bounds for α, β, λ and θ for $(n = 60, r = 42)$.

Tables 2- 5 present the Bayes averages, PRs and the 95% credible limits of the parameters and lengths using the SE and LINEX loss functions for given $(\nu = 0.5, and 1.5)$ concerning Priors I, II, III and IV, under two levels (30% and 0%). Whereas Tables 6 - 17 exhibit Bayes averages, PRs and the 95% CI limits of the rf, hrf and rhrf and lengths via LINEX loss functions and SE for given $(\nu = 0.5, and 1.5)$ with time equatl to $(x_0 = 0.5, 1.5, 2)$ according to Priors I, II, III and IV and under the pre-specified two levels of Type II censoring.
Figures 1– 3 illustrate the convergence diagnostics, including the trace plots, autocorrelation plots, and histograms with posterior densities—for two representative MCMC chains with parameters $n = 30$ and $r = 30$. Similarly, Figures 4– 6 present the corresponding diagnostics for the case $n = 60$ and $r = 42$.

Table 2. For the AFWE-L distribution parameters ($v = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss functions. The interval lengths and 95% credible intervals are determined according to Prior I.

n	r	$\underline{\psi}$	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	α	0.7478	0.0431	0.7373	0.0111	0.7176	0.0361	1.1931	0.4197	0.7734
		β	1.1703	0.0094	1.1679	0.0024	1.1633	0.0072	1.3805	1.0046	0.3759
		λ	1.0356	0.0222	1.0301	0.0057	1.0194	0.0177	1.3711	0.7672	0.6039
		θ	1.8301	0.0174	1.8257	0.0044	1.817	0.0133	2.092	1.5661	0.5259
	30	α	0.783	0.031	0.7753	0.0078	0.7599	0.0239	1.1203	0.4541	0.6662
		β	1.1649	0.0091	1.1627	0.0023	1.1582	0.007	1.3683	0.992	0.3762
		λ	0.9518	0.0208	0.9467	0.0053	0.9367	0.0165	1.2682	0.702	0.5662
		θ	1.8921	0.0152	1.8883	0.0038	1.8808	0.0117	2.1327	1.6693	0.4634
60	42	α	0.6691	0.0257	0.6627	0.0065	0.6501	0.0201	1.0039	0.3714	0.6325
		β	1.1826	0.0086	1.1805	0.0022	1.1763	0.0066	1.375	1.0092	0.3658
		λ	1.0924	0.0181	1.0879	0.0046	1.079	0.014	1.3748	0.8349	0.5399
		θ	1.8167	0.0137	1.8133	0.0035	1.8065	0.0106	2.0617	1.6085	0.4532
	60	α	0.7877	0.0216	0.7823	0.0054	0.7716	0.0166	1.0854	0.505	0.5804
		β	1.1525	0.0079	1.1505	0.002	1.1466	0.006	1.336	0.98	0.3559
		λ	0.9905	0.0172	0.9862	0.0044	0.978	0.0135	1.2843	0.7683	0.5159
		θ	1.8808	0.015	1.877	0.0038	1.8695	0.0114	2.0984	1.6503	0.448
100	70	α	0.6864	0.0298	0.679	0.0076	0.6647	0.0236	1.0173	0.389	0.6283
		β	1.195	0.0081	1.193	0.002	1.1889	0.0062	1.3677	1.022	0.3457
		λ	1.1167	0.0164	1.1127	0.0042	1.1049	0.013	1.3908	0.8927	0.4981
		θ	1.8153	0.0114	1.8125	0.0029	1.8069	0.0087	2.019	1.6151	0.4038
	100	α	0.9046	0.0079	0.9027	0.002	0.8987	0.006	1.0868	0.7304	0.3564
		β	1.1441	0.0036	1.1432	0.0009	1.1414	0.0028	1.2681	1.031	0.2371
		λ	1.0831	0.0124	1.0801	0.0031	1.074	0.0096	1.3144	0.8907	0.4236
		θ	1.7816	0.0046	1.7805	0.0012	1.7782	0.0035	1.9185	1.6648	0.2538

Table 14. For the AFWE-L distribution parameters ($v = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior III, and for $x_0 = 2$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.0306	0.0002	0.2111	0.0004	0.0305	0.0002	0.0604	0.0126	0.0479
		$h(x_0;\underline{\psi})$	1.2267	0.0157	1.2228	0.0019	1.2153	0.0171	1.4998	1.009	0.4908
		$r(x_0;\underline{\psi})$	0.0381	0.0002	0.038	2.5024E-05	0.0379	0.0002	0.072	0.0169	0.055
	30	$R(x_0;\underline{\psi})$	0.0311	0.0001	0.2197	0.0003	0.031	0.0001	0.0582	0.0138	0.0444
		$h(x_0;\underline{\psi})$	1.2349	0.0186	1.2304	0.0023	1.2218	0.0198	1.5174	1.0266	0.4909
		$r(x_0;\underline{\psi})$	0.0391	0.0002	0.0391	2.3055E-05	0.039	0.0002	0.0709	0.019	0.0519
60	42	$R(x_0;\underline{\psi})$	0.0347	0.0001	0.2313	0.0002	0.0346	0.0001	0.0569	0.0169	0.04
		$h(x_0;\underline{\psi})$	1.2096	0.0145	1.206	0.0018	1.1991	0.0157	1.4832	1.0034	0.4798
		$r(x_0;\underline{\psi})$	0.0429	0.0001	0.0428	1.6941E-05	0.0428	0.0002	0.0673	0.0225	0.0448
	60	$R(x_0;\underline{\psi})$	0.0326	8.7688E-05	0.2307	0.0002	0.0326	9.8227E-05	0.0544	0.0175	0.0368
		$h(x_0;\underline{\psi})$	1.2476	0.0152	1.2438	0.0019	1.2366	0.0165	1.5278	1.0274	0.5004
		$r(x_0;\underline{\psi})$	0.0416	0.0001	0.0416	1.4242E-05	0.0415	0.0001	0.0664	0.0239	0.0425
100	70	$R(x_0;\underline{\psi})$	0.0315	8.7598E-05	0.2255	0.0002	0.0314	9.8235E-05	0.0517	0.0183	0.0334
		$h(x_0;\underline{\psi})$	1.2406	0.015	1.2369	0.0019	1.2295	0.0166	1.4863	1.0205	0.4658
		$r(x_0;\underline{\psi})$	0.0397	0.0001	0.0397	1.2763E-05	0.0396	0.0001	0.0614	0.0242	0.0372
	100	$R(x_0;\underline{\psi})$	0.0257	4.3839E-05	0.1949	0.0001	0.0257	4.9196E-05	0.041	0.0143	0.0267
		$h(x_0;\underline{\psi})$	1.2711	0.0171	1.2669	0.0021	1.259	0.0182	1.5707	1.0497	0.521
		$r(x_0;\underline{\psi})$	0.0332	5.7711E-05	0.0332	7.2070E-06	0.0331	6.4738E-05	0.0505	0.0203	0.0302

Table 3. For the AFWE-L distribution parameters ($\nu = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss functions. The interval lengths and 95% credible intervals are determined according to Prior II.

n	r	$\underline{\psi}$	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	α	0.681	0.0503	0.6688	0.013	0.6458	0.0423	1.183	0.3165	0.8665
		β	1.2043	0.0428	1.1937	0.011	1.1735	0.0351	1.6539	0.8416	0.8123
		λ	0.9491	0.0283	0.9422	0.0072	0.9288	0.0228	1.284	0.6712	0.6129
		θ	1.8532	0.0313	1.8455	0.0079	1.8303	0.0248	2.2238	1.5264	0.6974
	30	α	0.8361	0.044	0.825	0.0111	0.803	0.0344	1.2266	0.4177	0.809
		β	1.1978	0.0333	1.1896	0.0085	1.1736	0.0268	1.5866	0.8679	0.7186
		λ	1.0241	0.0297	1.0167	0.0076	1.0027	0.024	1.3932	0.7317	0.6614
		θ	1.7762	0.0311	1.7685	0.0079	1.7532	0.0244	2.1418	1.443	0.6988
60	42	α	0.6797	0.0417	0.6695	0.0107	0.6502	0.0347	1.1485	0.3324	0.8162
		β	1.2421	0.0356	1.2333	0.0091	1.2162	0.0284	1.6456	0.9064	0.7392
		λ	1.1089	0.0252	1.1027	0.0064	1.0908	0.0202	1.4584	0.8372	0.6212
		θ	1.7184	0.0219	1.713	0.0055	1.7022	0.017	2.0189	1.4323	0.5866
	60	α	0.8583	0.0269	0.8515	0.0067	0.8376	0.0202	1.1412	0.5186	0.6225
		β	1.1314	0.0255	1.1251	0.0065	1.1131	0.0205	1.4879	0.8528	0.6351
		λ	1.122	0.0247	1.1158	0.0063	1.1037	0.0193	1.4474	0.8336	0.6137
		θ	1.7262	0.0225	1.7206	0.0057	1.7097	0.0176	2.0477	1.4465	0.6012
100	70	α	0.5643	0.0262	0.5579	0.0067	0.5453	0.0208	0.8967	0.2875	0.6092
		β	1.3706	0.0331	1.3624	0.0084	1.3464	0.0265	1.7526	1.0394	0.7132
		λ	1.2455	0.0242	1.2395	0.0061	1.2277	0.019	1.5701	0.9713	0.5988
		θ	1.6231	0.0193	1.6183	0.0049	1.6087	0.0148	1.9035	1.3589	0.5445
	100	α	0.9178	0.0172	0.9135	0.0043	0.9048	0.013	1.1706	0.6387	0.5319
		β	1.213	0.0231	1.2073	0.0059	1.1961	0.0182	1.5387	0.9343	0.6044
		λ	1.1723	0.0217	1.167	0.0055	1.1564	0.017	1.4862	0.8937	0.5925
		θ	1.6871	0.0191	1.6823	0.0048	1.6727	0.0146	1.9565	1.412	0.5445

