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Research article

A New Shifted Lomax-X Family of Distributions : Properties and Applications to Actuarial and Financial Data

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Abstract: The Lomax distribution, which is often used to describe severe losses and financial risks because of its heavy tail features, is the basis distribution of the shifted Lomax (SHL-X) family of distributions that we propose in this study. The main objective is to increase the adaptability and accuracy of the traditional Lomax model in the representation of complex data sets. We explore various mathematical properties of the special member called the shifted Lomax Weibull (SHL-W), including its moments, quantile function and entropy measures. We employ the maximum likelihood estimation approach to estimate the parameters of the SHL-W distribution. Simulation studies are conducted to evaluate the estimators' accuracy and dependability. The practical applicability of the proposed model is demonstrated by its application to insurance data, highlighting its effectiveness in modeling claims and determining appropriate premium rates.The findings highlight the new model's potential for broader applications in risk management and financial analysis by showing that it is more data-adapted than competing models.

Keywords: Shifted Lomax ; Risk measures ; Lomax weibull; Family of distributions; Mathematical Properties
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1. Introduction

In probability theory, it has been a common practice in recent years to improve existing probability distributions to enhance the flexibility of classical models. These modifications involve various techniques, such as increasing the number of parameters, applying transformations to the original distribution, and mixing two or more distributions. The primary goal of these modifications is to improve the adaptability and accuracy of traditional models in representing complex data sets. Numerous scholars have contributed to the development of the Lomax distribution, having been inspired by these techniques. Extreme loss scenarios and financial hazards are modeled using the Lomax distribution. Its ability to represent heavy tails makes it possible to capture rare but significant events, such as financial crises. It is also applied to study the distribution of income and wealth, due to its ability to model extreme inequalities, which is common in financial data. In insurance, the Lomax distribution helps to model claims and assess the premiums needed to cover the risk of high claims. This continuous probability distribution, sometimes referred to as the Type II Pareto distribution, is mostly used to explain data that has a large tail. The following represents its cumulative distribution function (CDF):

$$H(y) = 1 - \left(1 + \frac{y}{\lambda}\right)^{-\alpha}\ ;\ y \geq 0, \tag{1.1}$$

where the parameters for shape and scale are, respectively, $\alpha > 0$ and $\lambda > 0$. With regard to (1.1), the probability density function (pdf) is provided by:

$$h(y) = \frac{\alpha}{\lambda}\left[1 + \frac{y}{\lambda}\right]^{-\alpha-1}$$

But inaccurate or biased modeling can have detrimental effects as well, such large financial losses or poor choices made in crucial situations. The robustness of risk management systems and their capacity to foresee emergencies can potentially be jeopardized by inaccurate portrayals of catastrophic events. Given these obstacles, it is crucial to make sure that these more adaptable models are applied ethically and with consideration for any potential biases when doing so in contexts where modeling errors could have serious repercussions.

Recently, some researchers [1, 2] introduced new weighted Lomax distributions, while [3] produced the three-parameter exponentiated Lomax distribution. In [4], authors transformed the Lomax distribution to obtain the transmuted exponential Lomax distribution. [5] proposed a mixture form of the Weibull distribution. [6] includes the power parameter β in the uniform lambert distribution to obtain the uniform power lambert distribution. Several distribution generators have also been studied using these different techniques, including the Marshall Olkin generator, the new Weibull-G [7], the beta-G family [8], the beta generalized Marshall-Olkin Kumaraswamy-G family [9], the shifted Gompertz-G [10], the Lomax generator [11], the New Lomax-G family [12], new and improved form of the Lomax model [13], the T-R{generalized lambda} families of distributions [14], the Weibull-G family [15], the Topp-Leone Cauchy family [16], the transmuted Weibull-G family [17], the transmuted odd log-logistic-G family [18], the new hyperbolic sine-generator[19], the Lomax tangent generalized family [20], the exponentiated Weibull-H family [21], the new Topp-Leone Kumaraswamy Marshall-Olkin generated family [22], the Inverse odd Weibull family [23], the exponentiated transmuted-G family [24], the new Power Topp–Leone [25], the transmuted exponentiated generalized-G family [26], the Weibull-X

family [27].
Regarding a random variable T, let $h(t)$ be its pdf, where $T \in [a, b]$ for $-\infty \leq a < b < \infty$. Similarly, let $W[G(x; \zeta)]$ be its CDF, contingent upon the vector parameter ζ meeting the following requirements:

- $W[G(x; \zeta)] \in [a, b]$.
- In addition to being monotonically rising, $W[G(x; \zeta)]$ is differentiable.
- $W[G(x; \zeta)] \to a$ as $x \to -\infty$ and $W[G(x; \zeta)] \to b$ as $x \to +\infty$.

The CDF associated with the T-X family of distributions was recently defined by [28]:

$$F(x) = \int_a^{W[G(x;\zeta)]} h(t)dt.$$

In this article, we propose the following staggered transformation:

$$W[G(x; \zeta)] = x\,(G(x) + m)\,,$$

where $x \geq 0$, $G(x; \zeta)$ is the CDF of one of the classical models and $m \geq 0$ is a constant representing the shift parameter. $W[G(x; \zeta)]$ on $[0, +\infty[$ satisfies the aforementioned requirements. We study here with $h(t)$ the density function of the Lomax function whose support is for $x \geq 0$. This gives us a new family of SHL-X shifted Lomax distributions whose CDF is defined by:

$$F(x; \theta) = 1 - \left(1 + \frac{x\,(G(x; \zeta) + m)}{\lambda}\right)^{-\alpha}, \tag{1.2}$$

where $x \geq 0$, $\theta = (\zeta, \alpha, \lambda, m) \geq 0$.
This distribution function is differentiated based on x. As a result, we get:

$$f(x; \theta) = \frac{\alpha}{\lambda}\left[G(x; \zeta) + m + xg(x; \zeta)\right]\left(1 + \frac{x\,(G(x; \zeta) + m)}{\lambda}\right)^{-\alpha-1}$$

where $g(x; \zeta)$ is the derivate of $G(x; \zeta)$.
Several motivations support our proposed transformation $W[G(x; \zeta)] = x(G(x; \zeta) + m)$, including increased flexibility to fit the distribution to specific data, natural extension of classical models, adaptability to atypical characteristics of real data, preservation of essential mathematical properties such as monotonicity and differentiability, ability to model shifts present in many practical applications, and compatibility with heavy-tailed distributions. In particular, the Lomax density serves as the basis for this new family of distributions, offering a robust framework for capturing heavy tails and data-specific features.

