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*Research article*

# Statistical Analysis of Inverse Weibull based on Step-Stress Partially Accelerated Life Tests with Unified Hybrid Censoring Data

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**Abstract:** Accelerated life testing (ALT) is a primary method for rapidly evaluating product reliability. This paper focuses on statistical inference for the inverted Weibull distribution under a step-stress partially ALT (SSPALT) model with a unified hybrid censoring scheme. This censoring scheme enhances the efficiency of statistical analysis and reduces overall test time. The inverted Weibull distribution is widely used in reliability engineering to model various failure mechanisms, including infant mortality, wear-out periods, and general life testing scenarios. For the purpose of estimating the model parameters and acceleration factor, the maximum likelihood approach is applied along with the maximum product of the spacing procedure to generate point and interval estimates. The squared error loss function is used to calculate the Bayes point estimates based on the assumption of independent gamma priors. Since Bayesian estimators cannot be derived analytically, we employ the Markov chain Monte Carlo technique. This method allows us to construct credible intervals for the involved parameters of interest and the Bayesian estimates themselves. Asymptotic confidence intervals and confidence intervals using bootstrap-p and bootstrap-t methods are constructed. Moreover the performances of the various estimators of the SSPALT are compared through the simulation study. Based on the numerical results, the Bayes estimates are better than the corresponding other estimates with respect to smallest precision measures. The lengths of creditable interval for Bayesian estimates less than the approximate and Bootstrap confidence intervals for different sample sizes, observed failures, and censoring schemes, in most cases. Additionally, given different sample sizes, reported failures, and censoring techniques, the percentile Bootstrap confidence intervals give more accurate outcomes than Bootstrap-t in most cases. A real data set is used to illustrate the results derived.

**Keywords:** Step-Stress partially accelerated life tests; unified hybrid censoring; inverse Weibull distribution; maximum product of spacing estimation; Bayesian estimation
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## 1. Introduction

It is crucial to make products with a longer lifetime because in many industrial domains, rapid industrial progress, a variety of production techniques, and company competitiveness are necessary for enhancing product quality. Technological innovations have consistently improved the manufacturing of products. Consequently, gathering failure data for high reliability components under normal operating circumstances is difficult. Under normal conditions, lifelong testing is costly and time-consuming as a result. In order to speed up the assessment of product reliability in accelerated environmental circumstances, accelerated life tests (ALTs) and partial ALTs (PALTs) have gained widespread acceptance.

Test units are subjected to considerably more stress during ALTs in order to increase the possibility of noticing failures through step-stress ALTs (SSALTs). Test units are put through higher than normal operating or design stress levels, such as temperature, voltage, humidity, vibration, dust, mechanical stress, pressure, force, and use-rate, in order to minimise the test period. These observed accelerated failure times can then be used to draw conclusions about the real lifetime distribution under typical use situations. When performing ALTs, various stress types, including SSPALTs, random stress ALTs, and constant stress ALTs, are used. A test unit is stressed at progressively higher levels during SSALTs. A test unit goes through a series of stress procedures, beginning with a low stress level that is maintained for a predetermined amount of time. The stress level is raised and maintained for an additional predetermined amount of time if the unit survives this initial stress. Until the unit collapses, this procedure is repeated with ever higher stress levels. Notably, the stress levels and durations in each test unit usually follow a consistent pattern. SSPALTs have been extensively studied under a variety of distributional assumptions. Table 1 lists the research related to SSPALT that uses various censoring approaches for various failure models.

Assume $n$ identical items are the independent and identically distributed lifetimes and $y_{1:n}, y_{2:n}, ..., y_{n:n}$ items that are ordered according to failure times. Epstein [22] introduced the hybrid censoring scheme (HCS); the test is completed if a predetermined number of items, $1 \leq r \leq n$, of $n$, fail or when a predetermined period $T = (0, \infty)$ is reached. If the experiment ends at a random time, $T_1^* = min(y_{r:n}, T)$ this is type I (T-I) HCS (T-I HCS). The main issue with this type is a relatively low failure rate up to the predefined time $T_1^*$.

Childs et al. [17] presented type II (T-II) HCS (T-II HCS), which ensures a fixed number of failures and has a termination time $T_2^* = max(y_{r:n}, T)$. This method's potential for long testing periods, however, is a major disadvantage because it could take a long time to notice the necessary number of failures and finish the life test.

Chandrasekar et al. [15] introduced generalized T-I HCS (GT-I HCS) and generalized T-II HCS (GT-II HCS). Considering $k, r \in (1, 2, ..., n)$ and time $T \in (0, \infty)$: The two schemes have the following description:

- GT-I HCS with $k < r$ ; $T^* = min(y_{r:n}, T)$ if the $k^{th}$ failure happens before the time $T$ and $T^* = y_{k:n}$ if the $k^{th}$ failure happens after the time $T$.
- GT-II HCS, with $T_1, T_2 \in (0, \infty)$ where $T_1 < T_2$. If the $r^{th}$ failure occurs before time, $T_1$; so $T^* = T_1$

**Table 1.** Review of the literature related to proposed work

| Author(s) name | Method | Scheme | Failure model |
|---|---|---|---|
| Bai et al. [11] | PALT | Type I censoring | Log Normal distribution |
| Bai and Chung [12] | CSPALT | Type I censoring | Exponential distribution |
| Hassan et al. [26] | CSPALT | Type I and II censoring | Weibull distribution |
| Abdel Ghaly et al. [3] | SSPALT | Type I and Type II censoring data | Weibull distribution |
| Abdel Ghaly et al. [2] | SSPALT | Type II censoring data | Pareto distribution |
| Abdel-Ghani [4] | SSPALT | Type I censoring data | log logistic distribution |
| Ismail[35] | SSPALT | Type I censoring data | Gompertz distribution |
| Abd-Elfattah et al. [1] | SSPALT | type I censoring | Burr Type-XII distribution |
| Ling et al. [38] | SSALT | Progressive Type-I hybrid censoring scheme | Exponential distribution |
| Ismail and Aly [35] | SSPALT | Type - II censoring | Weibull distribution |
| Hassan, and Al-Thobety [25] | SSPALT | Type - II censoring | Inverted weibull distribution |
| Ismail [33] | SSPALT | Adaptive Type-I progressive hybrid censoring scheme | Weibull distribution |
| Ismail [32] | SSPALT | Adaptive Type-II progressive hybrid censoring scheme | Weibull distribution |
| El-Din et al. [20] | SSPALT | Progressive Type-II censoring | Extension of Exponential distribution |
| Hassan et al. [24] | SSPALT | progressive censoring with random removal | Pareto distribution |
| Ismail [31] | SSPALT | Progressive Type-II censoring scheme | Generalized Rayleigh distribution |
| Nassar et al. [42] | SSPALT | Adaptive Type-I/Type-II progressive hybrid censored schemes | Burr Type-XII distribution |
| Xiaolin et al.[45] | SSPALT | Progressive Type-II hybrid censored | Modified Weibull distribution |
| Alam and Ahmed [9] | SSPALT | Adaptive Type-II progressive hybrid censored | Exponentiated Pareto distribution |
| Hassan et al. [27] | SSPALT | Adaptive Type-II progressive hybrid censored | Lomax distribution |
| Bantan et al. [14] | SSPALT | Progressive type-II censored | Weighted Lomax distribution |
| Alam et al.[8] | SSPALT | Adaptive progressively hybrid censored schemes | Odd Lindley Half-Logistic distribution |
| Lone and Panahi [39] | CSPALT | Unified hybrid censored | Gompertz distribution |
| Alotaibi et al. [41] | CSPALT | Type-I Progressive Censored Data | Alpha Power Exponential |

and $T^* = y_{r:n}$ if the $r^{th}$ failure between $T_1$ and $T_2$; otherwise, $T^* = T_2$.

Balakrishnan et al. [13] presented UHC scheme (UHCS) that combines GT-I HCS and GT-II HCS in order to overcome the limitations of both types. Consider $r, k \in (1, 2, ..., n)$ and the time points $T_1, T_2 \in (0, \infty)$ in this scheme. $T^* = min(max(y_{r:n}, T_1), T_2)$ if the $k^{th}$ failure occurred prior to time $T_1$. $T^* = min(y_{r:n}, T_2)$ if the $k^{th}$ failure happens between $T_1$ and $T_2$, and $T^* = y_{(k:n)}$, if the $k^{th}$ failure happens after $T_2$. Using this censoring strategy, we can guarantee that the experiment will be completed in time $T_2$ with at least $k$ failures, and if not, exactly $k$ failures. As a result, we have the six cases under this UHCS (see Table 2). For more studies about UHCS, refer to Rad and Izanlo [44], Panahi and Sayyareh [43], Joen and Kang [36], Helmy et al. [29], Abo-Kasem et al.[5], Abushal [6]

**Table 2.** Test completion cases under UHCS

| Cases | UHCS | Terminated time | Number of failure units |
|---|---|---|---|
| 1 | $0 < y_{k:n} < y_{r:n} < T_1 < T_2$ | $T_1$ | $d_1$ |
| 2 | $0 < y_{k:n} < T_1 < y_{r:n} < T_2$ | $y_{r:n}$ | $r$ |
| 3 | $0 < y_{k,:n} < T_1 < T_2 < y_{r:n}$ | $T_2$ | $d_2$ |
| 4 | $0 < T_1 < y_{k:n} < y_{r:n} < T_2$ | $y_{r:n}$ | $r$ |
| 5 | $0 < T_1 < y_{k:n} < T_2 < y_{r:n}$ | $T_2$ | $d_2$ |
| 6 | $0 < T_1 < T_2 < y_{k:n} < y_{r:n}$ | $y_{k:n}$ | $k$ |

Despite the popularity of the inverse Weibull (IW) distribution and the flexibility of the UHCS in modeling lifespan data, no study has yet investigated parameter estimation for the IW distribution under SSPALT using UHCS. In order to fill this gap, this work uses UHCS samples to propose maximum likelihood (ML), maximum product spacing (MPS), and Bayesian estimate methods for the IW distribution under SSPALT. The following are this paper's main goals:

- Employ ML and MPS methods to derive point and interval estimators for the model parameters.
- Implement Bayesian estimation techniques to obtain Bayesian point and interval estimates.
- Construct bootstrap confidence intervals (CIs) based on the ML and MPS methods. In addition to, provide credible intervals based on the Bayesian method.
- Conduct a simulation study to compare the efficiency of the derived estimators in terms of mean squared error and bias.
- Examine actual data to demonstrate the usefulness of the suggested estimators and the acceleration factor.