Table 15. For the AFWE-L distribution parameters ($\nu = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior IV, and for $x_0 = 0.5$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.285	0.0043	0.2839	0.0005	0.2818	0.0048	0.4284	0.1698	0.2586
		$h(x_0;\underline{\psi})$	1.7318	0.0679	1.7149	0.0084	1.682	0.0747	2.2362	1.2534	0.9827
		$r(x_0;\underline{\psi})$	0.6795	0.0168	0.6753	0.0021	0.6671	0.0185	0.9564	0.4389	0.5175
	30	$R(x_0;\underline{\psi})$	0.226	0.003	0.2253	0.0004	0.2238	0.0033	0.3363	0.1267	0.2096
		$h(x_0;\underline{\psi})$	1.9297	0.0583	1.9153	0.0072	1.8874	0.0635	2.4353	1.4829	0.9524
		$r(x_0;\underline{\psi})$	0.5566	0.0138	0.5532	0.0017	0.5463	0.0155	0.7841	0.3387	0.4453
60	42	$R(x_0;\underline{\psi})$	0.2431	0.0019	0.2426	0.0002	0.2417	0.0021	0.3369	0.1674	0.1695
		$h(x_0;\underline{\psi})$	1.9084	0.0444	1.8974	0.0055	1.8754	0.0495	2.3221	1.5036	0.8186
		$r(x_0;\underline{\psi})$	0.6068	0.0077	0.6049	0.001	0.6011	0.0086	0.7892	0.4462	0.343
	60	$R(x_0;\underline{\psi})$	0.1947	0.0013	0.1944	0.0002	0.1937	0.0015	0.2673	0.1239	0.1434
		$h(x_0;\underline{\psi})$	1.6355	0.0296	1.6281	0.0037	1.6136	0.0329	1.9895	1.3038	0.6857
		$r(x_0;\underline{\psi})$	0.504	0.0059	0.5025	0.0007	0.4996	0.0066	0.6622	0.355	0.3072
100	70	$R(x_0;\underline{\psi})$	0.2595	0.0015	0.2592	0.0002	0.2584	0.0017	0.3586	0.1876	0.171
		$h(x_0;\underline{\psi})$	1.7817	0.037	1.7726	0.0046	1.7546	0.0406	2.1943	1.3935	0.8008
		$r(x_0;\underline{\psi})$	0.6198	0.0061	0.6183	0.0008	0.6153	0.0068	0.8005	0.4813	0.3192
	100	$R(x_0;\underline{\psi})$	0.1916	0.0009	0.1913	0.0001	0.1909	0.001	0.2531	0.1383	0.1148
		$h(x_0;\underline{\psi})$	2.0927	0.0352	2.084	0.0044	2.0669	0.0388	2.5074	1.7482	0.7592
		$r(x_0;\underline{\psi})$	0.4927	0.0042	0.4916	0.0005	0.4896	0.0047	0.625	0.3736	0.2515

Table 4. For the AFWE-L distribution parameters ($\nu = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior III

n	r	$\underline{\psi}$	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	α	0.505	0.0039	0.5041	0.001	0.5022	0.0029	0.6395	0.3954	0.2441
		β	4.0105	0.0346	4.0018	0.0087	3.9841	0.0263	4.3549	3.6367	0.7181
		λ	0.259	0.0057	0.2576	0.0015	0.2549	0.0045	0.4438	0.1433	0.3005
		θ	1.4918	0.028	1.4849	0.0071	1.4716	0.0223	1.8311	1.2242	0.6068
	30	α	0.5056	0.0036	0.5047	0.0009	0.5029	0.0028	0.6343	0.3995	0.2348
		β	4.0062	0.0324	3.9981	0.0081	3.9816	0.0247	4.3181	3.641	0.6771
		λ	0.2998	0.007	0.2981	0.0018	0.2947	0.0054	0.4945	0.1851	0.3094
		θ	1.5312	0.0263	1.5247	0.0067	1.512	0.0207	1.8619	1.2576	0.6043
60	42	α	0.509	0.0032	0.5082	0.0008	0.5066	0.0024	0.6251	0.4006	0.2245
		β	4.0048	0.0268	3.9981	0.0068	3.9848	0.0208	4.343	3.6715	0.6716
		λ	0.2914	0.0045	0.2902	0.0011	0.288	0.0035	0.4431	0.1876	0.2555
		θ	1.4606	0.0304	1.4531	0.0077	1.4383	0.0238	1.8077	1.1956	0.6121
	60	α	0.5156	0.0035	0.5148	0.0009	0.5131	0.0027	0.6418	0.4106	0.2313
		β	3.971	0.0262	3.9645	0.0066	3.9516	0.0205	4.3003	3.6454	0.6548
		λ	0.3095	0.0044	0.3084	0.0011	0.3063	0.0033	0.4585	0.1974	0.2611
		θ	1.5191	0.0223	1.5136	0.0056	1.5026	0.0173	1.8198	1.2408	0.5791
100	70	α	0.5054	0.0033	0.5046	0.0008	0.503	0.0024	0.6262	0.4008	0.2254
		β	3.9848	0.0323	3.9767	0.0081	3.9606	0.025	4.3353	3.6516	0.6836
		λ	0.3088	0.0033	0.308	0.0008	0.3064	0.0025	0.4376	0.2134	0.2242
		θ	1.5456	0.026	1.5391	0.0066	1.5263	0.0202	1.8352	1.2466	0.5886
	100	α	0.5266	0.0036	0.5257	0.0009	0.5239	0.0028	0.6546	0.4129	0.2417
		β	3.9433	0.0281	3.9363	0.0071	3.9225	0.0219	4.2785	3.6308	0.6477
		λ	0.3661	0.0044	0.3651	0.0011	0.3629	0.0034	0.5223	0.2516	0.2707
		θ	1.5438	0.0222	1.5382	0.0056	1.5275	0.0174	1.8454	1.2825	0.5628

Table 16. For the AFWE-L distribution parameters ($\nu = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior IV, and for $x_0 = 0.5$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.0808	0.0009	0.0805	0.0001	0.0801	0.0011	0.1528	0.035	0.1178
		$h(x_0;\underline{\psi})$	1.1578	0.0202	1.1528	0.0025	1.1431	0.0221	1.4776	0.8972	0.5805
		$r(x_0;\underline{\psi})$	0.0995	0.0011	0.0992	0.0001	0.0987	0.0013	0.1802	0.0482	0.132
	30	$R(x_0;\underline{\psi})$	0.0572	0.0004	0.0571	5.4460E-05	0.0569	0.0005	0.1046	0.0241	0.0805
		$h(x_0;\underline{\psi})$	1.2246	0.0189	1.2199	0.0023	1.2107	0.0208	1.5132	0.9745	0.5387
		$r(x_0;\underline{\psi})$	0.0729	0.0006	0.0728	7.0403E-05	0.0725	0.0006	0.1252	0.0342	0.091
60	42	$R(x_0;\underline{\psi})$	0.0609	0.0003	0.0608	4.2697E-05	0.0606	0.0004	0.1052	0.032	0.0732
		$h(x_0;\underline{\psi})$	1.2267	0.0175	1.2224	0.0022	1.2138	0.0193	1.5118	0.9805	0.5313
		$r(x_0;\underline{\psi})$	0.0781	0.0004	0.078	5.1030E-05	0.0778	0.0005	0.1247	0.0454	0.0794
	60	$R(x_0;\underline{\psi})$	0.0435	0.0002	0.0435	2.1977E-05	0.0434	0.0002	0.0732	0.0201	0.0531
		$h(x_0;\underline{\psi})$	1.3128	0.0197	1.3079	0.0024	1.2986	0.0213	1.6197	1.0639	0.5558
		$r(x_0;\underline{\psi})$	0.0587	0.0002	0.0586	2.9377E-05	0.0585	0.0003	0.0926	0.0306	0.0621
100	70	$R(x_0;\underline{\psi})$	0.0467	0.0002	0.0467	2.5289E-05	0.0466	0.0002	0.0787	0.0242	0.0545
		$h(x_0;\underline{\psi})$	1.236	0.0202	1.231	0.0025	1.2211	0.0223	1.5418	0.9702	0.5716
		$r(x_0;\underline{\psi})$	0.0592	0.0002	0.0592	2.6049E-05	0.0591	0.0002	0.0898	0.0349	0.0549
	100	$R(x_0;\underline{\psi})$	0.0431	0.0001	0.043	1.5758E-05	0.043	0.0001	0.0694	0.0243	0.0451
		$h(x_0;\underline{\psi})$	1.3059	0.0201	1.3009	0.0025	1.2912	0.022	1.6176	1.0484	0.5693
		$r(x_0;\underline{\psi})$	0.0578	0.0002	0.0577	1.8897E-05	0.0577	0.0002	0.0856	0.0363	0.0492

Table 5. For the AFWE-L distribution parameters ($\nu = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior IV

n	r	ψ	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	α	0.5116	0.0057	0.5102	0.0014	0.5074	0.0044	0.6728	0.3782	0.2946
		β	4.0348	0.0915	4.012	0.0234	3.9669	0.0748	4.5724	3.4702	1.1022
		λ	0.3994	0.0128	0.3962	0.0032	0.3902	0.0101	0.6626	0.2076	0.455
		θ	1.5406	0.0603	1.5256	0.0154	1.4966	0.0491	2.0228	1.0967	0.9262
	30	α	0.5182	0.0063	0.5166	0.0016	0.5135	0.0049	0.6897	0.3761	0.3136
		β	3.946	0.0757	3.9273	0.0195	3.8911	0.0633	4.5117	3.4427	1.0691
		λ	0.3163	0.0085	0.3142	0.0022	0.3101	0.0066	0.5029	0.1598	0.3431
		θ	1.5615	0.0477	1.5496	0.0121	1.5257	0.0375	2.0075	1.1333	0.8742
60	42	α	0.5053	0.006	0.5038	0.0015	0.5009	0.0046	0.6724	0.3723	0.3001
		β	4.013	0.0731	3.9947	0.0186	3.9582	0.059	4.5451	3.4794	1.0657
		λ	0.3678	0.0069	0.3661	0.0017	0.3628	0.0053	0.5536	0.2227	0.3308
		θ	1.6486	0.055	1.6349	0.0141	1.6084	0.0452	2.151	1.1992	0.9519
	60	α	0.5199	0.0059	0.5184	0.0015	0.5155	0.0045	0.6921	0.3768	0.3153
		β	3.9144	0.0729	3.8963	0.0186	3.8601	0.0597	4.4663	3.3931	1.0732
		λ	0.3049	0.0047	0.3037	0.0012	0.3014	0.0036	0.4625	0.1894	0.2731
		θ	1.5633	0.0485	1.5514	0.0125	1.5287	0.04	2.0364	1.195	0.8414
100	70	α	0.5077	0.0058	0.5062	0.0015	0.5034	0.0045	0.6659	0.3748	0.2912
		β	4.0147	0.0722	3.9967	0.0185	3.9613	0.0592	4.5682	3.4966	1.0716
		λ	0.2767	0.0048	0.2755	0.0012	0.2732	0.0037	0.4306	0.1488	0.2818
		θ	1.5833	0.0528	1.5702	0.0135	1.5449	0.0433	2.0639	1.1513	0.9126
	100	α	0.5477	0.0068	0.546	0.0017	0.5426	0.0052	0.7214	0.3953	0.3261
		β	3.8848	0.0661	3.8683	0.0169	3.8359	0.0537	4.4155	3.393	1.0225
		λ	0.2901	0.0043	0.289	0.0011	0.2869	0.0033	0.4364	0.1765	0.26
		θ	1.6735	0.048	1.6616	0.0123	1.6389	0.0394	2.1504	1.2858	0.8646

Concluding remarks

From Tables 2 – 17, it is noticeable that:

- With an increase in the sample size n and a decrease in the censoring level, the averages of the Bayes estimates of the AFWE-L distribution parameters converge more closely to the actual population parameter values for both loss functions considered. This reflects greater knowledge about the population under study, resulting in more accurate and stable estimates. Similarly, a lower censoring level ensures that the available data provides a clearer representation of the distribution's characteristics. Consequently, the estimation process becomes more accurate. Thus, with larger sample sizes and lower censoring levels, the Bayes estimates of the distribution's characteristics become increasingly accurate and reliable, regardless of whether symmetric (SE) or asymmetric (LINEX) loss function is used.
- In most instances presented in the table, the PRs associated with the Bayes estimates of the AFWE-L distribution parameters and rf, hrf, and rhrf tend to decrease as the sample size n increases and the censoring level decreases. This implies a reduction in the error associated with the Bayes estimates, thereby highlighting the increased precision achieved through larger sample sizes and lower censoring levels. This improved precision of estimates and reduced uncertainty contribute to more reliable and robust statistical inferences.
- As the sample size increases and the censoring level decreases, the lengths of the CIs of parameters and rf, hrf and rhrf of the AFWE-L distribution decrease, resulting in narrower CIs in most cases. This signifies an enhancement in the precision of the Bayes estimates for the key reliability characteristics of the distribution. This reduction in the width of the CIs implies a decreased

Table 6. For the AFWE-L distribution parameters ($v = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior I, and for $x_0 = 1.5$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.4187	0.002	0.4182	0.0003	0.4172	0.0023	0.5099	0.3306	0.1793
		$h(x_0;\underline{\psi})$	1.9797	0.0441	1.9688	0.0054	1.9477	0.0479	2.4153	1.5946	0.8208
		$r(x_0;\underline{\psi})$	1.4236	0.0276	1.4168	0.0034	1.4034	0.0303	1.7446	1.1116	0.633
	30	$R(x_0;\underline{\psi})$	0.3857	0.0022	0.3852	0.0003	0.3841	0.0024	0.4797	0.2932	0.1865
		$h(x_0;\underline{\psi})$	2.1127	0.0436	2.1018	0.0054	2.0805	0.0483	2.5386	1.7291	0.8094
		$r(x_0;\underline{\psi})$	1.324	0.026	1.3176	0.0032	1.3048	0.0289	1.6516	1.0186	0.633
60	42	$R(x_0;\underline{\psi})$	0.4391	0.0015	0.4126	0.0001	0.4122	0.001	0.4639	0.3494	0.1145
		$h(x_0;\underline{\psi})$	1.8674	0.0261	1.9798	0.0028	1.9693	0.0241	2.3488	1.7462	0.6027
		$r(x_0;\underline{\psi})$	1.461	0.019	1.3916	0.0017	1.3846	0.0157	1.6248	1.1566	0.4682
	60	$R(x_0;\underline{\psi})$	0.3968	0.0016	0.3964	0.0002	0.3956	0.0018	0.4754	0.3292	0.1462
		$h(x_0;\underline{\psi})$	2.0795	0.0297	2.0721	0.0037	2.0574	0.0333	2.3994	1.7507	0.6487
		$r(x_0;\underline{\psi})$	1.3671	0.0213	1.3618	0.0026	1.3514	0.0235	1.6597	1.1309	0.5288
100	70	$R(x_0;\underline{\psi})$	0.4489	0.0002	0.4488	1.9939E-05	0.4488	0.0002	0.4701	0.4198	0.0502
		$h(x_0;\underline{\psi})$	1.9065	0.0123	1.9035	0.0015	1.8976	0.0134	2.1521	1.731	0.4211
		$r(x_0;\underline{\psi})$	1.5542	0.013	1.551	0.0016	1.5445	0.0146	1.7679	1.3374	0.4304
	100	$R(x_0;\underline{\psi})$	0.4323	0.0003	0.4323	3.409E-05	0.4321	0.0003	0.464	0.4009	0.0631
		$h(x_0;\underline{\psi})$	2.0012	0.0037	2.0003	0.0005	1.9985	0.0041	2.1154	1.8826	0.2327
		$r(x_0;\underline{\psi})$	1.524	0.0033	1.5232	0.0004	1.5216	0.0037	1.633	1.415	0.218

Table 7. For the AFWE-L distribution parameters ($v = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior I, and for $x_0 = 0.5$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.0489	0.0005	0.0487	6.0228E-05	0.0485	0.0005	0.0955	0.0123	0.0832
		$h(x_0;\underline{\psi})$	2.7226	1.2713	2.4901	0.1163	2.255	0.7015	5.5875	1.5113	4.0762
		$r(x_0;\underline{\psi})$	0.1191	0.0009	0.1188	0.0001	0.1184	0.001	0.18	0.0616	0.1184
	30	$R(x_0;\underline{\psi})$	0.0386	0.0003	0.0385	3.7011E-05	0.0383	0.0003	0.0777	0.0124	0.0653
		$h(x_0;\underline{\psi})$	2.8578	0.7186	2.6991	0.0793	2.47	0.5817	4.8335	1.6404	3.1931
		$r(x_0;\underline{\psi})$	0.1024	0.0007	0.1022	8.2309E-05	0.1019	0.0007	0.1565	0.055	0.1015
60	42	$R(x_0;\underline{\psi})$	0.0483	0.0003	0.0482	4.1705E-05	0.0481	0.0004	0.0848	0.0123	0.0725
		$h(x_0;\underline{\psi})$	2.6258	0.9655	2.4504	0.0877	2.2557	0.5552	5.1638	1.5127	3.6512
		$r(x_0;\underline{\psi})$	0.1174	0.0005	0.1173	5.9986E-05	0.1171	0.0005	0.1515	0.0645	0.087
	60	$R(x_0;\underline{\psi})$	0.0394	0.0002	0.0394	2.6301E-05	0.0392	0.0002	0.0723	0.0159	0.0564
		$h(x_0;\underline{\psi})$	2.8258	0.5072	2.713	0.0564	2.5414	0.4267	4.521	1.7506	2.7703
		$r(x_0;\underline{\psi})$	0.1073	0.0005	0.1071	6.2075E-05	0.1069	0.0006	0.1542	0.0685	0.0857
100	70	$R(x_0;\underline{\psi})$	0.0623	0.0003	0.0622	3.9719E-05	0.0621	0.0004	0.0887	0.0232	0.0655
		$h(x_0;\underline{\psi})$	2.4181	0.6665	2.2845	0.0668	2.1207	0.4462	4.5833	1.5165	3.0668
		$r(x_0;\underline{\psi})$	0.1462	0.0002	0.1462	2.3054E-05	0.1461	0.0002	0.1632	0.1072	0.056
	100	$R(x_0;\underline{\psi})$	0.0352	6.3990E-05	0.0352	7.9937E-06	0.0352	7.1852E-05	0.0533	0.0211	0.0323
		$h(x_0;\underline{\psi})$	0.5643	0.0262	0.5579	0.0067	0.5453	0.0208	0.8967	0.2875	0.6092
		$r(x_0;\underline{\psi})$	1.3706	0.0331	1.3624	8.4209E-03	1.3464	0.0265	1.7526	1.0394	0.7132

Table 8. For the AFWE-L distribution parameters ($\nu = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior I, and for $x_0 = 2$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0; \underline{\psi})$	0.5116	0.0057	0.5102	1.4317E-03	0.5074	0.0044	0.6728	0.3782	0.2946
		$h(x_0; \underline{\psi})$	4.0348	9.1518E-02	4.012	0.0234	3.9669	7.4825E-02	4.5724	3.4702	1.1022
		$r(x_0; \underline{\psi})$	0.3994	0.0128	0.3962	0.0032	0.3902	0.0101	0.6626	0.2076	0.455
	30	$R(x_0; \underline{\psi})$	1.5406	0.0603	1.5256	1.5411E-02	1.4966	0.0491	2.0228	1.0967	0.9262
		$h(x_0; \underline{\psi})$	0.5182	0.0063	0.5166	1.5930E-03	0.5135	0.0049	0.6897	0.3761	0.3136
		$r(x_0; \underline{\psi})$	3.946	7.5729E-02	3.9273	0.0195	3.8911	6.3308E-02	4.5117	3.4427	1.0691
60	42	$R(x_0; \underline{\psi})$	0.3163	0.0085	0.3142	0.0022	0.3101	0.0066	0.5029	0.1598	0.3431
		$h(x_0; \underline{\psi})$	1.5615	0.0477	1.5496	1.2055E-02	1.5257	0.0375	2.0075	1.1333	0.8742
		$r(x_0; \underline{\psi})$	0.5053	0.006	0.5038	1.4996E-03	0.5009	0.0046	0.6724	0.3723	0.3001
	60	$R(x_0; \underline{\psi})$	4.013	7.3052E-02	3.9947	0.0186	3.9582	5.8996E-02	4.5451	3.4794	1.0657
		$h(x_0; \underline{\psi})$	0.3678	0.0069	0.3661	0.0017	0.3628	0.0053	0.5536	0.2227	0.3308
		$r(x_0; \underline{\psi})$	1.6486	0.055	1.6349	1.4095E-02	1.6084	0.0452	2.151	1.1992	0.9519
100	70	$R(x_0; \underline{\psi})$	0.0161	0.0001	0.4357	3.2134E-05	0.016	0.0001	0.0412	0.0009	0.0403
		$h(x_0; \underline{\psi})$	3.3736	3.0896	2.9201	0.2267	2.5226	1.2765	7.7879	1.5029	6.285
		$r(x_0; \underline{\psi})$	0.0411	0.0002	0.041	3.0258E-05	0.0409	0.0003	0.0657	0.0068	0.059
	100	$R(x_0; \underline{\psi})$	0.0055	9.4550E-06	0.4323	3.4090E-05	0.0055	1.0616E-05	0.0134	0.0013	0.0121
		$h(x_0; \underline{\psi})$	4.7812	1.0486	4.5498	0.1157	4.2002	0.8715	7.1752	3.0983	4.0769
		$r(x_0; \underline{\psi})$	0.0238	6.9064E-05	0.0238	8.6278E-06	0.0238	7.7556E-05	0.0423	0.0091	0.0332