2. Special members of the SHL-X distribution family

Many unique members with extremely intriguing traits have been deduced from this new family of distributions. This section will provide a few unique models that were derived from the SHL-X family by substituting CDFs of various classical distributions for the $G(x)$. The shifted parameters of the Lomax generator are α and λ in the examples below.

1. SHL-Lomax distribution
   To obtain the SHL-Lomax distribution, we take the Lomax distribution for $G(x;\zeta) = 1-\left(1+\frac{x}{k}\right)^{-\gamma}$ in equation (1.2). After that, the equation becomes

$$F_{SHL-L}(x;\theta) = 1-\lambda^{\alpha}\left[\lambda + x(1-\left(1+\frac{x}{k}\right)^{-\gamma}) + m\right]^{-\alpha},$$

   where $(\gamma, k, \alpha, \lambda, m) > 0$.
2. **SHL-Gompertz distribution**
   The two-parameter Gompertz model's CDF, $G(x;\zeta) = 1 - e^{-\eta\left(e^{bx-1}\right)}$, is now being examined. It has shape parameter $\eta > 0$ and scale parameter $b > 0$. Then, the CDF of the SHL-G distribution is:

$$F_{SHL-G}(x;\theta) = 1-\lambda^{\alpha}\left[\lambda + x\left(\left(1-e^{-\eta\left(e^{bx-1}\right)}\right)+m\right)\right]^{-\alpha}.$$

3. **SHL-Rayleigh**
   Based on the Rayleigh distribution's pdf and CDF, which are provided by $g(x;\zeta) = \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$ and $G(x;\zeta) = 1 - e^{-\frac{x^2}{2\sigma^2}}$, respectively, the CDF and pdf of the SHL-R model are defined as :

$$F_{SHL-R}(x;\theta) = 1-\lambda^{\alpha}\left[\lambda + x\left(1-e^{-\frac{x^2}{2\sigma^2}}+m\right)\right]^{-\alpha},$$

$$f_{SHL-R}(x;\theta) = \frac{\alpha\lambda^{\alpha}\left[e^{-\frac{x^2}{2\sigma^2}}\left(\frac{x^2}{\sigma^2}-1\right)+m+1\right]}{\left[\lambda + x\left(1-e^{-\frac{x^2}{2\sigma^2}}+m\right)\right]^{\alpha+1}}.$$

Likewise, we can have the following special members: SHL-Exponential, SHL-Weibull, SHL-Log logistic, SHL-Burr.

Figure 1 demonstrates the density function representation of some special members, reflecting the flexibility of the family, as we note the variety of curve shapes, parameter tuning, adaptability to several types of data.

3. The Shifted Lomax-Weibull distribution

We are mainly interested in the Weibull special member. The CDF of the Weibull distribution can be defined as follows:

$$G(x;\nu) = 1 - e^{-\left(\frac{x}{k}\right)^{a}},$$

where $\nu = (a, k) \geq 0$ and $x \geq 0$.
The linked pdf, hazard rate function (hrf), and survival function (suf) are provided, in that order, by:

$$g(x;\nu) = \frac{ax^{a-1}}{k^{a}}e^{-\left(\frac{x}{k}\right)^{a}},$$

$$suf(x;\nu) = e^{-\left(\frac{x}{k}\right)^{a}}$$

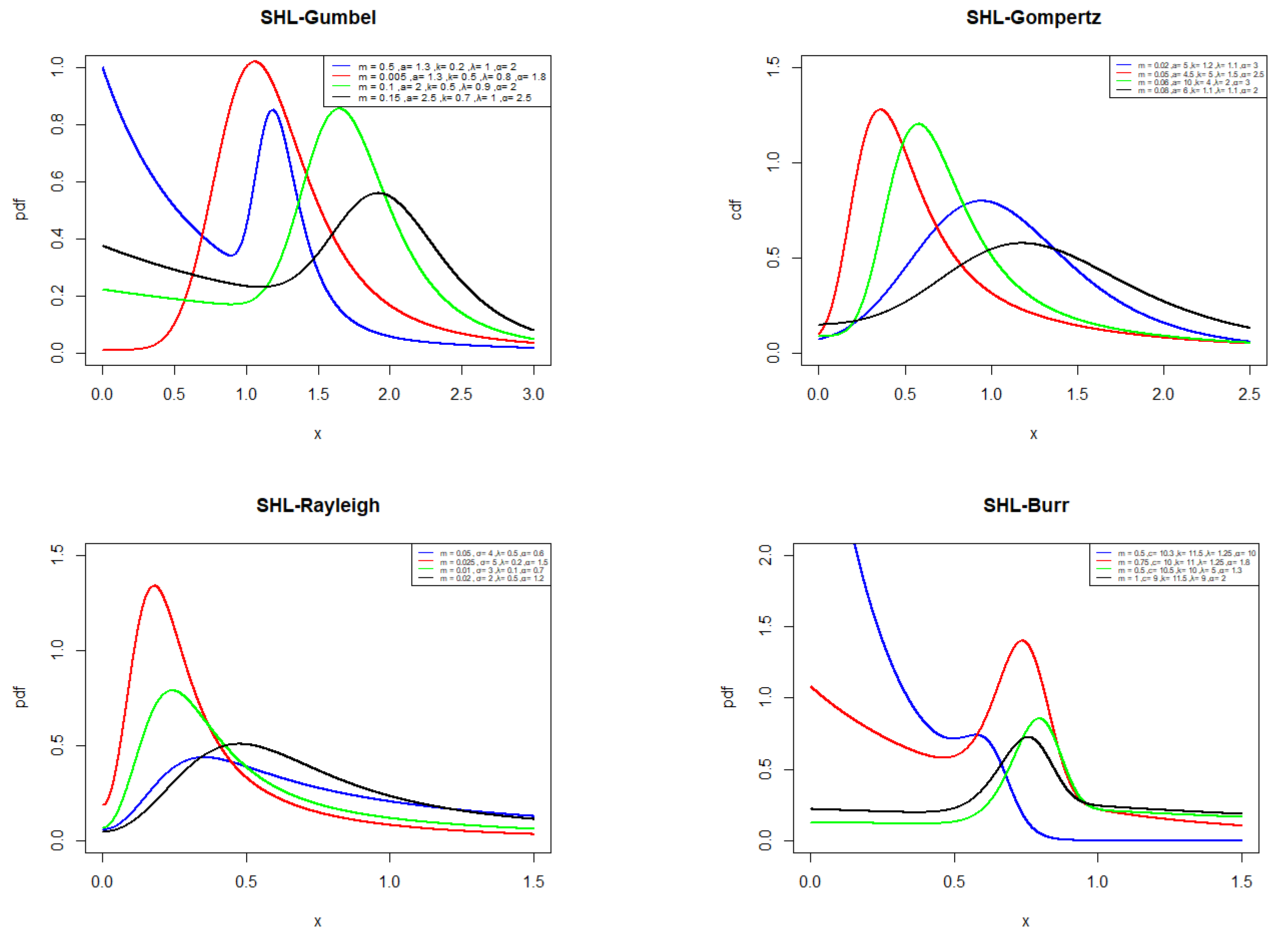


Figure 1. Density function plots of some special members of the SHL-X family

and

$$hrf(x;\nu) = \frac{ax^{a-1}}{k^a}.$$

This is the expression for the CDF that defines the SHL-W distribution:

$$F(x;\theta) = 1 - \left[1 + \frac{x\left(1 - e^{-\left(\frac{x}{k}\right)^a} + m\right)}{\lambda}\right]^{-\alpha}, \tag{3.1}$$

where $\theta = (\lambda, \alpha, a, k, m) \geq 0$, $x \geq 0$.
The associated pdf of (3.1) is:

$$\begin{aligned} f(x;\theta) =& \frac{\alpha}{\lambda}\left[1 - e^{-\left(\frac{x}{k}\right)^a} + m + \frac{a}{k^a}x^a e^{-\left(\frac{x}{k}\right)^a}\right] \\ & \times \left[1 + \frac{x\left(1 - e^{-\left(\frac{x}{k}\right)^a} + m\right)}{\lambda}\right]^{-\alpha-1} \end{aligned} \tag{3.2}$$

The 3D depiction of the CDF and pdf of the SHL-W distribution for various values of λ, m, a, k is shown in figure 2, which was created using Mathematica software.

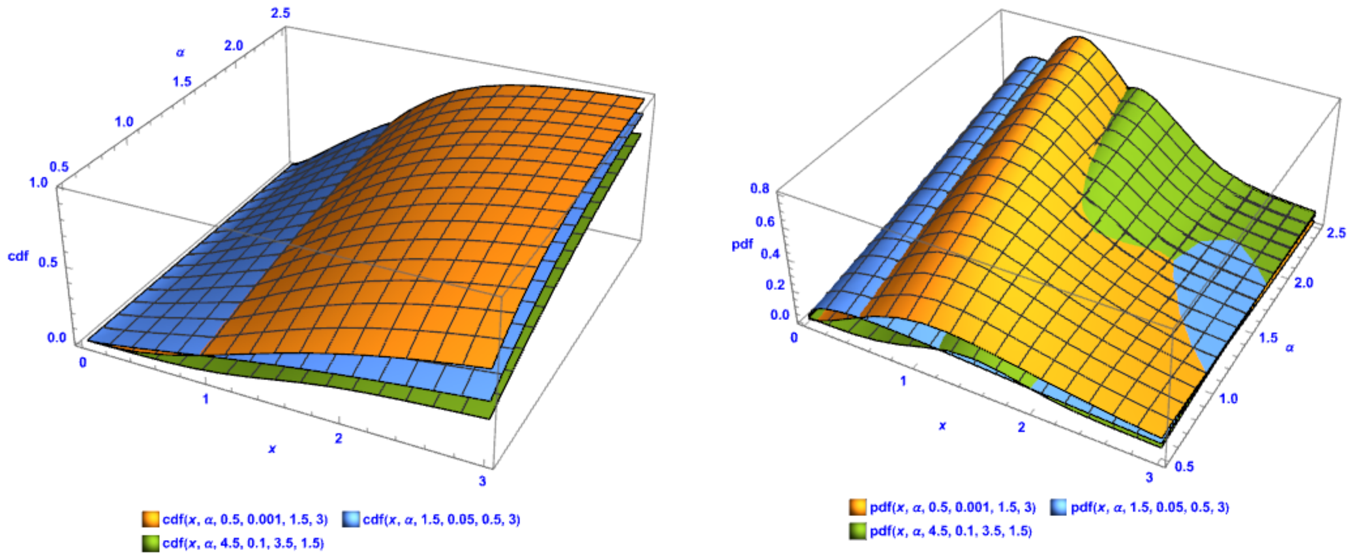


Figure 2. Distribution and density function for the Shifted Lomax-Weibull distribution