The layout of the article is as follows: The model and assumptions are described in detail in Section 2. The ML estimators and CIs of the model parameters are given in Section 3. The MPS estimators for the SSPALT model based on UHCS are given in Section 4. In Section 5, the Bayesian estimator of the model parameters are discussed. The simulation study as well as application with real data are given, respectively, in Sections 6 and 7. Finally, the conclusions are given in Section 8.

## 2. Inverted Weibul Distribution under Assumptions of SSPALT

The IW distribution is an important lifetime model and has many applications, including reliability, life testing, and survival analysis. The IW distribution is crucial because of the non-monotonicity of the hazard rate; it has applications in the research of mortality and breast cancer. Keller et al. [37] showed that the IW distribution fits failure data of pistons, crankshafts, main bearings, etc. The IW distribution is more closely related than the exponential and Weibull distributions in the degradation of mechanical components of diesel engines. According to research by Akgul et al. [7], the IW distribution fits the wind speed data more closely than the Weibull distribution does. The IW distribution has several uses in network design, risk management, financial issues, data analysis related to earthquakes and flood levels, and other extreme value distributions. The probability density function (PDF) of the IW distribution, with scale ( $\theta$ ) and shape ($\alpha$) parameters, is given below as

$$f(y;\theta,\alpha) = \theta\alpha y^{-\alpha-1} e^{-\theta y^{-\alpha}}; \quad y,\alpha,\theta > 0, \tag{2.1}$$

The following is the cumulative distribution function (CDF):

$$f(y;\theta,\alpha) = e^{-\theta y^{-\alpha}};\ y,\alpha,\theta, > 0. \tag{2.2}$$

The survival and hazard rate functions are as follows:

$$f(y;\theta,\alpha) = 1 - e^{-\theta y^{-\alpha}}, \tag{2.3}$$

and,

$$f(y;\theta,\alpha) = \frac{\theta\alpha y^{-\alpha-1} e^{-\theta y^{-\alpha}}}{1 - e^{-\theta y^{-\alpha}}}. \tag{2.4}$$

DeGroot and Goel [19] introduced the concept of SSPALT, the test procedure and its assumptions are described as follows:

$$Y = \begin{cases} T; & 0 < T < \tau \\ \tau + (T-\tau)/\beta; & T \geq \tau \end{cases} \tag{2.5}$$

where $T$ is the lifetime of an item under usual operating conditions, $\tau$ is the stress change time, and $\beta(> 1)$ is the acceleration factor. The PDF and CDF of $Y$ under the SSPALT model are given by

$$f(y) = \begin{cases} f_1(y) = \theta\alpha y^{-\alpha-1}(e^{-\theta y^{-\alpha}}); & 0 < y < \tau \\ f_2(y) = \theta\alpha\beta[\tau + \beta(x-\tau)]^{-\alpha-1}(e^{-\theta[\tau+\beta(y-\tau)]^{-\alpha}}); & y \geq \tau, \end{cases} \tag{2.6}$$

and CDF is given as follows:

$$f(y) = \begin{cases} F_1(y) = e^{-\theta y^{-\alpha}}; & 0 < y < \tau \\ F_2(y) = e^{-\theta[\tau+\beta(y-\tau)]^{-\alpha}}; & y \geq \tau. \end{cases} \tag{2.7}$$

The main assumptions of the test procedure in SSPALT, based on UHCS, are considered as follows:

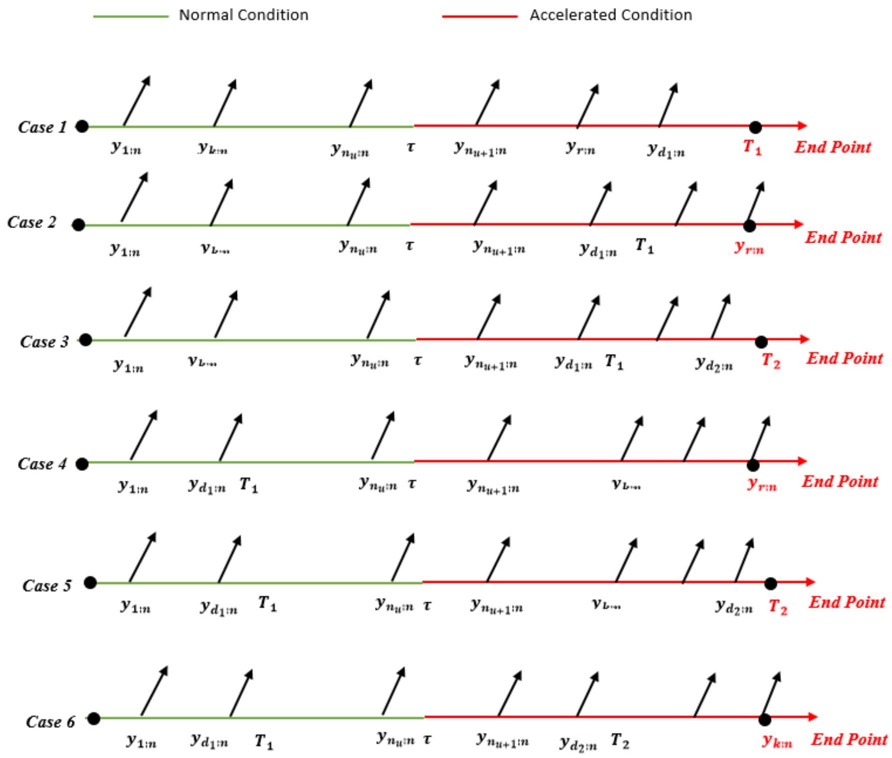


**Figure 1.** Description of the UHCS within the SSPALT Framework

1. $n$ identical and independent units, with a common IW distribution, are put on the life test.

2. The test is terminated at the integers $r, k \in (1, 2, ..., n)$ and the time points $T_1, T_2 \in (0, \infty)$ in this scheme. $T^* = min(max(y_{r:n}, T_1), T_2)$ if the $k^{th}$ failure occurred prior to time $T_1$. $T^* = min(y_{r:n}, T_2)$ if the $k^{th}$ failure happens between $T_1$ and $T_2$, and $T^* = y_{k:n}$, if the $k^{th}$ failure happens after $T_2$.

3. Each of the $n$ units is first operated under normal operating conditions. If it does not fail from the test by a pre-specified time, it is put under an accelerated condition (stress).

4. Some failures may arise under normal operating conditions, while others may occur during periods of accelerated stress. In other words, at least one failure must occur under normal conditions before time $\tau$, and at least one failure must occur under accelerated conditions. As a result, under the UHCS, we observe the six cases illustrated in Figure 1. One of these cases will be examined in detail using a specific sample:

Case 1: $0 < y_{1:n} < y_{2:n} < \ ... \ < y_{n_u:n} < \tau < y_{n_u+1:n} < \ ... \ < y_{d1:n}, ; if; y_{k:n} < y_{r:n} < \ T_1,$
Case 2: $0 < y_{1:n} < y_{2:n} < \ ... \ < y_{n_u:n} < \tau < y_{n_u+1:n} < \ ... \ < y_{r:n}, ; if; y_{k:n} < T_1 < y_{r:n},$
Case 3: $0 < y_{1:n} < y_{2:n} < \ ... \ < y_{n_u:n} < \tau < y_{n_u+1:n} < \ ... \ < y_{d2:n}, ; if; y_{k:n} < T_1 < \ T_2 \ ,$
Case 4: $0 < y_{1:n} < y_{2:n} < \ ... \ < y_{n_u:n} < \tau < y_{n_u+1:n} < \ ... \ < y_{r:n}, ; if \ T_1 < y_{k:n} < y_{r:n},$
Case 5: $0 < y_{1:n} < y_{2:n} < \ ... \ < y_{n_u:n} < \tau < y_{n_u+1:n} < \ ... \ < y_{d2:n}, ; if \ T_1 < y_{k:n} < \ T_2,$
Case 6: $0 < y_{1:n} < y_{2:n} < \ ... \ < y_{n_u:n} < \tau < y_{n_u+1:n} < \ ... \ < y_{k:n}, ; if \ T_1 < T_2 < y_{k:n}.$

## 3. Maximum Likelihood Estimators

In this section, the model parameters and accelerated factor are estimated via ML method. The likelihood function of the parameters is given by:

$$L(data|\Psi) = \frac{n!}{(n-D)!}\left[1 - F(C)\right]^{n-D}\left[\prod_{i=1}^{n_u} f_1(y)\right]\left[\prod_{i=n_u+1}^{D} f_2(y)\right], \tag{3.1}$$

where $\Psi = (\alpha, \beta, \theta)^T$ is the set of parameters, $D$ denotes the total number of unsuccessful experiments up to time $C$ and as stated by

$$(D, C) = \begin{cases} d_1, T_1 & for; case\ 1 \\ r, y_r & for; case\ 2, 4 \\ d_2, T_2 & for; case\ 3, 5 \\ k, y_k & for; case\ 6, \end{cases}$$

where the number of failures prior to $T_1$ and $T_2$ is indicated by $d_1$ and $d_2$. One may obtain the unified likelihood function by substituting Equations (2.6) and (2.7) in Equation (3.1)

$$L(\Psi) \propto \prod_{i=1}^{n_u} \theta\alpha y_i^{-\alpha-1} e^{-\theta y_i^{-\alpha}} \prod_{i=n_u+1}^{D} \theta\alpha\beta[A_1]^{-\alpha-1} e^{-\theta[A_1]^{-\alpha}} \ \left[1 - e^{-\theta[A_2]^{-\alpha}}\right]^{n-D}, \tag{3.2}$$

where $y_i$ is written instead of $y_{i:n}$ for simplified from which represents the units for the items obtained from IW distribution, $i = 1, 2, ..., n_i$, $A_1 = \tau + \beta(y_i - \tau)$, $A_2 = \tau + \beta(C - \tau)$ . Thus, the log-likelihood function, represented by $lnL$, can be given as:

$$\begin{aligned} lnL = {} & D\log\theta + D\log\alpha + (D - n_u)\log\beta - (\alpha+1)\sum_{i=1}^{n_u}\log y_i - \theta\sum_{i=1}^{n_u} y_i^{-\alpha} \\ & - (\alpha+1)\sum_{i=n_u+1}^{D}\log A_1 - \theta\sum_{i=n_u+1}^{D} A_1^{-\alpha} + (n-D)\log[1 - e^{-\theta[A_2]^{-\alpha}}]. \end{aligned} \tag{3.3}$$