Table 9. For the AFWE-L distribution parameters ($\nu = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior II, and for $x_0 = 0.5$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0; \underline{\psi})$	0.394	0.0034	0.3931	0.0004	0.3914	0.0038	0.5168	0.2796	0.2372
		$h(x_0; \underline{\psi})$	2.0281	0.0873	2.0068	0.0107	1.9669	0.0918	2.6991	1.5147	1.1845
		$r(x_0; \underline{\psi})$	1.3105	0.0377	1.3012	0.0047	1.283	0.0413	1.7268	0.9501	0.7767
	30	$R(x_0; \underline{\psi})$	0.3466	0.0029	0.3459	0.0004	0.3444	0.0032	0.4577	0.2476	0.2101
		$h(x_0; \underline{\psi})$	2.2429	0.0814	2.2228	0.01	2.1839	0.0885	2.83	1.7067	1.1233
		$r(x_0; \underline{\psi})$	1.1837	0.0328	1.1755	0.0041	1.1597	0.036	1.5683	0.8638	0.7045
60	42	$R(x_0; \underline{\psi})$	0.4623	0.0021	0.4617	0.0003	0.4607	0.0023	0.5515	0.3738	0.1776
		$h(x_0; \underline{\psi})$	1.7662	0.0482	1.7543	0.006	1.7313	0.0523	2.2193	1.3676	0.8517
		$r(x_0; \underline{\psi})$	1.5143	0.0298	1.507	0.0037	1.4928	0.0323	1.885	1.2124	0.6726
	60	$R(x_0; \underline{\psi})$	0.4676	0.0018	0.4672	0.0002	0.4663	0.002	0.5482	0.3784	0.1698
		$h(x_0; \underline{\psi})$	1.7879	0.0364	1.7789	0.0045	1.7616	0.0394	2.1992	1.4499	0.7493
		$r(x_0; \underline{\psi})$	1.5687	0.0273	1.5619	0.0034	1.5487	0.03	1.9333	1.2617	0.6716
100	70	$R(x_0; \underline{\psi})$	0.5257	0.0014	0.5253	0.0002	0.5246	0.0015	0.5984	0.4538	0.1446
		$h(x_0; \underline{\psi})$	1.4698	0.0228	1.4641	0.0028	1.4529	0.0254	1.7683	1.1848	0.5835
		$r(x_0; \underline{\psi})$	1.6281	0.0224	1.6225	0.0028	1.6116	0.0248	1.943	1.3536	0.5893
	100	$R(x_0; \underline{\psi})$	0.4728	0.0013	0.4725	0.0002	0.4718	0.0015	0.5483	0.4012	0.1471
		$h(x_0; \underline{\psi})$	1.8351	0.0282	1.8281	0.0035	1.814	0.0317	2.1718	1.5083	0.6635
		$r(x_0; \underline{\psi})$	1.6451	0.0231	1.6394	0.0029	1.628	0.0256	1.975	1.3538	0.6212

Table 10. For the AFWE-L distribution parameters ($v = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior II, and for $x_0 = 1.5$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.0507	0.0006	0.0006	7.9259E-05	0.0502	0.0007	0.1069	0.0088	0.0981
		$h(x_0;\underline{\psi})$	2.5062	1.2461	2.2836	0.1113	2.0619	0.6665	5.4773	1.3501	4.1272
		$r(x_0;\underline{\psi})$	0.1123	0.0012	0.112	0.0002	0.1114	0.0014	0.189	0.0448	0.1443
	30	$R(x_0;\underline{\psi})$	0.0429	0.0005	0.0427	6.4610E-05	0.0425	0.0006	0.0969	0.0095	0.0874
		$h(x_0;\underline{\psi})$	3.096	1.3082	2.8279	0.134	2.4949	0.9017	5.9656	1.5348	4.4308
		$r(x_0;\underline{\psi})$	0.1174	0.0013	0.1171	0.0002	0.1165	0.0014	0.1981	0.0536	0.1445
60	42	$R(x_0;\underline{\psi})$	0.0688	0.0008	0.0686	9.8974E-05	0.0682	0.0009	0.1302	0.0173	0.1129
		$h(x_0;\underline{\psi})$	2.3315	0.8946	2.1647	0.0834	1.9808	0.526	4.9439	1.2822	3.6617
		$r(x_0;\underline{\psi})$	0.1501	0.0011	0.1499	0.0001	0.1493	0.0013	0.2187	0.0822	0.1366
	60	$R(x_0;\underline{\psi})$	0.0607	0.0004	0.0606	4.68401E-05	0.0605	0.0004	0.1037	0.0273	0.0764
		$h(x_0;\underline{\psi})$	2.4932	0.3375	2.415	0.0391	2.2869	0.3094	3.873	1.5789	2.2941
		$r(x_0;\underline{\psi})$	0.1517	0.0008	0.1515	0.0001	0.1511	0.0009	0.2099	0.0965	0.1134
100	70	$R(x_0;\underline{\psi})$	0.1122	0.0008	0.112	0.0001	0.1116	0.0009	0.1688	0.0577	0.1111
		$h(x_0;\underline{\psi})$	1.6905	0.194	1.6468	0.0219	1.5799	0.1658	2.797	1.1193	1.6777
		$r(x_0;\underline{\psi})$	0.2025	0.0009	0.2022	0.0001	0.2018	0.001	0.2626	0.1479	0.1148
	100	$R(x_0;\underline{\psi})$	0.0432	0.0002	0.0432	2.9065E-05	0.0431	0.0003	0.0789	0.0188	0.0602
		$h(x_0;\underline{\psi})$	3.2933	0.5434	3.1672	0.063	2.9558	0.5063	4.9789	2.0366	2.9423
		$r(x_0;\underline{\psi})$	0.1391	0.0007	0.1389	9.1247E-05	0.1386	0.0008	0.1981	0.0899	0.1081

Table 11. For the AFWE-L distribution parameters ($v = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior II, and for $x_0 = 2$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.0178	0.0002	0.3931	0.0004	0.0177	0.0002	0.0528	0.0003	0.0525
		$h(x_0;\underline{\psi})$	3.2901	5.3137	2.6195	0.3353	2.1863	1.6557	9.8023	1.2558	8.5466
		$r(x_0;\underline{\psi})$	0.0381	0.0004	0.038	5.1210E-05	0.0378	0.0005	0.0804	0.0024	0.078
	30	$R(x_0;\underline{\psi})$	0.0111	0.0001	0.4215	0.0004	0.011	0.0001	0.0419	0.0002	0.0418
		$h(x_0;\underline{\psi})$	4.5131	6.0224	3.5713	0.4709	2.8094	2.5555	10.7517	1.515	9.2367
		$r(x_0;\underline{\psi})$	0.0311	0.0004	0.031	5.0717E-05	0.0308	0.0005	0.0761	0.0019	0.0742
60	42	$R(x_0;\underline{\psi})$	0.0245	0.0003	0.4617	0.0003	0.0243	0.0003	0.0676	0.0007	0.0669
		$h(x_0;\underline{\psi})$	3.0625	3.8415	2.5443	0.2591	2.1571	1.3581	8.3737	1.2172	7.1564
		$r(x_0;\underline{\psi})$	0.053	0.0005	0.0529	6.6177E-05	0.0526	0.0006	0.0976	0.0057	0.0919
	60	$R(x_0;\underline{\psi})$	0.0172	0.0001	0.4672	0.0002	0.0171	0.0001	0.0445	0.0026	0.0419
		$h(x_0;\underline{\psi})$	3.3103	1.2812	3.0431	0.1336	2.6879	0.9335	6.1029	1.661	4.4419
		$r(x_0;\underline{\psi})$	0.0476	0.0003	0.0475	3.7873E-05	0.0473	0.0003	0.0835	0.0153	0.0682
100	70	$R(x_0;\underline{\psi})$	0.0497	0.0005	0.5244	0.0002	0.0493	0.0005	0.0949	0.0111	0.0837
		$h(x_0;\underline{\psi})$	1.958	0.6338	1.8316	0.0632	1.678	0.4201	4.0009	1.0407	2.9602
		$r(x_0;\underline{\psi})$	0.0873	0.0004	0.0872	5.1894E-05	0.087	0.0005	0.1245	0.044	0.0805
	100	$R(x_0;\underline{\psi})$	0.0076	4.1145E-05	0.4725	0.0002	0.0075	4.6003E-05	0.0255	0.0007	0.0248
		$h(x_0;\underline{\psi})$	4.9598	2.5728	4.4368	0.2615	3.7511	1.8131	8.7634	2.4063	6.3571
		$r(x_0;\underline{\psi})$	0.0297	0.0002	0.0296	2.8845E-05	0.0295	0.0003	0.0647	0.0061	0.0586

Table 12. For the AFWE-L distribution parameters ($v = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior III, and for $x_0 = 0.5$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.2118	0.0029	0.2111	0.0004	0.2097	0.0033	0.3348	0.1205	0.2144
		$h(x_0;\underline{\psi})$	1.9337	0.0487	1.9216	0.0061	1.8978	0.0539	2.3785	1.5123	0.8662
		$r(x_0;\underline{\psi})$	0.514	0.0137	0.5106	0.0017	0.5041	0.015	0.7705	0.3193	0.4512
	30	$R(x_0;\underline{\psi})$	0.2204	0.0027	0.2197	0.0003	0.2184	0.003	0.334	0.1364	0.1976
		$h(x_0;\underline{\psi})$	1.9325	0.0423	1.9219	0.0053	1.9009	0.0473	2.3447	1.5235	0.8212
		$r(x_0;\underline{\psi})$	0.5414	0.0128	0.5382	0.0016	0.532	0.0141	0.7785	0.3595	0.419
60	42	$R(x_0;\underline{\psi})$	0.2318	0.0017	0.2313	0.0002	0.2305	0.0019	0.3173	0.1526	0.1646
		$h(x_0;\underline{\psi})$	1.8556	0.034	1.8472	0.0042	1.8309	0.037	2.2499	1.5459	0.704
		$r(x_0;\underline{\psi})$	0.5556	0.0074	0.5538	0.0009	0.5501	0.0082	0.7377	0.3917	0.346
	60	$R(x_0;\underline{\psi})$	0.2311	0.0015	0.2307	0.0002	0.2299	0.0017	0.3163	0.162	0.1543
		$h(x_0;\underline{\psi})$	1.8886	0.0264	1.882	0.0033	1.8689	0.0295	2.2238	1.5689	0.6549
		$r(x_0;\underline{\psi})$	0.5643	0.007	0.5626	0.0009	0.5591	0.0078	0.749	0.4146	0.3344
100	70	$R(x_0;\underline{\psi})$	0.2258	0.0013	0.2255	0.0002	0.2248	0.0015	0.2986	0.1724	0.1262
		$h(x_0;\underline{\psi})$	1.9206	0.0298	1.9132	0.0037	1.8985	0.0331	2.2534	1.6103	0.6431
		$r(x_0;\underline{\psi})$	0.5563	0.0055	0.5549	0.0007	0.5522	0.0061	0.7089	0.4384	0.2705
	100	$R(x_0;\underline{\psi})$	0.1951	0.0009	0.1949	0.0001	0.1944	0.001	0.2585	0.1407	0.1179
		$h(x_0;\underline{\psi})$	2.0047	0.0224	1.9991	0.0028	1.9881	0.0249	2.3222	1.7203	0.6019
		$r(x_0;\underline{\psi})$	0.4838	0.0043	0.4827	0.0005	0.4806	0.0048	0.623	0.3651	0.2579