The suf and hrf of the SHL-W distribution are respectively given by:

$$suf(x;\theta) = \left[1 + \frac{x\left(1 - e^{-\left(\frac{x}{k}\right)^a} + m\right)}{\lambda}\right]^{-\alpha}.$$

The survival function $\rightarrow 1$, when $x \rightarrow 0$ and $\rightarrow 0$, when $x \rightarrow +\infty$.

$$hrf(x;\theta) = \frac{\alpha}{\lambda}\left[1 - e^{-\left(\frac{x}{k}\right)^a} + m + \frac{a}{k^a}x^a e^{-\left(\frac{x}{k}\right)^a}\right] \left[1 + \frac{x\left(1 - e^{-\left(\frac{x}{k}\right)^a} + m\right)}{\lambda}\right]^{-1}.$$

The risk function $\rightarrow \frac{m\alpha}{\lambda}$, when $x \rightarrow 0$ and 0, when $x \rightarrow +\infty$.

The following expression defines the cumulative hazard function:

$$Cf(x;\theta) = -\log suf(x;\theta).$$

Thus, for SHL-Weibull distibution, it is defined by:

$$Cf(x) = \alpha\left[\log\lambda\left(1 + x\left(1 - e^{-\left(\frac{x}{k}\right)^a} + m\right)\right) - \log\lambda\right].$$

The reserve risk function is found using the following expression:

$$Rf(x;\theta) = \frac{f(x;\theta)}{F(x;\theta)}.$$

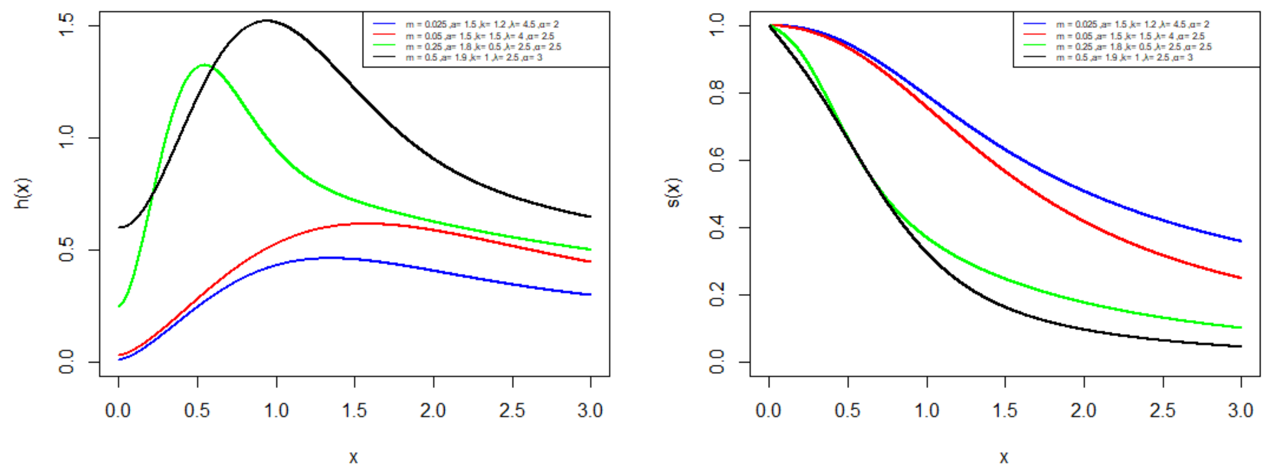


Figure 3. Plot of risk and survival functions

Therefore, that of the SHL-W distribution is:

$$Rf(x;\theta) = \frac{f(x;\theta)}{1 - \left[1 + \frac{x\left(1-e^{-\left(\frac{x}{k}\right)^{a}}+m\right)}{\lambda}\right]^{-\alpha}}.$$

4. SHL-W distribution's several mathematical characteristics

Several significant mathematical properties of the SHL-W distribution are highlighted in this section.

4.1. The quantile function

In probabilistic inference, the quantile function is crucial, especially when dealing with the SHL-W distribution. It is very helpful for figuring out crucial values in finance and insurance uncertainty analyses. Finding critical values, which act as cutoff points or thresholds in a variety of statistical and economic analyses, is one application for which it is especially helpful.
The SHL-Weibull distribution's quantile function is determined by

$$Q(y;\theta) = P^{-1}\left[\lambda\left((1-y)^{-\frac{1}{\alpha}} - 1\right)\right],$$

where $P(x;\mu) = x\left(1 - e^{-\left(\frac{x}{k}\right)^{a}} + m\right)$ and $\mu = (a, k, m)$.
Proof
Let $x_y = Q(y;v)$; for all, $y \in [0,1]$.
Then, x_y fulfills the nonlinear equation, as per the quantile function definition:

$$y = F(x;\tau).$$

Thus,

$$y = 1 - \left[1 + \frac{P(x;\mu)}{\lambda}\right]^{-\alpha}$$

$$\Rightarrow (1-y) = \left[1 + \frac{P(x;\mu)}{\lambda}\right]^{-\alpha}$$

Raising each member of the above equation to the power $\frac{1}{\alpha}$, we obtain:

$$(1-y)^{-\frac{1}{\alpha}} = \left[1 + \frac{P(x;\mu)}{\lambda}\right]$$

$$\Rightarrow \lambda\left((1-y)^{-\frac{1}{\alpha}} - 1\right) = P(x;\mu).$$

Let's set $P(x;\mu) = y$,

$$x = P^{-1}(y;\mu),$$

and we have:

$$x = P^{-1}\left[\lambda\left((1-y)^{-\frac{1}{\alpha}} - 1\right)\right].$$

As a result, the Shifted Lomax-Weibull distribution's first, median, and third quartiles are provided by:

$$Q_1 = P^{-1}\left[\lambda\left(\left(\frac{3}{4}\right)^{-\frac{1}{\alpha}} - 1\right)\right],$$

$$Q_2 = P^{-1}\left[\lambda\left(\left(\frac{1}{2}\right)^{-\frac{1}{\alpha}} - 1\right)\right]$$

and

$$Q_3 = P^{-1}\left[\lambda\left(\left(\frac{1}{4}\right)^{-\frac{1}{\alpha}} - 1\right)\right].$$