To obtain the ML estimators for the unknown parameters $\theta, \alpha$ and $\beta$, we maximize the likelihood function given in Equation (3.3). By partially differentiating this function with respect to each parameter, we obtain the following system of equations:

$$\frac{\partial lnL}{\partial\theta} = \frac{D}{\theta} - \sum_{i=1}^{n_u} y_i^{-\alpha} - \sum_{i=1}^{D} A_1^{-\alpha} + \frac{(n-D)\,[A_2]^{-\alpha}}{e^{\theta[A_2]^{-\alpha}} - 1}, \tag{3.4}$$

$$\begin{aligned} \frac{\partial lnL}{\partial\alpha} = {} & \frac{D}{\alpha} - \sum_{i=1}^{n_u}\log y_i + \theta\sum_{i=1}^{n_u} y_i^{-\alpha}\log y_i + \theta\sum_{i=1}^{D} A_1^{-\alpha}\log A_1 \\ & - \sum_{i=1}^{D}\log A_1 - \frac{\theta(n-D)\,[A_2]^{-\alpha}\log[A_2]}{e^{\theta[A_2]^{-\alpha}} - 1}, \end{aligned} \tag{3.5}$$

and,

$$\frac{\partial lnL}{\partial\beta} = \frac{D-n_u}{\beta} - (\alpha+1)\sum_{i=1}^{D}\frac{(y_i-\tau)}{A_1} + \theta\alpha\sum_{i=1}^{D}A_1^{-\alpha-1}(y_i-\tau) - \frac{\theta\alpha(n-D)(C-\tau)\left[A_2\right]^{-\alpha-1}}{e^{\theta[A_2]^{-\alpha}}-1}. \tag{3.6}$$

To obtain the ML estimators of $\theta$, $\alpha$ and $\beta$, the nonlinear system of Equations (3.4), (3.5) and (3.6) are to be solved numerically using R 4.1.2 software with the 'maxLik' package introduced by Henningsen and Toomet [30].

### 3.1. Asymptotic Confidence Interval

In this subsection, the asymptotic CIs (ACIs) of the parameters $\theta$, $\alpha$ and the acceleration factor $\beta$ based on the ML method are derived. The Fisher information matrix (FIM) is used to obtain the asymptotic variances of ML estimators (see Cohen [18]). The expected FIM is given by:

$$F = \begin{pmatrix} \frac{\partial^2 lnL}{\partial\theta^2} & \frac{\partial^2 lnL}{\partial\theta\partial\alpha} & \frac{\partial^2 lnL}{\partial\theta\partial\beta} \\ \frac{\partial^2 lnL}{\partial\alpha\partial\theta} & \frac{\partial^2 lnL}{\partial\alpha^2} & \frac{\partial^2 lnL}{\partial\alpha\partial\beta} \\ \frac{\partial^2 lnL}{\partial\alpha\partial\beta} & \frac{\partial^2 lnL}{\partial\theta\partial\beta} & \frac{\partial^2 lnL}{\partial\beta^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

The second partial derivative for $\theta, \alpha$ and $\beta$ are provided as follows:

$$I_{11} = \frac{-D}{\theta^2} - \frac{(n-D)\left[A_2\right]^{-2\alpha}\left[e^{\theta[A_2]^{-\alpha}}\right]}{\left[e^{\theta[A_2]^{-\alpha}}-1\right]^2},$$

$$I_{22} = -\frac{D}{\alpha^2} - \theta\alpha\sum_{i=1}^{n_u} y_i^{-\alpha}\left(\log y_i\right)^2 - \theta\sum_{i=1}^{D}\left[\tau+\beta(x-\tau)\right]^{-\alpha}\log\left[\tau+\beta(x-\tau)\right]^2 + \frac{(n-D)[A_2]^{-\alpha}\left(\log A_2\right)^2\left[\left(e^{\theta[A_2]^{-\alpha}}-1\right)-\theta e^{\theta[A_2]^{-\alpha}}\right]}{[e^{\theta[A_2]^{-\alpha}}-1]^2},$$

$$I_{33} = -\frac{D-n_u}{\beta^2} + (\alpha+1)\sum_{i=1}^{D}\frac{(y_i-\tau)^2}{A_1^2} - \theta\alpha\sum_{i=1}^{D}A_1^{-\alpha-1}(y_i-\tau)^2 + \frac{\theta\alpha(n-D)(C-\tau)^2[A_2]^{-\alpha-2}\left[(\alpha+1)\left(e^{\theta[A_2]^{-\alpha}}-1\right)-\theta\alpha e^{\theta[A_2]^{-\alpha}}[A_2]^{-\alpha}\right]}{[e^{\theta[A_2]^{-\alpha}}-1]^2},$$

$$I_{12}=I_{21} = \sum_{i=1}^{n_u} y_i^{-\alpha}\log y_i + \sum_{i=1}^{D}A_1^{-\alpha}\log A_1 + \frac{(n-D)(A_2)^{-\alpha}\log[A_2]\left[\left(e^{\theta[A_2]^{-\alpha}}-1\right)-\theta e^{\theta[A_2]^{-\alpha}}\right]}{\left[e^{\theta[A_2]^{-\alpha}}-1\right]^2},$$

$$I_{13} = I_{31} = \alpha\sum_{i=1}^{D}\left[\tau+\beta(x-\tau)\right]^{-\alpha-1}; (y_i-\tau) - \frac{(n-D)\alpha[A_2]^{-\alpha-1}(C-\tau)\left[\left(e^{\theta[A_2]^{-\alpha}}-1\right)-\theta[A_2]^{-\alpha}e^{\theta[A_2]^{-\alpha}}\right]}{\left[e^{\theta[A_2]^{-\alpha}}-1\right]^2},$$

$$I_{23} = I_{32} = -\sum_{i=1}^{D}\frac{(y_i-\tau)}{\tau+\beta(y_i-\tau)} + \theta\sum_{i=1}^{D}\left[\left(\tau+\beta(y_i-\tau)\right)^{-\alpha-1}(y_i-\tau)\left[1-\alpha\log\left(\tau+\beta(y_i-\tau)\right)\right]\right] - \frac{(n-D)\theta(C-\tau)[A_2]^{-\alpha-1}\left[\left[1-\alpha\log(A_2)\right]\left(e^{\theta[A_2]^{-\alpha}}-1\right)-\theta\alpha[A_2]^{-\alpha}\log[A_2]e^{\theta[A_2]^{-\alpha}}\right]}{[e^{\theta[A_2]^{-\alpha}}-1]^2}.$$

Consequently, the ML estimators of $\Psi = (\theta, \alpha, \beta\ )$ have an asymptotic variance covariance matrix defined by inverting the FIM and substituting $\hat{\theta}, \hat{\alpha}$ and $\hat{\beta}$ respectively. Therefore, the two-sided approximate $100(1-\gamma)\%$ percent limits for the ML estimators of $\Psi = (\theta, \alpha, \beta\ )$ can be obtained as follows:

$$\widehat{\Psi}_k \pm Z_{\gamma/2}\sqrt{Var(\widehat{\Psi}_k)}, \quad k = 1, 2, 3, \tag{3.7}$$

where, $z_{\gamma/2}$ is the standard normal percentile and var is the variance of $\widehat{\Psi} = (\hat{\theta}, \hat{\alpha}, \hat{\beta})$. Note that 95% ACIs of $\theta, \alpha$, and $\beta$ were calculated via the Newton-Raphson method.

### 3.2. *Bootstrap confidence intervals*

In this subsection, we propose to use bootstrap CIs. For this purpose, we generate parametric bootstrap samples and obtained two different bootstrap CIs. Initially, we used Efron's [21] percentile bootstrap (boot-P) technique. Next, we looked at the bootstrap-t (boot-t) method, which is based on Hall's [23] methodology. These bootstrap CIs work as follows:

#### 3.2.1. Boot-P method

1. Based on the UHCS, compute the ML and MPS estimates $\hat{\theta}, \hat{\alpha}$ and $\hat{\beta}$
2. Generate random samples from proposed moedel, then generate a bootstrap unified hybrid censored sample.
3. Calculate bootstrap estimates $\hat{\theta}, \hat{\alpha}$ and $\hat{\beta}$ say, $\hat{\theta}^{*}, \hat{\alpha}^{*}$ and $\hat{\beta}^{*}$.
4. To get B bootstrap samples, repeat Steps 2-3 B several times.
5. Arrange all $\hat{\theta}^{*}, \hat{\alpha}^{*}$ and $\hat{\beta}^{*}$ in ascending order as $(\hat{\theta}_k^{*[1]}, \hat{\theta}_k^{*[2]}, ..., \hat{\alpha}_k^{*[B]})$ $(\hat{\alpha}_k^{*[1]}, \hat{\alpha}_k^{*[2]}, ..., \hat{\beta}_k^{*[B]})$ $(\hat{\beta}_k^{*[1]}, \hat{\beta}_k^{*[2]}, ..., \hat{\beta}_k^{*[B]})$
6. Then, the $100(1-\gamma)\%$ Boot-P CI for $\Psi = (\theta, \alpha\, \beta)$ is given by:

$$(\widehat{\Psi}_k^{*}[B_{\gamma/2}], \widehat{\Psi}_k^{*}[B_{1-\gamma/2}]), \quad k = 1, 2, 3. \tag{3.8}$$

#### 3.2.2. Boot-T method

1. Run the Boot-p method's Steps 1 through 3 again.
2. Compute the t-statistic for parameter as: $T_1^{*} = \frac{\hat{\theta}^{*} - \hat{\theta}}{\sqrt{Var(\hat{\theta}^{*})}}$ , $T_2^{*} = \frac{\hat{\alpha}^{*} - \hat{\alpha}}{\sqrt{Var(\hat{\alpha}^{*})}}$ , $T_3^{*} = \frac{\hat{\beta}^{*} - \hat{\beta}}{\sqrt{Var(\hat{\beta}^{*})}}$
3. Obtain $T_k{*}^{(1)}, T_k{*}^{(2)}, ..., T_k^{*}(B)$ where $k = 1, 2, 3$ after repeating Steps 1-2 B many times
4. Arrange all $T_k{*}^{(1)}, T_k{*}^{(2)}, ..., T_k{*}^{(B)}$ in ascending order and denote $T_k{*}^{[1]}, T_k{*}^{[2]}, ..., T_k{*}^{[B]}$.
5. Then, the $100(1-\gamma)\%$ Boot-t CI for $\Psi = (\theta, \alpha, \beta)$ is given by:

$$(\widehat{\Psi}_k - \ T_L, \widehat{\Psi}_k + T_U), \quad k = 1, 2, 3, \tag{3.9}$$

where $T_L = T_k *^{[B_{(\gamma/2)}]} \sqrt{Var(\widehat{\Psi}_k)}$, and $T_U = T_k *^{[B_{(1-\gamma/2)}]} \sqrt{Var(\widehat{\Psi}_k)}$.