Table 13. For the AFWE-L distribution parameters ($v = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior III, and for $x_0 = 1.5$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.0549	0.0004	0.0548	5.0823E-05	0.0546	0.0005	0.1033	0.0241	0.0792
		$h(x_0;\underline{\psi})$	1.1812	0.0101	1.1786	0.0013	1.1737	0.0112	1.3883	1.0005	0.3878
		$r(x_0;\underline{\psi})$	0.0677	0.0005	0.0676	6.5735E-05	0.0673	0.0006	0.1223	0.0323	0.09
	30	$R(x_0;\underline{\psi})$	0.0562	0.0004	0.0561	4.5433E-05	0.056	0.0004	0.1018	0.0271	0.0747
		$h(x_0;\underline{\psi})$	1.1933	0.0107	1.1906	0.0013	1.1854	0.0118	1.4062	1.019	0.3872
		$r(x_0;\underline{\psi})$	0.0704	0.0005	0.0702	6.2298E-05	0.07	0.0006	0.1217	0.0369	0.0848
60	42	$R(x_0;\underline{\psi})$	0.0618	0.0003	0.0618	3.4917E-05	0.0616	0.0003	0.0971	0.0323	0.0648
		$h(x_0;\underline{\psi})$	1.1585	0.0096	1.1561	0.0012	1.1515	0.0106	1.3624	0.9973	0.3651
		$r(x_0;\underline{\psi})$	0.0755	0.0003	0.0754	4.0905E-05	0.0752	0.0004	0.1147	0.043	0.0717
	60	$R(x_0;\underline{\psi})$	0.0593	0.0002	0.0593	2.8648E-05	0.0591	0.0003	0.0943	0.0337	0.0605
		$h(x_0;\underline{\psi})$	1.1933	0.0081	1.1913	0.001	1.1872	0.0091	1.3828	1.0188	0.364
		$r(x_0;\underline{\psi})$	0.0746	0.0003	0.0745	3.6632E-05	0.0744	0.0003	0.1137	0.0459	0.0678
100	70	$R(x_0;\underline{\psi})$	0.0571	0.0002	0.0571	2.7381E-05	0.057	0.0002	0.0889	0.0351	0.0539
		$h(x_0;\underline{\psi})$	1.1992	0.0095	1.1968	0.0012	1.1921	0.0107	1.3637	1.0178	0.3458
		$r(x_0;\underline{\psi})$	0.0717	0.0002	0.0717	3.1137E-05	0.0715	0.0003	0.1052	0.0482	0.057
	100	$R(x_0;\underline{\psi})$	0.0474	0.0001	0.0473	1.4509E-05	0.0473	0.0001	0.0719	0.0284	0.0435
		$h(x_0;\underline{\psi})$	1.2201	0.0085	1.218	0.0011	1.2138	0.0094	1.4133	1.049	0.3643
		$r(x_0;\underline{\psi})$	0.0602	0.0001	0.0601	1.8586E-05	0.0601	0.0002	0.0864	0.0391	0.0473

Table 17. For the AFWE-L distribution parameters ($v = 0.5, 1.5$), Bayes estimates and PRs are calculated using the SE loss and LINEX loss function. The interval lengths and 95% credible intervals are determined according to Prior IV, and for $x_0 = 2$

n	r	function	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.0456	0.0004	0.2839	0.0005	0.0454	0.0004	0.0925	0.0168	0.0757
		$h(x_0;\underline{\psi})$	1.2235	0.0331	1.2154	0.004	1.2004	0.0347	1.6532	0.9217	0.7315
		$r(x_0;\underline{\psi})$	0.0569	0.0005	0.0568	6.0567E-05	0.0565	0.0005	0.1095	0.0245	0.0849
	30	$R(x_0;\underline{\psi})$	0.0313	0.0002	0.2253	0.0004	0.0312	0.0002	0.0606	0.0119	0.0487
		$h(x_0;\underline{\psi})$	1.2834	0.0355	1.2747	0.0043	1.2587	0.037	1.7098	0.9763	0.7335
		$r(x_0;\underline{\psi})$	0.0403	0.0002	0.0402	2.7081E-05	0.0401	0.0002	0.0741	0.017	0.057
60	42	$R(x_0;\underline{\psi})$	0.0333	0.0001	0.2426	0.0002	0.0332	0.0002	0.0618	0.0155	0.0463
		$h(x_0;\underline{\psi})$	1.2711	0.0313	1.2634	0.0038	1.2491	0.033	1.6855	0.9794	0.7061
		$r(x_0;\underline{\psi})$	0.0427	0.0002	0.0426	2.0970E-05	0.0426	0.0002	0.0728	0.0225	0.0503
	60	$R(x_0;\underline{\psi})$	0.0228	6.4258E-05	0.1944	0.0002	0.0227	7.2067E-05	0.0405	0.0093	0.0312
		$h(x_0;\underline{\psi})$	1.3671	0.0378	1.358	0.0046	1.341	0.0392	1.8247	1.0468	0.7779
		$r(x_0;\underline{\psi})$	0.0311	8.9310E-05	0.031	1.1155E-05	0.031	0.0001	0.0526	0.0144	0.0383
100	70	$R(x_0;\underline{\psi})$	0.0255	8.4740E-05	0.1946	0.0001	0.0255	9.4933E-05	0.0466	0.0114	0.0352
		$h(x_0;\underline{\psi})$	1.2731	0.0324	1.2651	0.004	1.2502	0.0343	1.6765	0.97	0.7065
		$r(x_0;\underline{\psi})$	0.0324	9.2751E-05	0.0324	1.1581E-05	0.0323	0.0001	0.0537	0.0172	0.0365
	100	$R(x_0;\underline{\psi})$	0.0225	5.0769E-05	0.1913	0.0001	0.0225	5.6950E-05	0.0394	0.0108	0.0286
		$h(x_0;\underline{\psi})$	1.3808	0.0393	1.3713	0.0047	1.3539	0.0403	1.8602	1.0643	0.7959
		$r(x_0;\underline{\psi})$	0.0308	6.0849E-05	0.0307	7.6007E-06	0.0307	6.8310E-05	0.0488	0.0177	0.0311

uncertainty surrounding the true values of the rf, hrf, and rhrf. The improved precision, stemming from a larger and more complete dataset, allows for more confident and informative inferences regarding the reliability behavior modeled by the AFWE-L distribution, providing a more reliable range of values for these parameters.

- Across all results, the Bayes estimates obtained via the LINEX loss generally outperform the ones derived via the SE loss in most instances. Moreover, the results obtained with the LINEX loss for $v = 0.5$ outperform those with $v = 1.5$, as evidenced by lower PRs and shorter 95% CI lengths. Additionally, the Bayes estimates obtained with the LINEX loss for $v = 0.5$ are notably approximately equal to those derived under the SE loss. This observation aligns with the theoretical behavior that as v approaches zero, the LINEX-based Bayes estimates converge to the ones with the SE loss.
- Across all considered findings, the Bayes estimates derived using Priors I and III, characterized by their small variances, consistently demonstrated superior performance compared to those obtained with Priors II and IV, which exhibited larger variances.

4. Some Applications

This section evaluates the effectiveness of the Bayes estimates using three applications of COVID-19 data from countries previously analyzed in Salem et al. [25] to demonstrate the theoretical results of the Bayesian inference.

Salem et al. [25] conducted three applications on COVID-19 data from several countries to establish the superior fit of the AFWE-L distribution compared to established distributions, specifically L-Weibull, new modified Weibull, additive Weibull, FWE, and L distributions. To compare the fit of these competing distributions, the authors employed maximum likelihood estimation of the parameters under two levels of Type II censoring (30% and 0%), along with the standard errors (SE), the Kolmogorov-Smirnov (K-S) statistic and its associated p-value, the $-2log$ likelihood (-2ℓ) statistic, Akaike information criterion (AIC), Bayesian information criterion (BIC), and consistent Akaike information criterion (CAIC).

They demonstrated, across three distinct applications, that the AFWE-L distribution yielded the lowest K-S statistic values and the highest corresponding p-values. These findings suggest that the AFWE-L distribution offers the best representation of the data compared to the alternative distributions considered. Furthermore, the AFWE-L distribution exhibited the minimum values for the -2ℓ statistic, AIC, BIC and CAIC, suggesting that the AFWE-L model outperformed the alternative competing distributions.

Application 1

This dataset, provided by Mubarak and Almetwally [47] offer an analysis of drought-related mortality rates in the United Kingdom over a 76-day, from between April 15 and June 30, 2020. The data are given as: 0.0587, 0.0863, 0.1165, 0.1247, 0.1277, 0.1303, 0.1652, 0.2079, 0.2395, 0.2751, 0.2845, 0.2992, 0.3188, 0.3317, 0.3446, 0.3553, 0.3622, 0.3926, 0.3926, 0.4110, 0.4633, 0.4690, 0.4954, 0.5139, 0.5696, 0.5837, 0.6197, 0.6365, 0.7096, 0.7193, 0.7444, 0.8590, 1.0438, 1.0602, 1.1305, 1.1468, 1.1533, 1.2260, 1.2707, 1.3423, 1.4149, 1.5709, 1.6017, 1.6083, 1.6324, 1.6998, 1.8164, 1.8392, 1.8721, 1.9844, 2.1360, 2.3987, 2.4153, 2.5225, 2.7087, 2.7946, 3.3609, 3.3715, 3.7840, 3.9042, 4.1969, 4.3451, 4.4627, 4.6477, 5.3664, 5.4500, 5.7522, 6.4241, 7.0657, 7.4456, 8.2307, 9.6315, 10.1870, 11.1429, 11.2019 and 11.4584.

The suitability of the AFWE-L distribution for this data set was evaluated using the K-S test which resulted in a statistic of 0.0790 with an associated p-value of 0.9735. These results indicate that the AFWE-L distribution is an appropriate model for this dataset.

Figure 7 illustrates the empirical scaled TTT-transform of the United Kingdom COVID-19 data, suggesting a modified bathtub-shaped hazard function. The data exhibits the right skewness. Furthermore, Q-Q and P-P plots, along with the fitted AFWE-L distribution plots indicate a superior goodness-of-fit of the AFWE-L distribution to the observed data.

Table 18 displays the Bayes estimates and PRs for the parameters and rf, hrf and rhrf of AFWE-L distribution applied to the United Kingdom COVID-19 data via both the SE and LINEX loss, with the LINEX loss is evaluated at $v = 1.5$ and 0.5 and time ($x_0 = 0.5, 1.5, 2$). For the Bayesian analysis, the hyper-parameters of the joint prior are determined using the following sets of means and variances: (0.5,0.5,0.25,0.5) and (0.005,0.005,0.05,0.03), respectively. Additionally, two levels of Type II censoring are implemented, corresponding to 0% and 30%. Also, Figures 8 to 10 present the trace and autocorrelation plots, along with the histograms and posterior density estimates for each parameter, based on the United Kingdom COVID-19 data with 30% Type II censoring applied.