With the quantile function, SHL-W's skewness and kurtosis are simply calculated. A distribution's asymmetry with respect to its mean is measured by its skewness and kurtosis calculates a distribution's "tail" by contrasting its tails with those of a normal distribution. Indeed, the Moors kurtosis [29] and the Bowley skewness [30] are provided by:

$$BS = \frac{Q_Y(3/4) + Q_Y(1/4) - 2Q_Y(1/2)}{Q_Y(3/4) - Q_Y(1/4)}$$

and

$$MK = \frac{Q_Y(7/8) - Q_Y(5/8) + Q_Y(3/8) - Q_Y(1/8)}{Q_Y(6/8) - Q_Y(2/8)}$$

The SHL-Weibull distribution's first, median, and third quartiles are shown in Table 1 for a range of parameter values (m, α, λ). The evolution of these quantiles with respect to parameter variations is shown in Table 1. The quartile values tend to go down as λ and α rise, suggesting that the distribution

is increasingly concentrated around lower values. This pattern clarifies if the model is appropriate for a given set of data. Larger values, for instance, could correspond with certain economic phenomena in financial applications like stock market data, enabling parameter estimate that takes this behavior into account.
Additionally, the values for kurtosis (MK) and skewness (BS) are included in Table 1. As the parameters α and λ grow, their values tend to decrease. This suggests that the distribution gets less skewed and shows lower kurtosis as the parameters increase, suggesting a decrease in asymmetry and a drop in the distribution's tail heaviness. As a result, we may deduce that larger parameter values result in a less peaked and more symmetrical model, which would be more suitable for datasets exhibiting these traits.

Table 1. Quantiles, skewness, and kurtosis simulation table for various values of m, α, λ, $a = 1.2$, and $k = 1.5$.

		$m = 0.25$					$m = 1.2$				
λ	α	$Q1$	$Q2$	$Q3$	BS	MK	$Q1$	$Q2$	$Q3$	BS	MK
0.5	1.5	0.2814	0.5674	1.0452	0.2511	1.489	0.0858	0.2262	0.5215	0.3608	1.5681
	2.7	0.1742	0.3548	0.6188	0.1876	1.311	0.0462	0.1174	0.2556	0.3198	1.435
	4.00	0.1246	0.2594	0.4501	0.1715	1.258	0.0308	0.077	0.1634	0.3029	1.3929
	5.4	0.0956	0.2033	0.3548	0.1692	1.234	0.0226	0.0561	0.1174	0.2934	1.371
1.0	1.5	0.4564	0.8865	1.6464	0.2770	1.599	0.1666	0.4209	0.9316	0.3351	1.560
	2.7	0.2943	0.5661	0.9652	0.1896	1.348	0.0911	0.2255	0.4722	0.2946	1.395
	4.00	0.217	0.4234	0.7089	0.1608	1.280	0.061	0.15	0.3092	0.2831	1.3554
	5.4	0.1705	0.3387	0.5661	0.1497	1.247	0.045	0.1102	0.2255	0.278	1.339
1.7	1.5	0.647	1.2465	2.3983	0.3153	1.7440	0.272	0.6605	1.4273	0.3275	1.6098
	2.7	0.42263	0.7973	1.3598	0.2051	1.3943	0.1514	0.364	0.7373	0.2745	1.3819
	4	0.3207	0.602	0.9954	0.1659	1.3073	0.1023	0.2459	0.4921	0.2631	1.331
	5.4	0.2562	0.4868	0.7973	0.1478	1.2678	0.0758	0.182	0.364	0.260	1.313
2.2	1.5	0.7637	1.4766	2.9266	0.3408	1.8277	0.3429	0.8166	1.7582	0.3306	1.6555
	2.7	0.5058	0.9401	1.6143	0.2175	1.426	0.193	0.4557	0.9099	0.2669	1.386
	4	0.384	0.7112	1.1748	0.1726	1.325	0.131	0.3106	0.612	0.253	1.325
	5.4	0.309	0.577	0.9401	0.1508	1.280	0.09731	0.2316	0.4557	0.2505	1.3035

To see how skewness and kurtosis behave, we have plotted them in 3D as a function of α and a with matlab.

4.2. Serial expansion of f

Local approximations of the density function around a specific point are obtained through series development of the density function $f(x)$, an essential tool in statistics and probability theory. This is very useful for studying the local characteristics of the probability distribution, evaluating parameters and solving various practical problems. The SHL-W distribution's serial development is determined by:

$$f(x;\theta) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} S_{n,i} r(x), \tag{4.1}$$