## 4. Maximum Product of Spacing Estimator

Cheng and Amin [16] presented the MPS approach as an alternative to the ML method. In small sample sizes, the MPS approach outperforms the ML method for heavy-tailed or skewed distributions. The parameter values that maximize the product of the distances between the distribution function values at neighboring ordered points are selected in order to evaluate the MPS estimators. This section proposes the MPS procedure to provide the point estimates and ACIs for the IW distribution parameters and acceleration factor under UCHS samples, assuming a SSPALT model. This novel approach extends the applicability of the MPC method to this specific censoring scheme and lifetime model. The first-order partial derivatives of Equation (3.1), with respect to $\theta, \alpha$, and $\beta$, are derived as follows:

$$M(\Psi) \propto \prod_{i=1}^{n_u}[F_1(y_i;\theta,\alpha) - F_1(y_{i-1};\theta,\alpha)] \prod_{i=n_u+1}^{D+1}[F_2(y_i;\Psi) - F_2(y_{i-1};\Psi)] \\ [1-F_2(C;\Psi)]^{n-D}. \tag{4.1}$$

Thus,

$$M(\Psi) \propto \prod_{i=1}^{n_u}[e^{-\theta y_i^{-\alpha}} - e^{-\theta y_{i-1}^{-\alpha}}] \prod_{i=n_u+1}^{D+1}[e^{-\theta A_1^{-\alpha}} - e^{-\theta A_3^{-\alpha}}] \\ [1-e^{-\theta[A_2]^{-\alpha}}]^{n-D};, \tag{4.2}$$

where $A_3 = [\tau + \beta(y_{i-1} - \tau)]$. The natural logarithm of Equation (4.2) is as follows:

$$lnM(\Psi) = \sum_{i=1}^{n_u}\log[e^{-\theta y_i^{-\alpha}} - e^{-\theta y_{i-1}^{-\alpha}}] + \sum_{i=n_u+1}^{D+1}\log[e^{-\theta A_1^{-\alpha}} - e^{-\theta A_3^{-\alpha}}] \\ + (n-D)\log[1-e^{-\theta[A_2]^{-\alpha}}];. \tag{4.3}$$

The MPS estimators of $\theta$, $\alpha$ and $\beta$, can be computed by differentiating Equation (4.3) with respect to $\theta$, $\alpha$ and $\beta$,

$$\frac{\partial lnM(\Psi)}{\partial\theta} = \sum_{i=1}^{n_u}\frac{e^{-\theta y_i^{-\alpha}}y_{i-1}^{-\alpha} - e^{-\theta y_{i-1}^{-\alpha}}y_i^{-\alpha}}{e^{-\theta y_i^{-\alpha}} - e^{-\theta y_{i-1}^{-\alpha}}} + (n-D)\frac{e^{-\theta[A_2]^{-\alpha}}[A_2]^{-\alpha}}{1-e^{-\theta[A_2]^{-\alpha}}} \\ + \sum_{i=n_u+1}^{D+1}\frac{e^{-\theta A_3^{-\alpha}}A_3^{-\alpha} - e^{-\theta A_1^{-\alpha}}A_1^{-\alpha}}{e^{-\theta A_1^{-\alpha}} - e^{-\theta A_3^{-\alpha}}}, \tag{4.4}$$

$$\frac{\partial lnM(\Psi)}{\partial\alpha} = \sum_{i=1}^{n_u}\frac{\theta e^{-\theta y_i^{-\alpha}}y_i^{-\alpha}\log y_i - \theta e^{-\theta y_i^{-\alpha}}y_{i-1}^{-\alpha}\log y_{i-1}}{e^{-\theta y_i^{-\alpha}} - e^{-\theta y_{i-1}^{-\alpha}}} - \frac{(n-D)\theta e^{-\theta[A_2]^{-\alpha}}\log[A_2]}{1-e^{-\theta[A_2]^{-\alpha}}}; \\ + \sum_{i=n_u+1}^{D+1}\frac{\theta e^{-\theta A_1^{-\alpha}}A_1^{-\alpha}logA_1 - \theta e^{-\theta A_3^{-\alpha}}logA_3}{e^{-\theta A_1^{-\alpha}} - e^{-\theta A_3^{-\alpha}}}, \tag{4.5}$$

$$\frac{\partial lnM(\Psi)}{\partial\beta} = \sum_{i=n_u+1}^{D+1}\frac{\theta\alpha(y_i-\tau)e^{-\theta A_1^{-\alpha}}A_1^{-\alpha-1} - \theta\alpha(y_{i-1}-\tau)e^{-\theta A_3^{-\alpha}}A_3^{-\alpha-1}}{e^{-\theta A_1^{-\alpha}} - e^{-\theta A_3^{-\alpha}}} \\ - (n-D)\frac{\theta\alpha(C-\tau)e^{-\theta[A_2]^{-\alpha}}[A_2]^{-\alpha-1}}{1-e^{-\theta[A_2]^{-\alpha}}}. \tag{4.6}$$

The nonlinear system of Equations (4.4), (4.5), and (4.5) is solved numerically using a nonlinear optimization algorithm. By setting these equations to zero, we obtain the MPS estimators for $\theta$, $\alpha$ and $\beta$.

## 5. Bayesian Estimation

This section focuses on Bayesian estimation and credible interval construction for the parameters $\theta$, $\alpha$ and $\beta$. Bayesian method has gained significant popularity for analyzing failure time data, as they incorporate prior knowledge about the parameters and utilize the information provided by the observed data. In this study, we employ a Bayesian approach to obtain point and interval estimators for the unknown parameters and acceleration factor. Based on gamma priors for $\theta \sim Gamma(a_1, b_1)$ and $\alpha \sim Gamma(a_2, b_2)$ and Jeffery prior density function for $\beta(\beta \propto 1/\beta)$, which are given by:

$$\pi_1(\theta) \propto \theta^{a_1-1}e^{-\theta b_1}, \theta > 0, \tag{5.1}$$

$$\pi_2(\alpha) \propto \alpha^{a_2-1}e^{-\alpha b_2}, \alpha > 0, \tag{5.2}$$

$$\pi_3(\beta) \propto 1/\beta, \beta > 1, \tag{5.3}$$

where $a_1, b_1, a_2$, and $b_2$ are the hyper-parameters. The joint prior distribution for $\theta, \alpha$ and $\beta$ is given by

$$\pi(\Psi) \propto \frac{1}{\beta}\theta^{a_1-1}\alpha^{a_2-1}e^{-(\theta b_1+\alpha b_2)}. \tag{5.4}$$

The joint posterior density function of $\theta, \alpha$ and $\beta$, denoted by $\pi^*(\Psi|data)$, can be written as:

$$\pi^*(\Psi|data) = \frac{L(\Psi)\pi(\Psi)}{\int_1^\infty \int_0^\infty \int_0^\infty L(\Psi)\pi(\Psi)d\theta d\alpha d\beta}. \tag{5.5}$$

The joint posterior of the parameters $\theta$, $\alpha$ and $\beta$ for the SSPALT under the assumption of the IW distribution is given by combinning the likelihood function (3.1) and the joint prior density (5.4)

$$\begin{aligned}\pi\,(\Psi|data) \propto\, & e^{-(b_1\theta+b_2\alpha)}\theta^{D+a_1-1}\alpha^{D+a_2-1}\beta^{D-n_u-1}\prod_{i=1}^{n_u} y_i^{-\alpha-1}e^{-\theta\sum_{i=1}^{n_u} y_i^{-\alpha}} \\ & \prod_{i=n_u+1}^{D}\left[\tau+\beta(y_i-\tau)\right]^{-\alpha-1} e^{-\theta\sum_{i=n_u+1}^{D}\left[\tau+\beta(y_i-\tau)\right]^{-\alpha}}[1-e^{-\theta[\tau+\beta(C-\tau)]^{-\alpha}}]^{n-D}.\end{aligned} \tag{5.6}$$

Thus, under the squared error loss function (SELF), the Bayesian estimators $\tilde{\theta}, \tilde{\alpha}$ and $\tilde{\beta}$ of $\theta$, $\alpha$ and $\beta$, respectively, are obtained by minimizing the posterior expected loss. These estimators are equivalent to the posterior means, which are given by:

$$\tilde{\theta} = \int_0^\infty \theta\,\pi\,(\Psi|data)\,d\theta, \tag{5.7}$$

$$\tilde{\alpha} = \int_0^\infty \alpha\,\pi(\Psi|data)d\alpha, \tag{5.8}$$

and,

$$\tilde{\beta} = \int_1^\infty \beta\,\pi\,(\Psi|data)\,d\beta. \tag{5.9}$$

Since it is extremely difficult to calculate the integrals described in Equations (5.7), (5.8), and (5.9) analytically, a numerical evaluation is performed using the Markov chain Monte Carlo (MCMC) method. Gibbs sampling and a more generic Metropolis within Gibbs samplers are important subclasses of the MCMC method. Hastings [28] and Metropolis et al. [40] were the first to introduce the MCMC approach. The two most widely used MCMC method variations are the Gibbs sampling and the Metropolis-Hastings (MH) algorithm.