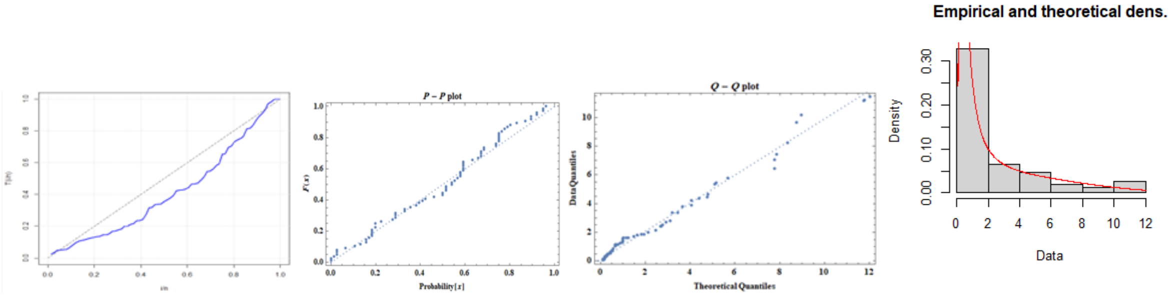

Figure 7. Empirical scaled TTT-transform, Q-Q, P-P plots, and the fitted pdf United Kingdom COVID-19 data

Application 2

This dataset, provided by Mubarak and Almetwally [47], represents COVID-19 mortality rates in Japan over a 38-day period, from September 4th to October 12th, 2020. The data are as follows: 0.1596, 0.2733, 0.1142, 0.0851, 0.1976, 0.2243, 0.1810, 0.0828, 0.1504, 0.2169, 0.0404, 0.1208, 0.1334, 0.1589, 0.1184, 0.1698, 0.0648, 0.1027, 0.0511, 0.1019, 0.1520, 0.1006, 0.0624, 0.0372, 0.1112, 0.0859, 0.0854, 0.0847, 0.1443, 0.0924, 0.0344 and 0.0228.

From the value of $K-S = 0.1316$ and its corresponding $p-value = 0.9033$, the AFWE-L distribution demonstrates a good fit to this dataset. Figure 11 demonstrates the empirical scaled TTT-transform of the Japan COVID-19 data, suggesting an increasing hazard function. Furthermore, the P-P plot, Q-Q plot, and the graphical representation of the fitted AFWE-L distribution show a superior fit of the AFWE-L distribution to the observed Japanese COVID-19 data.

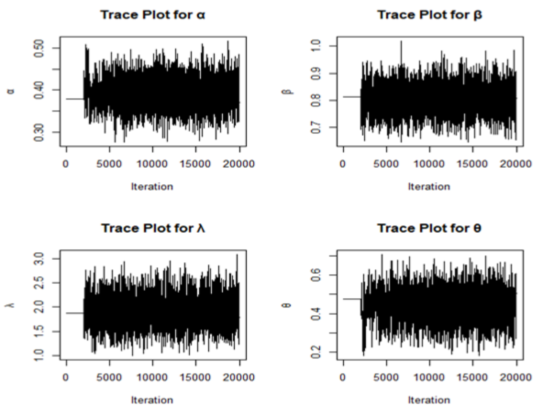

Figure 8. Trace plots of α, β, λ and θ for United Kingdom COVID-19 data with 30% of Type II censoring

Table 19 presents the Bayes estimates and corresponding PRs for the parameters and rf, hrf, and rhrf of the AFWE-L distribution, based on Japan COVID-19 data. The results are caculated via the SE and LINEX loss , with the LINEX loss determined at $v = 0.5$ and 1.5 and for time $(x_0 = 2, 1.5, 0.5)$. In the Bayesian analysis, the hyper-parameters of the joint prior were determined using the following sets of means and variances: (0.5,0.5,0.5,1.15) and (0.005,0.0015,0.008,0.002).Also, two levels of Type II censoring are applied (0%,30%). Additionally, Figures 12 to 14 illustrate the convergence diagnostics

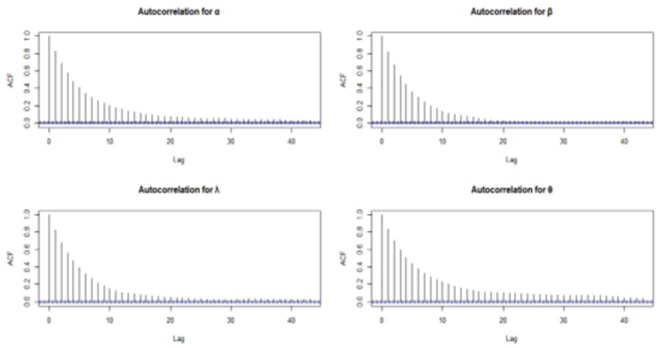

Figure 9. Autocorrelation plots of α, β, λ and θ for the United Kingdom COVID-19 data under 30% of Type II censoring

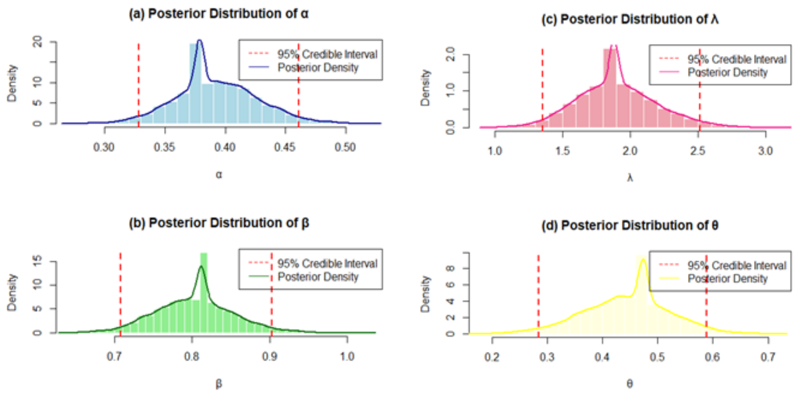

Figure 10. Posterior density and histogram with 95% credible interval bounds of α, β, λ and θ for the United Kingdom COVID-19 data with 30% of Type II censoring

and distributional characteristics of the parameters, including trace plots, autocorrelation functions, histograms, and posterior density estimates of each parameter for COVID-19 data of Japan under 30% of Type II censoring.

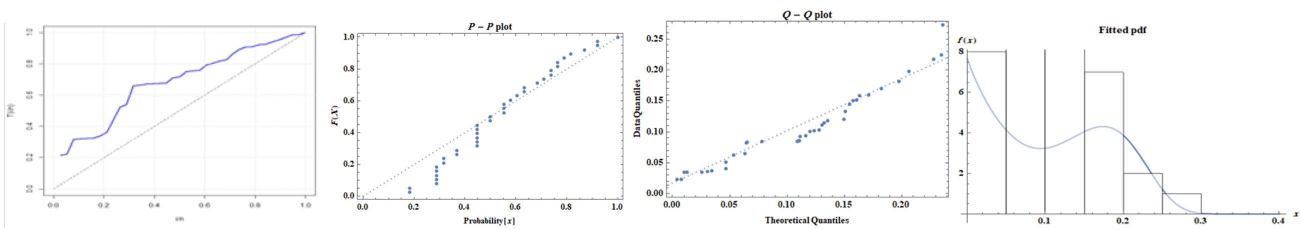

Figure 11. . Empirical scaled TTT-transform, Q-Q, P-P plots, and the fitted pdf for Japan COVID-19 data

Table 18. Bayes estimates and PRs via the SE and LINEX loss functions for ($v = 0.5, 1.5$) and time ($x_0 = 0.5, 1.5, 2$) of the rf, hrf and rhrf of the AFWE-L distribution, with 95% credible intervals and interval lengths for the United Kingdom COVID-19 data

n	r	$\underline{\psi}$	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
76	53	α	0.3922	0.0011	0.3919	0.0003	0.3914	0.0008	0.4611	0.3284	0.1327
		β	0.8032	0.0023	0.8026	0.0006	0.8014	0.0018	0.9029	0.708	0.1949
		λ	1.9	0.0826	1.8796	0.0214	1.8401	0.0703	2.5145	1.3561	1.1584
		θ	0.445	0.0057	0.4435	0.0014	0.4406	0.0043	0.5893	0.2847	0.3046
	76	α	0.1631	0.0002	0.1631	5.0698E-05	0.163	0.0002	0.1882	0.1354	0.0528
		β	0.8552	0.0088	0.853	0.0022	0.8487	0.0068	1.0328	0.6845	0.3483
		λ	1.4276	0.433	1.3287	0.1312	1.1764	0.7201	2.9981	0.4566	2.5415
		θ	0.3455	0.0109	0.3428	0.0028	0.3377	0.0086	0.5935	0.1877	0.4058
	t=0.5										
	53	$R(x_0,\underline{\psi})$	0.7035	0.001	0.7032	0.0001	0.7028	0.0011	0.7634	0.6378	0.1256
		$h(x_0,\underline{\psi})$	1.0707	0.0057	1.0693	0.0007	1.0665	0.0063	1.2287	0.9212	0.3075
		$r(x_0,\underline{\psi})$	2.5535	0.0422	2.5431	0.0052	2.5225	0.0466	2.9923	2.1587	0.8336
	76	$R(x_0,\underline{\psi})$	0.7271	0.0013	0.7268	0.0002	0.7262	0.0015	0.7957	0.6606	0.1351
		$h(x_0,\underline{\psi})$	0.9006	0.005	0.8993	0.0006	0.8968	0.0056	1.0303	0.7709	0.2594
		$r(x_0,\underline{\psi})$	2.4241	0.0655	2.4078	0.0081	2.3761	0.072	3.0033	1.98	1.0234
	t=1.5										
	53	$R(x_0,\underline{\psi})$	0.2677	0.0006	0.2675	7.8166E-05	0.2672	0.0007	0.3195	0.2183	0.1012
		$h(x_0,\underline{\psi})$	0.9258	0.0044	0.9247	0.0005	0.9225	0.0049	1.0714	0.8055	0.2659
		$r(x_0,\underline{\psi})$	0.3375	0.0008	0.3373	0.0001	0.3369	0.0009	0.3968	0.2809	0.1159
	76	$R(x_0,\underline{\psi})$	0.37	0.0012	0.3696	0.0002	0.369	0.0014	0.4346	0.3059	0.1287
		$h(x_0,\underline{\psi})$	0.515	0.0019	0.5145	0.0002	0.5135	0.0022	0.611	0.4412	0.1698
		$r(x_0,\underline{\psi})$	0.3018	0.0009	0.3016	0.0001	0.3012	0.001	0.3637	0.2487	0.115
	t=2										
	53	$R(x_0,\underline{\psi})$	0.1668	4.3397E-04	0.7032	0.0001	0.1665	0.0005	0.2104	0.1268	0.0836
		$h(x_0,\underline{\psi})$	0.9909	0.0097	0.9884	0.0012	0.9838	0.0107	1.2121	0.8192	0.3929
		$r(x_0,\underline{\psi})$	0.1968	0.0003	0.1967	4.0335E-05	0.1966	0.0004	0.2341	0.161	0.0731
	76	$R(x_0,\underline{\psi})$	0.2921	1.0234E-03	0.7268	0.0002	0.2914	1.1560E-03	0.3482	0.2316	0.1166
		$h(x_0,\underline{\psi})$	0.4447	0.0013	0.4444	0.0002	0.4437	0.0015	0.5118	0.3728	0.139
		$r(x_0,\underline{\psi})$	0.1829	0.0004	0.1828	4.5005E-05	0.1827	4.0491E-04	0.225	0.1508	0.0742