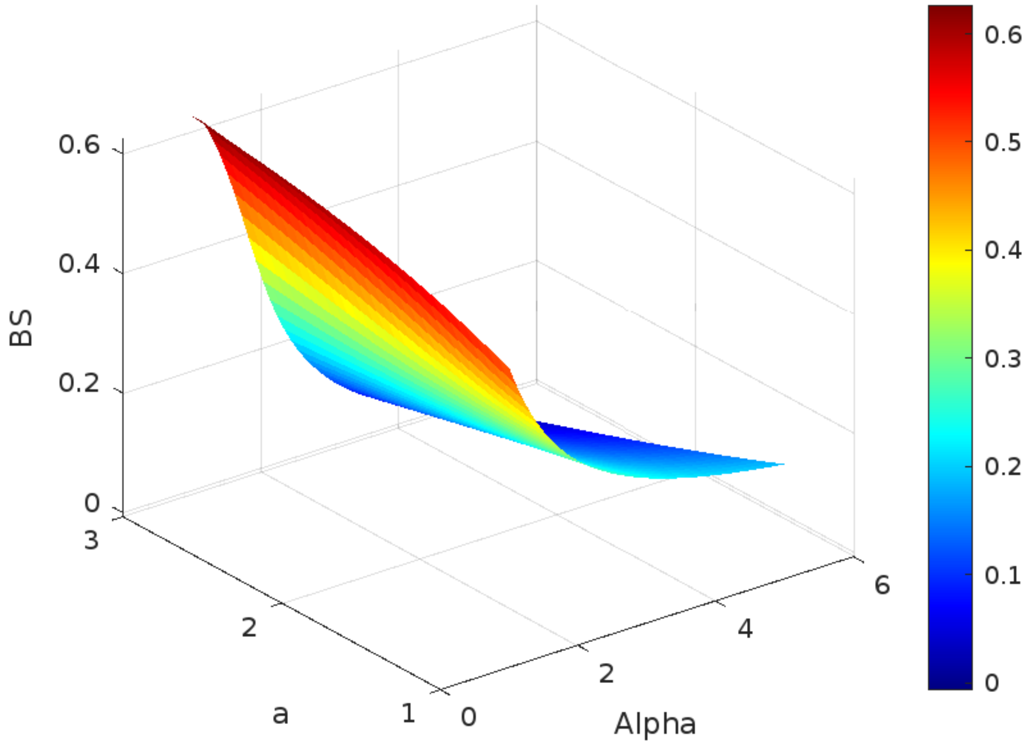


Figure 4. Skewness for $\lambda = 2.5$, $m = 0.25$, $k = 1.25$

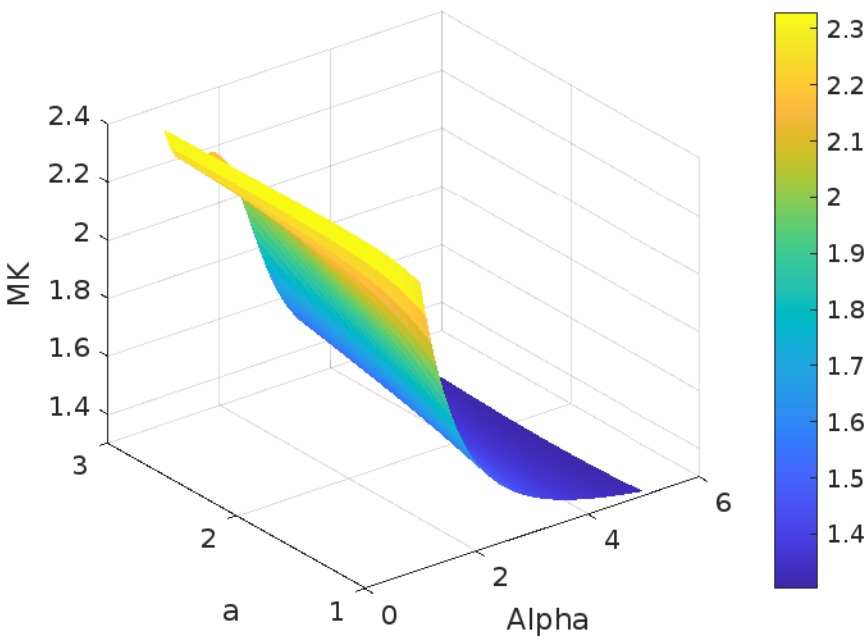


Figure 5. Kurtosis for $\lambda = 2.5$, $m = 0.25$, $k = 1.25$

with

$$S_{n,i} = \binom{\alpha+n-1}{n}\binom{n}{i}(-1)^{n+i+1}\lambda^{-n}(m+1)^{n-i}$$

and $r(x;\theta) = e^{-i\left(\frac{x}{k}\right)^a} x^{n-1}\left[n - \frac{ia}{k^a}x^a\right]$ is the derivative of $R(x;\theta) = x^n e^{-i\left(\frac{x}{k}\right)^a}$.

Proof

From (3.1), we have:

$$F(x;\theta) = 1 - \frac{1}{\left[1 + \frac{x\left(1-e^{-\left(\frac{x}{k}\right)^a}+m\right)}{\lambda}\right]^{\alpha}}.$$

Setting $z = \frac{x\left(1-e^{-\left(\frac{x}{k}\right)^a}+m\right)}{\lambda}$, we have:

$$F(x;\theta) = 1 - (1+z)^{-\alpha}.$$

The negative binomial series allows us to write the following:

$$(1+z)^{-\alpha} = \sum_{n=0}^{\infty}(-1)^n\binom{\alpha+n-1}{n}z^n.$$

So, we have:

$$F(x;\theta) = 1 - \sum_{n=0}^{\infty}(-1)^n\binom{\alpha+n-1}{n}z^n. \tag{4.2}$$

Let's calculate z^n.

$$\begin{aligned} z^n &= \left(\frac{x\left(1-e^{-\left(\frac{x}{k}\right)^a}+m\right)}{\lambda}\right)^n \\ &= \lambda^{-n}x^n\left[(m+1) - e^{-\left(\frac{x}{k}\right)^a}\right]^n. \end{aligned}$$