The conditional posterior distribution of parameters $\theta$, $\alpha$ and acceleration $\beta$ are given by:

$$\pi_1^*(\theta|\alpha,\beta,data) \propto \theta^{D+a_1-1} e^{-\theta(b_1+\sum_{i=1}^{n_u} y_i^{-\alpha}+\sum_{i=n_u+1}^{D}[\tau+\beta(y_i-\tau)]^{-\alpha}} e^{(n-D)log[1-e^{-\theta[\tau+\beta(C-\tau)]^{-\alpha}}]}, \tag{5.10}$$

$$\begin{aligned}\pi_2^*(\alpha|\theta,\alpha,data) \propto\ & \alpha^{D+a_2-1} e^{-(b_2\alpha+\theta\sum_{i=1}^{n} y_i^{-\alpha}} ) e^{-\theta\sum_{i=n_u+1}^{D}[\tau+\beta(y_i-\tau)]^{-\alpha}} \\ & e^{-(\alpha+1)\sum_{i=n_u+1}^{D} log[(\tau+\beta(y_i-\tau)]} ] e^{(n-D)log[1-e^{-\theta[\tau+\beta(C-\tau)]^{-\alpha}}]},\end{aligned} \tag{5.11}$$

$$\begin{aligned}\pi_3^*(\beta|\theta,\alpha,data) \propto\ & \beta^{D-n_u-1} e^{-(\alpha+1)\sum_{i=n_u+1}^{D} log[(\tau+\beta(y_i-\tau)]} ] \\ & e^{-\theta\sum_{i=n_u+1}^{D}[\tau+\beta(y_i-\tau)]^{-\alpha}} e^{(n-D)log[1-e^{-\theta[\tau+\beta(C-\tau)]^{-\alpha}}]}.\end{aligned} \tag{5.12}$$

It is clear that the full conditional distributions in Equations (5.10), (5.11), and (5.12) cannot be reduced to any well-known distributions. Like acceptance-rejection sampling, the MH technique takes into account the possibility that candidate values may be generated from a proposal distribution at each algorithm iteration. The following procedures are used by the MH algorithm to create a series of draws from this distribution:

1. Put the initial $\left(\Psi_k^{(0)}\right)$; where $k = 1,2,3$ satisfying the condition of $\pi\left(\Psi_k^{(0)}\right) > 0$.
2. Using the initial value, sample a candidate point $(\Psi^*)$from proposal $\delta(\Psi^*)$.
3. Given the candidate $(\Psi^*)$, the acceptance probability is calculated by the following formula:
$$\eta_k = min\left(1, \frac{L\left(\Psi_k^*|data\right)\pi\left(\Psi_k^*\right)\delta(\Psi_k)}{L(data|\Psi_k)\pi(\Psi_k)\delta\left(\Psi_k^*\right)}\right);\ k = 1,2,3.$$
4. Draw a value of $u$ from the uniform (0, 1) distribution; if $u \leq \eta_k$ accept $\Psi_k^*$ as $\Psi_k^{(j)}$.
5. Otherwise, reject $\Psi_k^*$ and set $\Psi_k^{(j)} = \Psi_k^{(j-1)}$.
6. To obtain $j$ draws, we repeat steps 2-5 $(j+1)$ times.
7. Obtain Bayes estimate (BE) for $\Psi_k$, using the SELF, given by; $\sum_{j=1}^{J} \frac{\left(\Psi_k^{(j-1)}\right)_j}{J}$.
8. Steps (1–7) have to be repeated $k$ times to get BE of $\Psi_k$.

According to Hastings [28], the Bayes credible intervals (BCIs) of $\Psi =; (\theta,;\alpha,\beta)$ can be obtained through the following steps:

1. Arrange $\Psi_k^{(j)}; k = 1,2,3$ as $\theta^{[1]},\theta^{[2]},...,;\theta^{[M]},\alpha^{[1]},\alpha^{[2]},...,\alpha^{[M]} and\ \beta^{[1]},\beta^{[2]},...,;\beta^{[M]}$,where $M$ denotes the length of the generated simulation.
2. The $100(1-\gamma)\%$ BCIs of $\Psi = (\theta,\alpha,\beta)$ are acquired as:
$$\left(\Psi_k^{\left[M\frac{\gamma}{2}\right]}, \Psi_k^{\left[M\left(1-\frac{\gamma}{2}\right)\right]}\right),\ k = 1,2,3.$$

## 6. Simulation Studies

This section presents simulation results comparing the performance of classical estimates (ML, MPS) and BEs under the UHCS with SSPALT. The BEs were obtained using gamma and uniform priors under SELF. The primary challenge with the Bayesian approach was determining the posterior distribution; deviations from the posterior density were simulated using the MH algorithm. The simulation steps were as follows:

1. The true parameter values of $(\theta, \alpha, \beta, \tau)$, are selected as;
   $(\theta = 1.5, \alpha = 0.5, \beta = 2, \tau = 3), (\theta = 1.5, \alpha = 0.5, \beta = 4, \tau = 3)$
2. Based on UHCS, select the sample size (failure items) and time at $n = 50$

$$\begin{aligned} k = 10, r = 25, T_1 = 30, T_2 = 40,\\ k = 15, r = 35, T_1 = 30, T_2 = 40,\\ k = 20, r = 45, T_1 = 30, T_2 = 40,\\ k = 25, r = 30, T_1 = 20, T_2 = 40,\\ k = 25, r = 45, T_1 = 20, T_2 = 40,\\ k = 30, r = 40, T_1 = 15, T_2 = 25, \end{aligned}$$

   and at $n = 100$;

$$\begin{aligned} k = 20, r = 50, T_1 = 60, T_2 = 80,\\ k = 30, r = 70, T_1 = 60, T_2 = 80,\\ k = 40, r = 90, T_1 = 60, T_2 = 80,\\ k = 50, r = 60, T_1 = 40, T_2 = 80,\\ k = 50, r = 90, T_1 = 40, T_2 = 80,\\ k = 60, r = 80, T_1 = 30, T_2 = 50, \end{aligned}$$

3. The associated values of the hyper-parameters $(a_1, b_1, a_2, b_2)$ were taken as (0.9, 1.7, 1.6, 1.5).
4. Using the SSPALT algorithm within UHCS, create 10,000 random samples of size $n = 50$, and $n = 100$ from the IW distribution.
5. Calculate the ML estimates (MLEs), MPS estimates (MPSEs), as well as the associated ACIs at a 95 % confidence level $\gamma = 0.05$. The BEs and associated credible intervals are at a 95% confidence level$\gamma = 0.05$.
6. To generate the posterior samples, using the 'coda' package, a total of 12,000 MCMC samples were generated. The first 2,000 samples were discarded as burn-in. The subsequent 10,000 samples were utilized to calculate Bayes point and interval estimates for $\theta, \alpha$ and $\beta$ employing ML methods.
7. The performance of the estimates is evaluated based on measures of accuracy, bias, mean squared error (MSE), and CI length (L.CI). Simulation results are shown in Tables 3, 4, 5, 6, 7, 8.

The following conclusions can be constructed in considering the simulation results that were obtained:

1. For fixed values of $n, k$ , $r, T_1$, and $T_2$, the MSE values of MLEs, MPSEs, and BEs decrease when $n$ increases.

2. For fixed values of $n, k$ , $r, T_1$, and $T_2$, the biases values of MLEs, MPSEs, and BEs decrease when $n$ increases.

3. For fixed values of $n, k$ , $r, T_1$, and $T_2$, the length of CIs values for Boot-P, Boot-t, and BCI decrease with increases values of $n$.

4. For the majority of parameter situations, for fixed values of $\theta, \alpha$ and $\tau$, the bias values of estimates of $\theta$ and $\alpha$ decrease for ML, MPS and Bayesian estimation methods with $\beta$ increases.

5. For the majority of parameter setting, the MSE values of $\theta$ and $\alpha$ estimates decrease for all estimation methods with increasing values of $\beta$ while keeping $\theta, \alpha$ and $\tau$ fixed.

6. For most parameter settings, the length of CI values of $\theta$ and $\alpha$ estimates decrease for ML, MPS and Bayesian estimation methods with an increasing value of $\beta$, given fixed values of $\theta, \alpha$ and $\tau$.

7. For all the UHCS, it is observed that the BEs perform better than classical methods (ML and MPS) based on bias, MSE and length of CI.

**Table 3.** The mean of all estimates in SSPALT under UHCS

| | | | | | $\beta = 2$ | | | | | | | | | $\beta = 4$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | ML | | | MPS | | | Bayesian | | | ML | | | MPS | | | Bayesian | | |
| $n$ | $k$ | $r$ | $T_1$ | $T_2$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
| 50 | 10 | 25 | 30 | 40 | 0.91 | 1.65 | 2.56 | 0.70 | 1.50 | 1.88 | 1.88 | 1.95 | 2.82 | 1.51 | 2.06 | 1.50 | 1.41 | 2.28 | 0.15 | 1.99 | 2.13 | 2.03 |
| | 15 | 35 | 30 | 40 | 1.07 | 1.68 | 2.65 | 0.86 | 1.55 | 2.30 | 1.93 | 2.04 | 2.90 | 1.59 | 2.08 | 1.69 | 1.49 | 2.33 | 0.99 | 1.95 | 2.15 | 2.20 |
| | 20 | 45 | 30 | 40 | 1.24 | 2.07 | 2.81 | 1.15 | 1.82 | 2.85 | 1.98 | 2.04 | 2.93 | 1.68 | 2.22 | 2.01 | 1.63 | 2.49 | 2.08 | 1.95 | 2.34 | 2.24 |
| | 25 | 30 | 20 | 40 | 1.37 | 2.13 | 2.86 | 1.26 | 1.84 | 3.08 | 1.99 | 2.16 | 2.96 | 1.74 | 2.23 | 2.10 | 1.69 | 2.52 | 2.54 | 1.96 | 2.37 | 2.31 |
| | 25 | 45 | 20 | 40 | 1.06 | 1.68 | 2.58 | 0.78 | 1.54 | 3.10 | 1.90 | 2.05 | 2.88 | 1.59 | 2.08 | 1.54 | 1.45 | 2.53 | 2.59 | 1.96 | 2.15 | 2.15 |
| | 30 | 40 | 15 | 25 | 1.11 | 1.70 | 2.73 | 0.89 | 1.57 | 3.11 | 1.91 | 2.18 | 2.93 | 1.61 | 2.09 | 1.85 | 1.50 | 2.55 | 2.61 | 1.96 | 2.16 | 2.24 |
| 100 | 20 | 50 | 60 | 80 | 1.72 | 2.45 | 2.90 | 1.69 | 2.37 | 3.09 | 1.97 | 2.41 | 3.08 | 1.72 | 2.27 | 3.16 | 1.64 | 2.31 | 2.83 | 1.97 | 2.46 | 2.91 |
| | 30 | 70 | 60 | 80 | 1.52 | 2.48 | 2.95 | 1.78 | 2.31 | 3.10 | 1.90 | 2.31 | 3.14 | 1.74 | 2.49 | 3.09 | 1.72 | 2.33 | 2.85 | 2.02 | 2.51 | 3.06 |
| | 40 | 90 | 60 | 80 | 1.54 | 2.43 | 2.72 | 1.61 | 2.16 | 3.15 | 1.95 | 2.49 | 3.15 | 1.66 | 2.40 | 3.12 | 1.50 | 2.38 | 2.97 | 2.05 | 2.53 | 3.32 |
| | 50 | 50 | 40 | 80 | 1.47 | 2.47 | 2.82 | 1.33 | 2.16 | 3.18 | 1.95 | 2.41 | 3.08 | 1.76 | 2.44 | 3.07 | 1.52 | 2.40 | 2.71 | 2.05 | 2.54 | 3.38 |
| | 50 | 90 | 40 | 80 | 1.72 | 2.43 | 2.97 | 1.60 | 2.40 | 3.11 | 1.91 | 2.43 | 3.10 | 1.86 | 2.48 | 3.11 | 1.67 | 2.34 | 3.13 | 2.06 | 2.55 | 3.40 |
| | 60 | 40 | 30 | 50 | 1.63 | 2.41 | 3.00 | 1.51 | 2.28 | 3.03 | 1.96 | 2.44 | 3.11 | 1.88 | 2.39 | 3.14 | 3.14 | 1.246 | 3.01 | 2.06 | 2.56 | 3.42 |