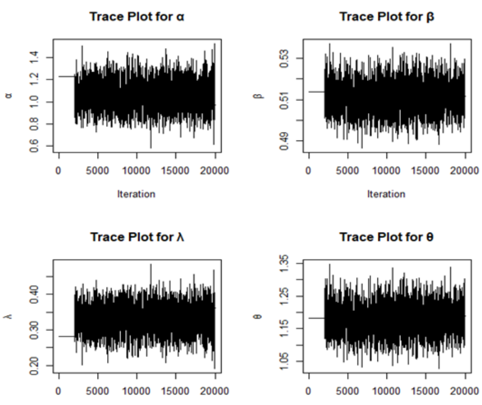

Figure 12. Trace plots of α, β, λ and θ for COVID-19 data of Japan under 30% of Type II censoring

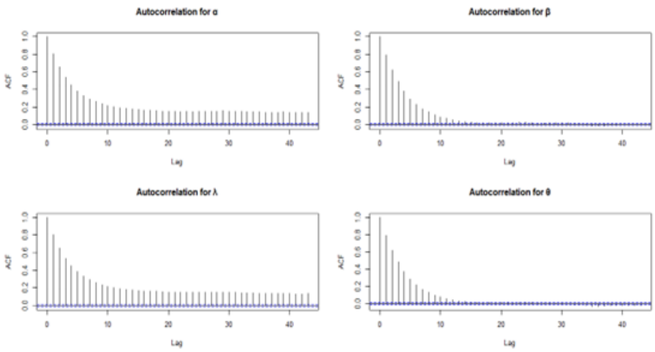

Figure 13. Autocorrelation plots of α, β, λ and θ for COVID-19 data of Japan under 30% of Type II censoring

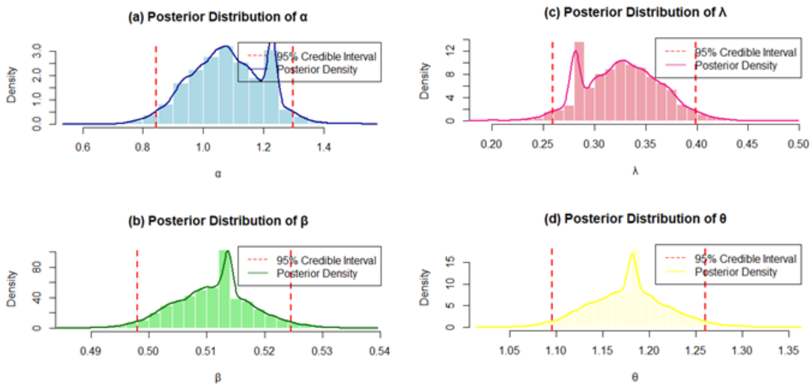

Figure 14. Posterior density and histogram with 95% credible interval bounds of α, β, λ and θ for COVID-19 data of Japan under 30% of Type II censoring

Application 3

This application is given by Liu et al. [48]. It analyzes the survival times of patients affected by the COVID-19 outbreak in China, specifically measuring the duration from hospital admission to death. The dataset includes 53 critically ill COVID-19 patients hospitalized between January and February 2020. The data are given by: 0.054, 0.064, 0.704, 0.816, 0.235, 0.976, 0.865, 0.364, 0.479, 0.568, 0.352, 0.978, 0.787, 0.976, 0.087, 0.548, 0.796, 0.458, 0.087, 0.437, 0.421, 1.978, 1.756, 2.089, 2.643, 2.869, 3.867, 3.890, 3.543, 3.079, 3.646, 3.348, 4.093, 4.092, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.274, 7.058, 8.273, 9.324, 10.827, 11.282, 13.324, 14.278, 15.287, 16.978, 17.209, 19.092 and 20.083.

According to the K-S statistic value of 0.0943 and its associated $p - value = 0.9747$, the AFWE-L distribution demonstrates a good fit to this real dataset. Figure 15 presents the plot of the empirical scaled TTT-transform of COVID-19 data of China, which indicates that this data has a bathtub hazard function. The P-P plot, Q-Q plot and the fitted AFWE-L distribution plots imply that AFWE-L distribution presents better fit for this data.

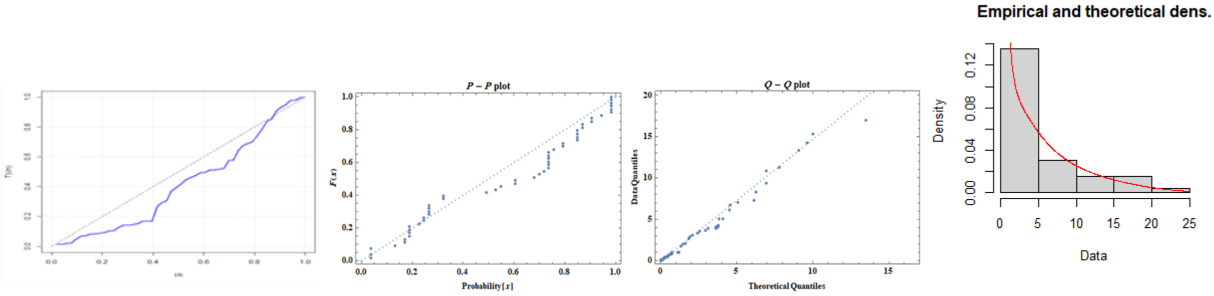

Figure 15. Empirical scaled TTT-transform, Q-Q, P-P plots and the fitted AFWE-L distribution for China COVID-19 data

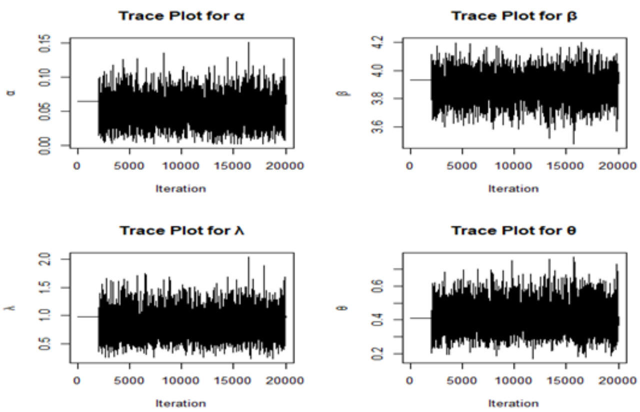

Figure 16. Trace plots of α, β, λ and θ for China COVID-19 data with 30% of Type II censoring

Table 20 provides the Bayes estimates and corresponding PRs for the parameters and rf, hrf, and rhrf of the AFWE-L distribution, based on China COVID-19 data. The results are computed via both the SE loss and LINEX loss function, with the LINEX loss obtained at asymmetry parameters $\nu = 0.5$ and 1.5 and time $(x_0 = 2, 1.5, 0.5)$.

For the Bayesian analysis, the hyper-parameters of the joint prior were determined using the following combinations of means and variances: (0.05,4,0.5,0.25) and (0.005,0.05,0.05,0.03). Also, two levels of Type II censoring are applied (0%,30%). Additionally, Figures 16 to 18 show the trace and autocorrelation plots, along with the histograms and posterior density estimates for each parameter for China COVID-19 data under 30% of Type II censoring.

The key difference between the three applications, as shown in the empirical scaled TTT-transform plots in Figures 7, 11 and 15, is the shape of their respective hazard functions. This distinction holds considerable implications for understanding the underlying distribution. These distinct hazard function shapes, derived from the TTT analysis, highlight the fundamental differences in the time-related patterns observed across the three applications. We believe this explicit clarification enhances the clarity and impact of our findings.

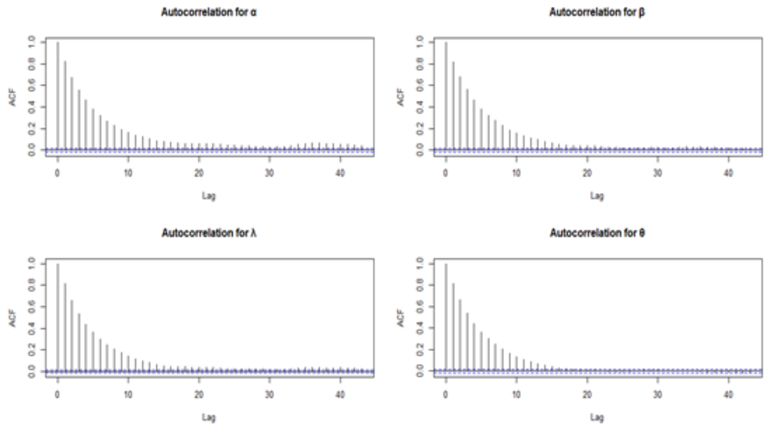

Figure 17. Autocorrelation plots of α, β, λ and θ for China COVID-19 data under 30% of Type II censoring

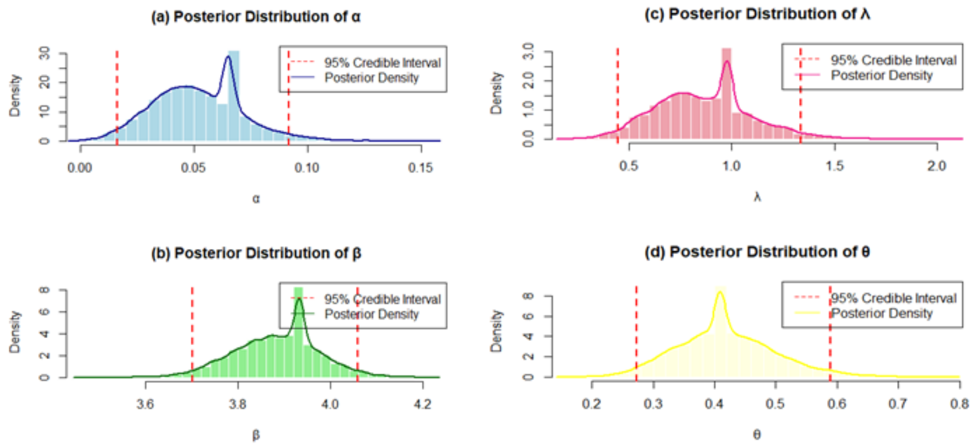

Figure 18. Posterior density and histogram with 95% credible interval bounds of α, β, λ and θ for COVID-19 data of China under 30% of Type II censoring

Concluding remarks

An examination of Tables 18, 19, and 20 reveals that:

- Employing the LINEX loss function, the Bayes estimates of the AFWE-L distribution parameters exhibit smaller PRs and CI lengths compared to those obtained under the SE loss.
- Also, the results obtained through applying the LINEX loss function with $v = 0.5$ are higher than those with $v = 1.5$, and they closely approximate the outcomes via the SE loss function.
- The findings of the PRs under 30% of Type II censoring scheme are greater than their corresponding values under 0% of Type II censoring.