Knowing that $(a-b)^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}(-b)^i$, we have:

$$\left[(m+1) - e^{-\left(\frac{x}{k}\right)^a}\right]^n = \sum_{i=0}^{n}\binom{n}{i}(m+1)^{n-i}(-1)^i\left[e^{-\left(\frac{x}{k}\right)^a}\right]^i.$$

So, we obtain

$$z^n = \lambda^{-n}x^n\sum_{i=0}^{n}(-1)^i\binom{n}{i}(m+1)^{n-i}e^{-i\left(\frac{x}{k}\right)^a}. \tag{4.3}$$

Replacing the expression (4.3) in (4.2), we obtain:

$$F(x;\theta) = 1 + \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{\alpha+n-1}{n}\binom{n}{i}(-1)^{n+i+1}\lambda^{-n}x^n$$

$$\times (m+1)^{n-i} e^{-i\left(\frac{x}{k}\right)^a}. \tag{4.4}$$

Therefore,

$$f(x;\theta) = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{\alpha+n-1}{n}\binom{n}{i}(-1)^{n+i+1}\lambda^{-n}(m+1)^{n-i}$$
$$\times e^{-i\left(\frac{x}{k}\right)^a} x^{n-1}\left[n - \frac{ia}{k^a}x^a\right].$$

4.3. Renyi Entropy

The Renyi entropy can be used to measure the uncertainty or diversity of a probability distribution, but with a α diversity parameter to adjust sensitivity to rare or frequent events. The Renyi Entropy is calculated by

$$ER(X) = \frac{1}{1-\gamma}\log\left(\int_{\mathbb{R}} f(x;\theta)^{\gamma} dx\right)$$

In the case of SHL-W distribution, it is defined by:

$$ER(X) = \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha}{\lambda}\right)^{\gamma}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{\gamma}\sum_{t=0}^{j} S_{n,i,j,t} I(x;\theta)\right],$$

where

$$S_{n,i,j,t} = (-1)^{n+i+t}\binom{\gamma(\alpha+1)+n-1}{n}\binom{\gamma}{j}\binom{j}{t}\lambda^{-n}(m+1)^{n-i+\gamma-j}a^{j-t-1}k^{n+1}.$$

and

$$I(x;\theta) = \Gamma\left(\frac{n+1}{a}+j-t\right).$$

Proof

According to the equation (3.2)

$$f(x;\theta)^{\gamma} = \left(\frac{\alpha}{\lambda}\right)^{\gamma}\left[1 - e^{-\left(\frac{x}{k}\right)^a} + m + \frac{a}{k^a}x^a e^{-\left(\frac{x}{k}\right)^a}\right]^{\gamma}$$
$$\times\left[1 + \frac{x\left(1 - e^{-\left(\frac{x}{k}\right)^a} + m\right)}{\lambda}\right]^{-\gamma(\alpha+1)}. \tag{4.5}$$

We may write the following by using the negative binomial series:

$$(1+z)^{-\gamma(\alpha+1)} = \sum_{n=0}^{\infty}(-1)^n\binom{\gamma(\alpha+1)+n-1}{n}z^n.$$

By introducing (4.3) into the previous equation, we have:

$$(1+z)^{-\gamma(\alpha+1)} = \sum_{n=0}^{\infty}(-1)^n\binom{\gamma(\alpha+1)+n-1}{n}\lambda^{-n}x^n\sum_{i=0}^{n}(-1)^i\binom{n}{i}(m+1)^{n-i}e^{-i\left(\frac{x}{k}\right)^a}$$

$$(1+z)^{-\gamma(\alpha+1)} = \sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^{n+i}\binom{\gamma(\alpha+1)+n-1}{n}\binom{n}{i}\lambda^{-n}x^{n}(m+1)^{n-i}e^{-i\left(\frac{x}{k}\right)^{a}}. \tag{4.6}$$

Let's develop $\left[1-e^{-\left(\frac{x}{k}\right)^{a}}+m+\frac{a}{k^{a}}x^{a}e^{-\left(\frac{x}{k}\right)^{a}}\right]^{\gamma}$.
If we put $a=(1+m)$ and $b=e^{-\left(\frac{x}{k}\right)^{a}}\left(\frac{a}{k^{a}}x^{a}-1\right)$, we have:

$$\begin{aligned}(a+b)^{\gamma} &= \sum_{j=0}^{\gamma}\binom{\gamma}{j}a^{\gamma-j}b^{j}\\ &= \sum_{j=0}^{\gamma}\binom{\gamma}{j}(m+1)^{\gamma-j}e^{-j\left(\frac{x}{k}\right)^{a}}\left(\frac{a}{k^{a}}x^{a}-1\right)^{j}\end{aligned}$$

and

$$\left(\frac{a}{k^{a}}x^{a}-1\right)^{j} = \sum_{t=0}^{j}\binom{j}{t}(-1)^{t}\left(\frac{a}{k^{a}}x^{a}\right)^{j-t}.$$

Consequently,

$$\left[1-e^{-\left(\frac{x}{k}\right)^{a}}+m+\frac{a}{k^{a}}x^{a}e^{-\left(\frac{x}{k}\right)^{a}}\right]^{\gamma} = \sum_{j=0}^{\gamma}\sum_{t=0}^{j}\binom{\gamma}{j}\binom{j}{t}(-1)^{t}(m+1)^{\gamma-j}\left(\frac{a}{k^{a}}x^{a}\right)^{j-t}e^{-j\left(\frac{x}{k}\right)^{a}}. \tag{4.7}$$

By substituting equation (4.6) and equation (4.7) into (4.5):

$$\begin{aligned}f(x;\theta)^{\gamma} = \left(\frac{\alpha}{\lambda}\right)^{\gamma}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{\gamma}\sum_{t=0}^{j}(-1)^{n+i+t}\binom{\gamma(\alpha+1)+n-1}{n}\binom{\gamma}{j}\binom{j}{t}\\ \times\lambda^{-n}(m+1)^{n-i+\gamma-j}x^{n+a(j-t)}e^{-(i+j)\left(\frac{x}{k}\right)^{a}}\left(\frac{a}{k^{a}}x^{a}\right)^{j-t}.\end{aligned}$$