**Table 4.** Comparative Analysis of the MSE in SSPALT under UHCS for Various Estimates

| | | | | | $\beta = 2$ | | | | | | | | | $\beta = 4$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | ML | | | MPS | | | Bayesian | | | ML | | | MPS | | | Bayesian | | |
| $n$ | $k$ | $r$ | $T_1$ | $T_2$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
| 50 | 10 | 25 | 30 | 40 | 1.184 | 0.948 | 1.042 | 1.398 | 1.096 | 1.72 | 0.221 | 0.648 | 0.778 | 0.585 | 0.541 | 2.098 | 0.692 | 0.317 | 3.444 | 0.111 | 0.467 | 1.57 |
| | 15 | 35 | 30 | 40 | 1.024 | 0.914 | 0.945 | 1.236 | 1.048 | 1.3 | 0.172 | 0.554 | 0.694 | 0.505 | 0.517 | 1.904 | 0.611 | 0.27 | 2.61 | 0.079 | 0.45 | 1.402 |
| | 20 | 45 | 30 | 40 | 0.86 | 0.524 | 0.789 | 0.944 | 0.778 | 0.75 | 0.116 | 0.563 | 0.67 | 0.423 | 0.382 | 1.592 | 0.465 | 0.108 | 1.522 | 0.051 | 0.255 | 1.354 |
| | 25 | 30 | 20 | 40 | 0.724 | 0.466 | 0.74 | 0.838 | 0.758 | 0.52 | 0.104 | 0.438 | 0.639 | 0.355 | 0.372 | 1.494 | 0.412 | 0.083 | 1.058 | 0.045 | 0.226 | 1.292 |
| | 25 | 45 | 20 | 40 | 1.038 | 0.92 | 1.021 | 1.318 | 1.054 | 0.5 | 0.198 | 0.546 | 0.716 | 0.512 | 0.52 | 2.056 | 0.652 | 0.066 | 1.006 | 0.042 | 0.453 | 1.446 |
| | 30 | 40 | 15 | 25 | 0.99 | 0.898 | 0.865 | 1.208 | 1.026 | 0.49 | 0.186 | 0.418 | 0.672 | 0.488 | 0.506 | 1.744 | 0.597 | 0.052 | 0.984 | 0.036 | 0.442 | 1.358 |
| 100 | 20 | 50 | 60 | 80 | 0.126 | 0.147 | 0.7 | 0.413 | 0.231 | 0.51 | 0.146 | 0.192 | 0.523 | 0.076 | 0.327 | 0.442 | 0.455 | 0.285 | 0.772 | 0.382 | 0.143 | 0.686 |
| | 30 | 70 | 60 | 80 | 0.095 | 0.114 | 0.652 | 0.323 | 0.29 | 0.5 | 0.145 | 0.284 | 0.454 | 0.056 | 0.11 | 0.513 | 0.38 | 0.269 | 0.751 | 0.275 | 0.091 | 0.539 |
| | 40 | 90 | 60 | 80 | 0.053 | 0.168 | 0.877 | 0.493 | 0.44 | 0.45 | 0.137 | 0.112 | 0.451 | 0.037 | 0.194 | 0.483 | 0.597 | 0.22 | 0.628 | 0.254 | 0.071 | 0.274 |
| | 50 | 50 | 40 | 80 | 0.048 | 0.127 | 0.781 | 0.172 | 0.434 | 0.42 | 0.136 | 0.188 | 0.515 | 0.036 | 0.162 | 0.526 | 0.575 | 0.201 | 0.887 | 0.233 | 0.054 | 0.222 |
| | 50 | 90 | 40 | 80 | 0.095 | 0.167 | 0.626 | 0.101 | 0.202 | 0.49 | 0.135 | 0.168 | 0.495 | 0.035 | 0.123 | 0.485 | 0.432 | 0.262 | 0.468 | 0.277 | 0.049 | 0.197 |
| | 60 | 40 | 30 | 50 | 0.037 | 0.185 | 0.598 | 0.191 | 0.315 | 0.57 | 0.131 | 0.162 | 0.487 | 0.023 | 0.211 | 0.459 | 0.351 | 0.246 | 0.584 | 0.269 | 0.043 | 0.175 |

**Table 5.** Evaluating the bias for different estimates in SSPALT using UHCS

| | | | | | $\beta = 2$ | | | | | | | | | $\beta = 4$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | ML | | | MPS | | | Bayesian | | | ML | | | MPS | | | Bayesian | | |
| $n$ | $k$ | $r$ | $T_1$ | $T_2$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
| 50 | 10 | 25 | 30 | 40 | 1.086 | 0.850 | 0.944 | 1.300 | 0.998 | 1.622 | 0.123 | 0.550 | 0.680 | 0.487 | 0.443 | 2.000 | 0.594 | 0.219 | 3.346 | 0.013 | 0.369 | 1.472 |
| | 15 | 35 | 30 | 40 | 0.926 | 0.816 | 0.847 | 1.138 | 0.950 | 1.202 | 0.074 | 0.456 | 0.596 | 0.407 | 0.419 | 1.806 | 0.513 | 0.172 | 2.512 | 0.048 | 0.352 | 1.304 |
| | 20 | 45 | 30 | 40 | 0.762 | 0.426 | 0.691 | 0.846 | 0.680 | 0.652 | 0.018 | 0.465 | 0.572 | 0.325 | 0.284 | 1.494 | 0.367 | 0.010 | 1.424 | 0.047 | 0.157 | 1.256 |
| | 25 | 30 | 20 | 40 | 0.626 | 0.368 | 0.642 | 0.740 | 0.660 | 0.422 | 0.006 | 0.340 | 0.541 | 0.257 | 0.274 | 1.396 | 0.314 | -0.015 | 0.960 | 0.039 | 0.128 | 1.194 |
| | 25 | 45 | 20 | 40 | 0.940 | 0.822 | 0.923 | 1.220 | 0.956 | 0.402 | 0.100 | 0.448 | 0.618 | 0.414 | 0.422 | 1.958 | 0.554 | -0.032 | 0.908 | 0.038 | 0.355 | 1.348 |
| | 30 | 40 | 15 | 25 | 0.892 | 0.800 | 0.767 | 1.110 | 0.928 | 0.392 | 0.088 | 0.320 | 0.574 | 0.390 | 0.408 | 1.646 | 0.499 | -0.046 | 0.886 | 0.037 | 0.344 | 1.260 |
| 100 | 20 | 50 | 60 | 80 | 0.284 | 0.049 | 0.602 | 0.315 | 0.133 | 0.412 | 0.028 | 0.094 | 0.425 | 0.278 | 0.229 | 0.344 | 0.357 | 0.187 | 0.674 | 0.033 | 0.045 | 0.588 |
| | 30 | 70 | 60 | 80 | 0.477 | 0.016 | 0.554 | 0.225 | 0.192 | 0.402 | 0.097 | 0.186 | 0.356 | 0.258 | 0.012 | 0.415 | 0.282 | 0.171 | 0.653 | -0.019 | -0.007 | 0.441 |
| | 40 | 90 | 60 | 80 | 0.456 | 0.070 | 0.779 | 0.395 | 0.342 | 0.352 | 0.055 | 0.014 | 0.353 | 0.339 | 0.096 | 0.385 | 0.499 | 0.122 | 0.530 | -0.047 | -0.027 | 0.176 |
| | 50 | 50 | 40 | 80 | 0.535 | 0.029 | 0.683 | 0.674 | 0.336 | 0.322 | 0.050 | 0.090 | 0.417 | 0.238 | 0.064 | 0.428 | 0.477 | 0.103 | 0.789 | -0.053 | -0.044 | 0.124 |
| | 50 | 90 | 40 | 80 | 0.279 | 0.069 | 0.528 | 0.403 | 0.104 | 0.392 | 0.087 | 0.070 | 0.397 | 0.137 | 0.025 | 0.387 | 0.334 | 0.164 | 0.370 | -0.056 | -0.049 | 0.099 |
| | 60 | 40 | 30 | 50 | 0.371 | 0.087 | 0.500 | 0.493 | 0.217 | 0.472 | 0.039 | 0.064 | 0.389 | 0.125 | 0.113 | 0.361 | 0.253 | 0.148 | 0.486 | -0.062 | -0.055 | 0.077 |