5. Conclusion

This paper considers point and interval Bayesian estimation for the model parameters and rf, hrf and rhrf of the AFWE-L distribution using the SE and the LINEX loss based on Type II censoring.

Table 19. For the AFWE-L distribution parameters ($\nu = 0.5, 1.5$) and time ($x_0 = 0.5, 1.5, 2$) Bayes estimates and PRs are calculated using the SE and LINEX loss functions. The interval lengths and 95% credible intervals are determined for Japan's COVID-19 data

n	r	$\underline{\psi}$	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
38	27	α	1.0806	0.0149	1.0769	0.0037	1.0693	0.0113	1.2977	0.8444	0.4533
		β	0.5112	4.3011E-05	0.5112	1.0753E-05	0.5111	3.2259E-05	0.5245	0.498	0.0265
		λ	0.3259	0.0014	0.3255	0.0003	0.3248	0.001	0.3989	0.2592	0.1397
		θ	1.1771	0.0017	1.1767	0.0004	1.1759	0.0012	1.2603	1.0955	0.1649
	38	α	0.9967	0.005	0.9954	0.0013	0.9929	0.0037	1.1324	0.8485	0.2839
		β	0.4938	0.0008	0.4936	0.0002	0.4932	0.0006	0.5448	0.4404	0.1044
		λ	0.262	0.0012	0.2617	0.0003	0.2611	0.0009	0.3322	0.1965	0.1356
		θ	1.2228	0.0005	1.2226	0.0001	1.2223	0.0004	1.2684	1.1785	0.0899
	t=0.5										
	27	$R(x_0,\underline{\psi})$	0.1804	0.0005	0.1803	0.000058783	0.1801	0.0005	0.2222	0.1429	0.0794
		$h(x_0,\underline{\psi})$	3.3657	0.0675	3.3489	0.0084	3.316	0.0746	3.8426	2.8902	0.9524
		$r(x_0,\underline{\psi})$	0.7355	0.0027	0.7349	0.0003	0.7335	0.003	0.8256	0.6393	0.1862
	38	$R(x_0,\underline{\psi})$	0.1468	0.0005	0.1467	0.000061659	0.1465	0.0006	0.1924	0.1037	0.0888
		$h(x_0,\underline{\psi})$	3.4323	0.0416	3.4219	0.0052	3.4014	0.0462	3.8491	3.0341	0.8151
		$r(x_0,\underline{\psi})$	0.5873	0.0049	0.586	0.0006	0.5836	0.0056	0.7269	0.4448	0.2821
	t=1.5										
	27	$R(x_0,\underline{\psi})$	0.0044	1.0924E-05	0.0044	1.3643E-06	0.0044	1.2257E-05	0.0127	0.0007	0.012
		$h(x_0,\underline{\psi})$	5.5076	1.8122	5.1046	0.2015	4.5401	1.4512	8.2533	3.3239	4.9295
		$r(x_0,\underline{\psi})$	0.0207	0.0001	0.0206	1.2834E-05	0.0206	0.0001	0.043	0.0061	0.0368
	38	$R(x_0,\underline{\psi})$	0.0043	4.9411E-06	0.0043	6.1730E-07	0.0043	5.5497E-06	0.01	0.0013	0.0087
		$h(x_0,\underline{\psi})$	4.642	0.4392	4.5352	0.0534	4.3356	0.4596	6.0533	3.4195	2.6339
		$r(x_0,\underline{\psi})$	0.0188	4.5160E-05	0.0188	5.6403E-06	0.0188	5.0679E-05	0.0345	0.0079	0.0266
	t=2										
	27	$R(x_0,\underline{\psi})$	0.0003	3.2473E-07	0.1803	5.8783E-05	0.0003	3.6470E-07	0.0018	2.5960E-06	0.0018
		$h(x_0,\underline{\psi})$	9.0752	8.6835	7.4893	0.7929	5.9881	4.6307	15.3225	4.5701	10.7524
		$r(x_0,\underline{\psi})$	0.0019	5.4605E-06	0.0019	6.8187E-07	0.0019	6.1246E-06	0.0083	3.9760E-05	0.0083
	38	$R(x_0,\underline{\psi})$	0.0003	1.4057E-07	0.1467	6.1659E-05	0.0003	1.5802E-07	0.0013	2.5976E-05	0.0013
		$h(x_0,\underline{\psi})$	7.0848	1.792	6.6704	0.2072	5.9829	1.6528	10.0119	4.6745	5.3374
		$r(x_0,\underline{\psi})$	0.002	2.5894E-06	0.002	3.2349E-07	0.002	2.9080E-06	0.0064	0.0003	0.0061

Informative gamma priors are adopted, based on the joint bivariate prior distribution proposed by Al-Hussaini and Jaheen [39]. A comprehensive simulation analysis evaluates the performance of the Bayes estimates employing two censoring levels (0% and 30%). Simulation results generally indicate that the efficiency of the AFWE-L distribution increases with sample size and decreases with censoring level. The results also indicate that the estimation efficiency is better by larger sample sizes and further observed failures. Moreover, a smaller shape parameter for the LINEX loss delivers better efficiency than larger values. Also, the LINEX loss generally outperforms the SE loss. In addition, as the shape parameter of the LINEX loss closes to zero the Bayes estimates are more nearby to those delivered via the SE loss. We also demonstrate that reducing the prior variance results in enhancing the efficiency of the Bayes estimates. The findings are illustrated through three applications to COVID-19 datasets.

6. Potential Avenues for Future Work

For future research on the AFWE-L distribution, several avenues exist. One direction involves exploring alternative estimation techniques for the model parameters and rf and hrf, such as the method

Table 20. Bayes estimates and PRs via the SE loss and LINEX loss for ($\nu = 1.5, 0.5$) and time ($x_0 = 0.5, 1.5, 2$) of the rf, hrf and rhrf of the AFWE-L distribution, with 95% credible intervals and interval lengths for China COVID-19 data

n	r	$\underline{\psi}$	SE		LINEX 0.5		LINEX 1.5		Credible Interval		
			Est.	PR	Est.	PR	Est.	PR	UL	LL	Length
53	37	α	0.0518	0.0004	0.0517	0.0000925	0.0515	0.0003	0.0918	0.0163	0.0755
		β	3.8851	0.0081	3.8831	0.002	3.879	0.0061	4.0586	3.7018	0.3568
		λ	0.8609	0.0514	0.8482	0.0131	0.8235	0.0419	1.336	0.4479	0.8881
		θ	0.4176	0.0062	0.416	0.0016	0.413	0.0047	0.5892	0.273	0.3161
	53	α	0.0538	0.0002	0.0538	3.9308E-05	0.0537	0.0001	0.0763	0.0294	0.0469
		β	3.9911	5.4349E-05	3.9911	1.3582E-05	3.9911	4.0712E-05	4.0045	3.9762	0.0283
		λ	0.7561	0.0211	0.7508	0.0053	0.7403	0.0162	1.031	0.4756	0.5554
		θ	0.4165	0.0077	0.4146	0.0019	0.4108	0.0059	0.5945	0.26	0.3345
	t=0.5										
	37	$R(x_0,\underline{\psi})$	0.8221	0.001	0.8219	0.0001	0.8214	0.0011	0.8779	0.7526	0.1253
		$h(x_0,\underline{\psi})$	0.317	0.003	0.3163	0.0004	0.3148	0.0033	0.4343	0.2196	0.2147
		$r(x_0,\underline{\psi})$	1.4691	0.0088	1.4669	0.0011	1.4623	0.0102	1.6205	1.2579	0.3627
	53	$R(x_0,\underline{\psi})$	0.8091	0.0008	0.8089	9.9352E-05	0.8085	0.0009	0.8615	0.7526	0.1089
		$h(x_0,\underline{\psi})$	0.3365	0.003	0.3357	0.0004	0.3343	0.0033	0.4443	0.2352	0.2092
		$r(x_0,\underline{\psi})$	1.4233	0.0039	1.4223	0.0005	1.4203	0.0045	1.5219	1.2812	0.2407
	t=1.5										
	37	$R(x_0,\underline{\psi})$	0.6033	0.0021	0.6028	0.0003	0.6018	0.0023	0.6894	0.5103	0.1791
		$h(x_0,\underline{\psi})$	0.3209	0.0009	0.3207	0.0001	0.3202	0.001	0.3824	0.2653	0.1171
		$r(x_0,\underline{\psi})$	0.4914	0.0035	0.4905	0.0004	0.4888	0.0039	0.6131	0.3787	0.2344
	53	$R(x_0,\underline{\psi})$	0.5897	0.0019	0.5892	0.0002	0.5883	0.0021	0.6751	0.5079	0.1672
		$h(x_0,\underline{\psi})$	0.3223	0.001	0.322	0.0001	0.3215	0.0011	0.3833	0.2636	0.1197
		$r(x_0,\underline{\psi})$	0.4646	0.002	0.4641	0.0003	0.463	0.0023	0.5579	0.3818	0.176
	t=2										
	37	$R(x_0,\underline{\psi})$	0.5156	0.002	0.8219	0.0001	0.5141	0.0022	0.6011	0.4273	0.1738
		$h(x_0,\underline{\psi})$	0.3083	0.0005	0.3082	6.2874E-05	0.308	0.0006	0.3542	0.2647	0.0894
		$r(x_0,\underline{\psi})$	0.3304	0.0019	0.3299	0.0002	0.3289	0.0021	0.4219	0.2492	0.1727
	53	$R(x_0,\underline{\psi})$	0.5037	0.0019	0.8089	9.9352E-05	0.5023	0.0021	0.5908	0.4227	0.1681
		$h(x_0,\underline{\psi})$	0.3097	0.0007	0.3096	8.3781E-05	0.3092	0.0008	0.3598	0.261	0.0988
		$r(x_0,\underline{\psi})$	0.3152	0.0012	0.3149	0.0001	0.3143	0.0013	0.3866	0.2539	0.1326

of moments, modified ML estimation or modified moments. Furthermore, within the Bayesian framework, investigating the impact of different loss functions, including general entropy, balanced SE, balanced LINEX, and precautionary loss functions, on parameter and reliability estimation warrants attention.

Future work could consider empirical and E-Bayesian estimation of the AFWE-L distribution's parameters, rf, and hrf, as well as the prediction of future observations. Investigating the ML and Bayesian estimation. Also, predictions under various censoring approaches, such as Type II, Type I, progressive, and hybrid censored data, would be valuable. Exciting directions for future research include generalizing this study to dual generalized order statistics or generalized order statistics, exploring optimal accelerated life testing models under different stress conditions, examining finite mixture and compound models involving the FWE and L distributions, developing new competing risks models with other lifetime distributions, and constructing new additive families of distributions based on classes of lifetime models.

Data Availability: The data that support the findings of this study are available upon request from the corresponding author.

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