**Table 6.** The 95 % CI for $\theta$ estimate of IW parameters and acceleration factor based on UHCS data

| | | | | | $\beta = 2$ | | | | | | | $\beta = 4$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | ACI | | Boot-p | | Boot-t | | BCI | ACI | | Boot-p | | Boot-t | | BCI |
| $n$ | $k$ | $r$ | $T_1$ | $T_2$ | ML | MPS | ML | MPS | ML | MPS | | ML | MPS | ML | MPS | ML | MPS | |
| 50 | 10 | 25 | 30 | 40 | 2.069 | 2.0582 | 1.366 | 1.454 | 1.764 | 1.7532 | 1.267 | 1.568 | 1.917 | 1.174 | 1.191 | 1.263 | 1.612 | 1.203 |
| | 15 | 35 | 30 | 40 | 1.699 | 1.7202 | 1.268 | 1.331 | 1.394 | 1.4152 | 0.982 | 1.53 | 1.491 | 0.951 | 0.978 | 1.225 | 1.186 | 0.911 |
| | 20 | 45 | 30 | 40 | 1.541 | 2.0482 | 1.044 | 1.687 | 1.236 | 1.7432 | 0.853 | 0.758 | 1.571 | 0.507 | 0.518 | 0.453 | 1.266 | 0.535 |
| | 25 | 30 | 20 | 40 | 1.541 | 1.2372 | 0.786 | 0.869 | 1.236 | 0.9322 | 0.505 | 0.447 | 1.455 | 0.501 | 0.46 | 0.142 | 1.15 | 0.459 |
| | 25 | 45 | 20 | 40 | 1.409 | 1.0332 | 0.85 | 0.636 | 1.104 | 0.7282 | 0.437 | 0.943 | 0.977 | 0.456 | 0.445 | 0.638 | 0.672 | 0.434 |
| | 30 | 40 | 15 | 25 | 1.345 | 0.413 | 0.786 | 0.604 | 1.04 | 0.108 | 0.405 | 0.823 | 1.361 | 0.424 | 0.413 | 0.518 | 1.056 | 0.402 |
| 100 | 20 | 50 | 60 | 80 | 0.987 | 0.842 | 0.461 | 0.503 | 0.682 | 0.537 | 0.326 | 0.541 | 1.055 | 0.223 | 0.26 | 0.236 | 0.75 | 0.314 |
| | 30 | 70 | 60 | 80 | 0.937 | 0.804 | 0.41 | 0.377 | 0.632 | 0.499 | 0.201 | 0.509 | 0.689 | 0.197 | 0.227 | 0.204 | 0.384 | 0.198 |
| | 40 | 90 | 60 | 80 | 0.638 | 0.632 | 0.313 | 0.334 | 0.333 | 0.327 | 0.215 | 0.463 | 0.688 | 0.154 | 0.15 | 0.158 | 0.383 | 0.153 |
| | 50 | 50 | 40 | 80 | 0.523 | 0.521 | 0.288 | 0.324 | 0.218 | 0.216 | 0.144 | 0.452 | 0.727 | 0.144 | 0.146 | 0.147 | 0.422 | 0.144 |
| | 50 | 90 | 40 | 80 | 0.601 | 0.697 | 0.278 | 0.318 | 0.296 | 0.392 | 0.238 | 0.446 | 0.546 | 0.138 | 0.14 | 0.141 | 0.241 | 0.139 |
| | 60 | 40 | 30 | 50 | 0.552 | 0.508 | 0.242 | 0.201 | 0.247 | 0.203 | 0.212 | 0.487 | 0.479 | 0.12 | 0.122 | 0.182 | 0.174 | 0.121 |

**Table 7.** The 95 % CI for $\alpha$ estimate of IW parameters and acceleration factor based on UHCS data

| | | | | | $\beta = 2$ | | | | | | | $\beta = 4$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | ACI | | Boot-p | | Boot-t | | BCI | ACI | | Boot-p | | Boot-t | | BCI |
| $n$ | $k$ | $r$ | $T_1$ | $T_2$ | ML | MPS | ML | MPS | ML | MPS | | ML | MPS | ML | MPS | ML | MPS | |
| 50 | 10 | 25 | 30 | 40 | 2.258 | 2.2472 | 1.555 | 1.643 | 1.953 | 1.9422 | 1.456 | 1.757 | 2.106 | 1.363 | 1.38 | 1.452 | 1.801 | 1.392 |
| | 15 | 35 | 30 | 40 | 1.888 | 1.9092 | 1.457 | 1.52 | 1.583 | 1.6042 | 1.171 | 1.719 | 1.68 | 1.14 | 1.167 | 1.414 | 1.375 | 1.1 |
| | 20 | 45 | 30 | 40 | 1.73 | 2.2372 | 1.233 | 1.876 | 1.425 | 1.9322 | 1.042 | 0.947 | 1.76 | 0.696 | 0.707 | 0.642 | 1.455 | 0.724 |
| | 25 | 30 | 20 | 40 | 1.73 | 1.4262 | 0.975 | 1.058 | 1.425 | 1.1212 | 0.694 | 0.636 | 1.644 | 0.69 | 0.649 | 0.331 | 1.339 | 0.648 |
| | 25 | 45 | 20 | 40 | 1.598 | 1.2222 | 1.039 | 0.825 | 1.293 | 0.9172 | 0.626 | 1.132 | 1.166 | 0.645 | 0.634 | 0.827 | 0.861 | 0.623 |
| | 30 | 40 | 15 | 25 | 1.534 | 0.602 | 0.975 | 0.793 | 1.229 | 0.297 | 0.594 | 1.012 | 1.55 | 0.613 | 0.602 | 0.707 | 1.245 | 0.591 |
| 100 | 20 | 50 | 60 | 80 | 1.176 | 1.031 | 0.65 | 0.692 | 0.871 | 0.726 | 0.515 | 0.73 | 1.244 | 0.412 | 0.449 | 0.425 | 0.939 | 0.503 |
| | 30 | 70 | 60 | 80 | 1.126 | 0.993 | 0.599 | 0.566 | 0.821 | 0.688 | 0.39 | 0.698 | 0.878 | 0.386 | 0.416 | 0.393 | 0.573 | 0.387 |
| | 40 | 90 | 60 | 80 | 0.827 | 0.821 | 0.502 | 0.523 | 0.522 | 0.516 | 0.404 | 0.652 | 0.877 | 0.343 | 0.339 | 0.347 | 0.572 | 0.342 |
| | 50 | 50 | 40 | 80 | 0.712 | 0.71 | 0.477 | 0.513 | 0.407 | 0.405 | 0.333 | 0.641 | 0.916 | 0.333 | 0.335 | 0.336 | 0.611 | 0.333 |
| | 50 | 90 | 40 | 80 | 0.79 | 0.886 | 0.467 | 0.507 | 0.485 | 0.581 | 0.427 | 0.635 | 0.735 | 0.327 | 0.329 | 0.33 | 0.43 | 0.328 |
| | 60 | 40 | 30 | 50 | 0.741 | 0.697 | 0.431 | 0.39 | 0.436 | 0.392 | 0.401 | 0.676 | 0.668 | 0.309 | 0.311 | 0.371 | 0.363 | 0.31 |

**Table 8.** The 95 % CI for $\beta$ estimate of IW parameters and acceleration factor based on UHCS data

| | | | | | $\beta = 2$ | | | | | | | $\beta = 4$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | ACI | | Boot-p | | Boot-t | | BCI | ACI | | Boot-p | | Boot-t | | BCI |
| $n$ | $k$ | $r$ | $T_1$ | $T_2$ | ML | MPS | ML | MPS | ML | MPS | | ML | MPS | ML | MPS | ML | MPS | |
| 50 | 10 | 25 | 30 | 40 | 1.878 | 2.227 | 1.484 | 1.501 | 1.573 | 1.922 | 1.513 | 2.379 | 2.3682 | 1.676 | 1.764 | 2.074 | 2.0632 | 1.577 |
| | 15 | 35 | 30 | 40 | 1.84 | 1.801 | 1.261 | 1.288 | 1.535 | 1.496 | 1.221 | 2.009 | 2.0302 | 1.578 | 1.641 | 1.704 | 1.7252 | 1.292 |
| | 20 | 45 | 30 | 40 | 1.068 | 1.881 | 0.817 | 0.828 | 0.763 | 1.576 | 0.845 | 1.851 | 2.3582 | 1.354 | 1.997 | 1.546 | 2.0532 | 1.163 |
| | 25 | 30 | 20 | 40 | 0.757 | 1.765 | 0.811 | 0.77 | 0.452 | 1.46 | 0.769 | 1.851 | 1.5472 | 1.096 | 1.179 | 1.546 | 1.2422 | 0.815 |
| | 25 | 45 | 20 | 40 | 1.253 | 1.287 | 0.766 | 0.755 | 0.948 | 0.982 | 0.744 | 1.719 | 1.3432 | 1.16 | 0.946 | 1.414 | 1.0382 | 0.747 |
| | 30 | 40 | 15 | 25 | 1.133 | 1.671 | 0.734 | 0.723 | 0.828 | 1.366 | 0.712 | 1.655 | 0.723 | 1.096 | 0.914 | 1.35 | 0.418 | 0.715 |
| 100 | 20 | 50 | 60 | 80 | 0.851 | 1.365 | 0.533 | 0.57 | 0.546 | 1.06 | 0.624 | 1.297 | 1.152 | 0.771 | 0.813 | 0.992 | 0.847 | 0.636 |
| | 30 | 70 | 60 | 80 | 0.819 | 0.999 | 0.507 | 0.537 | 0.514 | 0.694 | 0.508 | 1.247 | 1.114 | 0.72 | 0.687 | 0.942 | 0.809 | 0.511 |
| | 40 | 90 | 60 | 80 | 0.773 | 0.998 | 0.464 | 0.46 | 0.468 | 0.693 | 0.463 | 0.948 | 0.942 | 0.623 | 0.644 | 0.643 | 0.637 | 0.525 |
| | 50 | 50 | 40 | 80 | 0.762 | 1.037 | 0.454 | 0.456 | 0.457 | 0.732 | 0.454 | 0.833 | 0.831 | 0.598 | 0.634 | 0.528 | 0.526 | 0.454 |
| | 50 | 90 | 40 | 80 | 0.756 | 0.856 | 0.448 | 0.45 | 0.451 | 0.551 | 0.449 | 0.911 | 1.007 | 0.588 | 0.628 | 0.606 | 0.702 | 0.548 |
| | 60 | 40 | 30 | 50 | 0.797 | 0.789 | 0.43 | 0.432 | 0.492 | 0.484 | 0.431 | 0.862 | 0.818 | 0.552 | 0.511 | 0.557 | 0.513 | 0.522 |

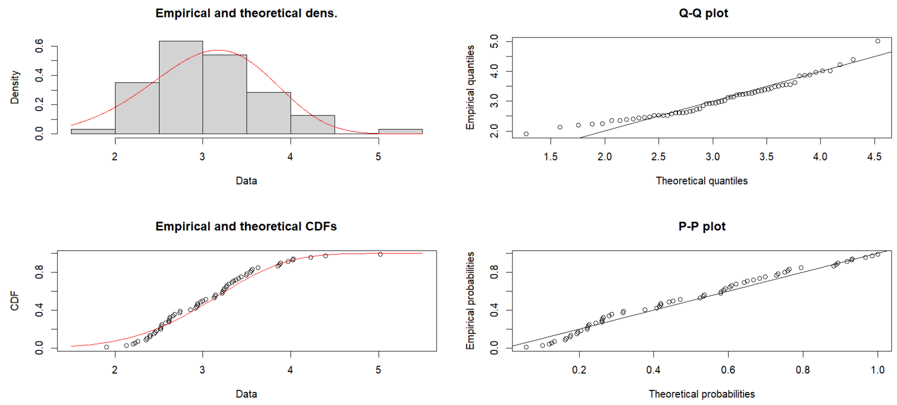


**Figure 2.** Empirical PDF, CDF, Q-Q plot,and P-P plot of IW distribution

## 7. Real Data Application

This section examines accelerated datasets to evaluate the performance of the proposed estimation approaches. The data set was introduced by Bader and Priest [10] as the tensile strength measurements on 1000 carbon fiber-impregnated tows at four different gauge lengths. The recorded data set is given as follows:

1.312 1.314 1.479 1.552 1.700 1.803 1.861 1.865 1.944 1.958 1.966 1.997 2.006 2.021 2.027 2.055 2.063 2.098 2.140 2.179 2.224 2.240 2.253 2.270 2.272 2.274 2.301 2.301 2.359 2.382 2.382 2.426 2.434 2.435 2.478 2.490 2.511 2.514 2.535 2.554 2.566 2.570 2.586 2.629 2.633 2.642 2.648 2.684 2.697 2.726 2.770 2.773 2.800 2.809 2.818 2.821 2.848 2.880 2.954 3.012 3.067 3.084 3.090 3.096 3.128 3.233 3.433 3.585 3.585

Using the Kolmogorov-Smirnov (K-S) goodness of the fit test. The estimated values of parameters for first real data are; $\hat{\theta}$ = 5.434376, $\hat{\alpha}$ = 2.721414 and the K-S distance is determined to be 0.10015, with an associated *pvalue* of 0.5524. Furthermore, multiple plots are analyzed for the goodness of fit test to check further if the data fits the IW distribution. The empirical CDF, the histogram of the PDF, probability-probability (P-P) plot, and quantile-quantile (Q-Q) plots are displayed in Figure 2. This figure indicates that the IW distribution provides an adequate fit to the dataset.

Under SSPALT, we consider that the test runs under normal condition until the high stress is utilized, and raised to make the test under accelerated condition. We assume that the stress change time $\tau$ is 2.360, and the normal and accelerated stress are provided in Table 9.

The SSPALT of the IW distribution under the UHCS with was taken into consideration based on the real data. The accelerated factor and parameter estimates from ML, MPS, and Bayesian analysis were computed. Six artificial UHCD sets are generated as follows:

**Table 9.** Real Data Selection under Different Conditions

| Normal condition | | | Accelerated condition | | | |
|---|---|---|---|---|---|---|
| 1.312 | 1.966 | 2.224 | 2.382 | 2.554 | 2.726 | 3.012 |
| 1.314 | 1.997 | 2.24 | 2.382 | 2.566 | 2.77 | 3.067 |
| 1.479 | 2.006 | 2.253 | 2.426 | 2.57 | 2.773 | 3.084 |
| 1.552 | 2.021 | 2.27 | 2.434 | 2.586 | 2.8 | 3.09 |
| 1.7 | 2.027 | 2.272 | 2.435 | 2.629 | 2.809 | 3.096 |
| 1.803 | 2.055 | 2.274 | 2.478 | 2.633 | 2.818 | 3.128 |
| 1.861 | 2.063 | 2.301 | 2.49 | 2.642 | 2.821 | 3.233 |
| 1.865 | 2.098 | 2.301 | 2.511 | 2.648 | 2.848 | 3.433 |
| 1.944 | 2.14 | 2.359 | 2.514 | 2.684 | 2.88 | 3.585 |
| 1.958 | 2.179 | | 2.535 | 2.697 | 2.954 | 3.585 |

$$\begin{aligned}
&Case\ 1: k = 20, r = 50, T_1 = 3, T_2 = 3.5,\ \text{where}, D = 59, c = T_1 = 3,\\
&Case\ 2: k = 20, r = 50, T_1 = 2.5, T_2 = 3.5,\ \text{where}, D = 50, c = \mathrm{x}_{r:n} = 2.726,\\
&Case\ 3: k = 20, r = 65, T_1 = 2.5, T_2 = 3,\ \text{where}, D = 59, c = T_2 = 2.954,\\
&Case\ 4: k = 20, r = 2.5, T_1 = 1.5, T_2 = 3,\ \text{where}\ D = 36, c = \mathrm{x}_{r:n} = 2.49,\\
&Case\ 5: k = 20, r = 60, T_1 = 1.5, T_2 = 2.7,\ \text{where}, D = 49, c = T_2 = 2.7,\\
&Case\ 6: k = 55, r = 60, T_1 = 1.5, T_2 = 2.5,\ \text{where}, D = 55, c = \mathrm{x}_{k:n} = 2.818,
\end{aligned}$$

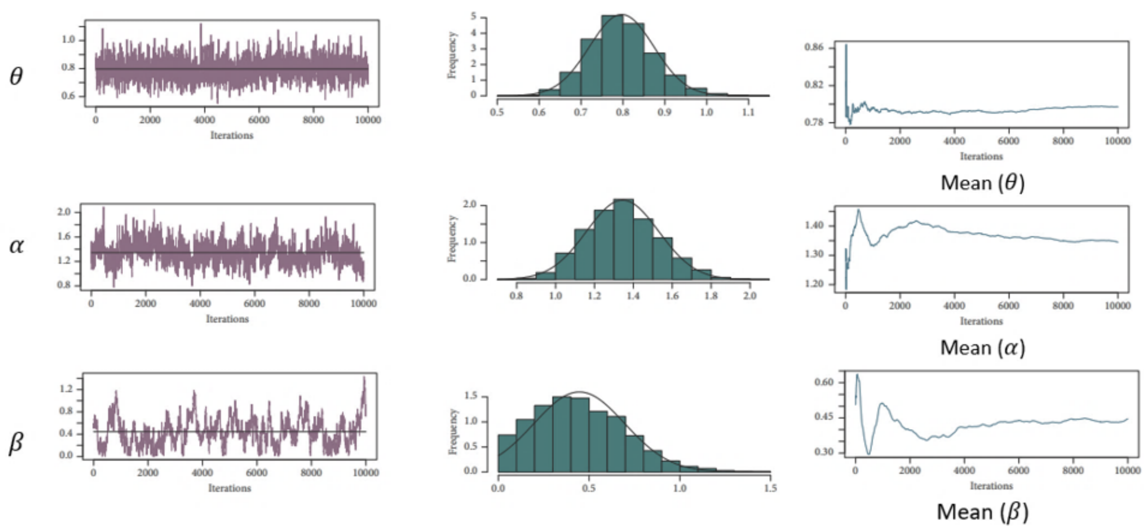


**Figure 3.** Iterations and Convergence of MCMC results for real data

Table 10 presented the standard errors (SEs) of MLEs, MPSEs and BEs for $\theta$, $\alpha$, and $\beta$. Given the lack of prior knowledge, the hyperparameters $a_1$, $b_1$, $a_2$, and $b_2$ are set to be 0.001. Figure 3 illustrates the history plots, MCMC convergence diagnostics, and estimated marginal posterior densities for $\theta$, $\alpha$, and $\beta$. Diagnostic plots indicate good convergence, as the generated posterior values closely match the theoretical posterior density function. The MCMC chains exhibit stability, with no apparent trends or significant autocorrelation.

**Table 10.** SSPALT-based estimates and standard errors for real UCHS data

| | $k$ | $r$ | $T_1$ | $T_2$ | ML | | | MPS | | | Bayesian | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
| Estimates | 20 | 40 | 3 | 3.5 | 9.45 | 25.87 | 9.74 | 8.24 | 24.67 | 14.51 | 5.2 | 8.93 | 17.39 |
| SE | | | | | 5.035 | 4.907 | 5.280 | 4.962 | 4.865 | 5.090 | 4.656 | 4.723 | 6.403 |
| Estimates | 20 | 50 | 2.5 | 3.5 | 5.035 | 4.907 | 5.28 | 4.962 | 4.865 | 5.09 | 4.656 | 4.723 | 6.403 |
| SE | | | | | 4.960 | 4.890 | 5.232 | 4.855 | 4.849 | 5.080 | 4.636 | 4.671 | 6.034 |
| Estimates | 20 | 65 | 2.5 | 3 | 13.39 | 41.26 | 9.79 | 12.19 | 40.06 | 14.56 | 5.69 | 10.94 | 18.93 |
| SE | | | | | 5.177 | 5.074 | 5.457 | 4.834 | 4.800 | 5.030 | 4.617 | 4.651 | 5.331 |
| Estimates | 20 | 2.5 | 1.5 | 3 | 4.96 | 4.89 | 5.232 | 4.855 | 4.849 | 5.08 | 4.636 | 4.671 | 6.034 |
| SE | | | | | 5.155 | 5.042 | 5.361 | 4.813 | 4.780 | 5.000 | 4.616 | 4.634 | 5.095 |
| Estimates | 20 | 60 | 1.5 | 2.7 | 4.81 | 8.09 | 14.07 | 5.61 | 8.89 | 18.84 | 4.82 | 8.1 | 24.77 |
| SE | | | | | 5.012 | 4.876 | 5.206 | 4.857 | 4.842 | 5.070 | 4.615 | 4.629 | 5.075 |
| Estimates | 55 | 60 | 1.5 | 2.5 | 5.177 | 5.074 | 5.457 | 4.834 | 4.8 | 5.03 | 4.617 | 4.651 | 5.331 |
| SE | | | | | 4.931 | 4.861 | 5.178 | 4.849 | 4.826 | 5.150 | 4.603 | 4.623 | 5.067 |

## 8. Conclusions

The analysis of SSPALT based on UHCS data is investigated in this article when the testing products' lifetimes have an IW distribution. The point and interval estimates for ML and MPS methods as classical methods are provided. The approximate confidence intervals are provided using the asymptotic properties of the classical estimates. In addition, the Boot-p and Boot-t CIs for both methods are provided. The point and the interval estimates for the Bayesian method is employed to obtain the unknown parameters. By using the likelihood function, the joint posterior distribution is derived and the Bayes estimates are calculated using the SELF. However,we employ MCMC techniques, specifically the MH algorithm since the difficult nature of the Bayesian estimators and their corresponding BCIs.

Additionally, simulation studies were conducted to investigate the impact of acceleration factor ($\beta$)

and sample size ($n$) on the accuracy of the estimators. The results indicate that the performance of all estimation methods improves as both $\beta$ and $n$ increase. Overall, Bayesian estimation consistently outperforms ML and MPS in terms of bias, MSE, and CI length. As the sample size increases, the CI lengths for all estimates decrease. For smaller sample sizes, boot-p CIs tend to outperform boot-t CIs. To prove the practicality of the techniques used in this work real data taken into consideration. Future research could extend this study to explore the analysis of data under progressive UHCS or other more complex censoring schemes, potentially involving varying distributions.

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