Then the $ER(X)$ is

$$\begin{aligned}ER(X) = \frac{1}{1-\gamma}\log\Bigg[\left(\frac{\alpha}{\lambda}\right)^{\gamma}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{\gamma}\sum_{t=0}^{j}(-1)^{n+i+t}\binom{\gamma(\alpha+1)+n-1}{n}\binom{\gamma}{j}\binom{j}{t}\lambda^{-n}\\ \times(m+1)^{n-i+\gamma-j}\left(\frac{a}{k^{a}}\right)^{j-t}\int_{0}^{\infty}\left(x^{n+a(j-t)}e^{-(i+j)\left(\frac{x}{k}\right)^{a}}\right)dx\Bigg].\end{aligned}$$

Assuming $h=(i+j)\left(\frac{x}{k}\right)^{a}$, we have $x=k\left(\frac{h}{i+j}\right)^{\frac{1}{a}}$ and $dx=\frac{k}{a(i+j)^{\frac{1}{a}}}h^{\frac{1}{a}-1}dh$.
So,

$$\int_{0}^{\infty}\left(x^{n+a(j-t)}e^{-(i+j)\left(\frac{x}{k}\right)^{a}}\right)dx = \frac{k^{n+a(j-t)+1}}{a(i+j)^{\frac{n+1}{a}+j-t}}\int_{0}^{\infty}h^{\frac{n+1}{a}+j-t-1}e^{-h}.dh$$

Given that: $\Gamma(z)=\int_{0}^{\infty}t^{z-1}e^{-t}dt$ where $\Gamma(.)$ represents the gamma function, then

$$\int_{0}^{\infty}\left(x^{n+a(j-t)}e^{-(i+j)\left(\frac{x}{k}\right)^{a}}\right)dx = \frac{k^{n+a(j-t)+1}}{a(i+j)^{\frac{n+1}{a}+j-t}}\Gamma\left(\frac{n+1}{a}+j-t\right).$$

Consequently, we obtain:

$$ER(X) = \frac{1}{1-\gamma} \log\left[\left(\frac{\alpha}{\lambda}\right)^{\gamma} \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{\gamma}\sum_{t=0}^{j}(-1)^{n+i+t}\binom{\gamma(\alpha+1)+n-1}{n}\binom{\gamma}{j}\binom{j}{t}\lambda^{-n} \right.$$
$$\left. \times(m+1)^{n-i+\gamma-j}a^{j-t-1}k^{n+1}\Gamma\left(\frac{n+1}{a}+j-t\right)\right].$$

4.4. Moment of order s

The s-order moment of the proposed SHL-W distribution is defined by:

$$M_s = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i} k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}}\left[-\left(\frac{s}{a}\right)\right]\Gamma\left(\frac{s+n}{a}\right). \tag{4.8}$$

Proof:

We compute the moment of order s as follows:

$$M_s = \mathbb{E}(X^s)$$
$$= \int_{\mathbb{R}} x^s f(x;\theta)dx.$$

Substituting equation (4.1),

$$M_s = \int_0^{\infty} x^s \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i} e^{-i\left(\frac{x}{k}\right)^a} x^{n-1}\left[n - \frac{ia}{k^a}x^a\right]dx$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}\int_0^{\infty} e^{-i\left(\frac{x}{k}\right)^a} x^{s+n-1}\left[n - \frac{ia}{k^a}x^a\right]dx.$$

Making substitution $i\left(\frac{x}{k}\right)^a = t$, we have $x = k\left(\frac{t}{i}\right)^{\frac{1}{a}}$ and $dx = \frac{k}{a}\left(\frac{1}{i}\right)^{\frac{1}{a}} t^{\frac{1}{a}-1}$.

$$M_s = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}\int_0^{\infty} e^{-t} k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}} t^{\frac{s+n}{a}-1}\left(\frac{n}{a}-t\right)dt$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i} k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}}\left[\frac{n}{a}\int_0^{\infty} e^{-t}t^{\frac{s+n}{a}-1}dt - \int_0^{\infty} e^{-t}t^{\frac{s+n}{a}}dt\right].$$
$$M_s = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i} k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}}\left[\frac{n}{a}\Gamma\left(\frac{s+n}{a}\right) - \Gamma\left(\frac{s+n}{a}+1\right)\right].$$

Given that $\Gamma(z+1) = z\Gamma(z)$, then: $\Gamma\left(\frac{s+n}{a}+1\right) = \left(\frac{s+n}{a}\right)\Gamma\left(\frac{s+n}{a}\right)$. Therefore,

$$M_s = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i} k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}}\left[-\left(\frac{s}{a}\right)\right]\Gamma\left(\frac{s+n}{a}\right).$$

4.5. Moment generating function

In probability and statistics, the moment generating function is a powerful tool for completely defining the distribution of a random variable. It is used to determine the moments of the distribution and simplify theoretical analyses. The moment generating function is given by:

$$M_x(t) = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{l=0}^{\infty} S_{n,i}\frac{t^l}{l!}k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}}\left[-\left(\frac{l}{a}\right)\right]\Gamma\left(\frac{l+n}{a}\right).$$

Proof:

The following determines the moment generating function:

$$M_X(t) = \mathbb{E}(e^{tX}).$$

Knowing that $e^{tx} = \sum_{l=0}^{\infty}\frac{(tx)^l}{l!}$, so:

$$\begin{aligned} M_X(t) &= \mathbb{E}\left(\sum_{l=0}^{\infty}\frac{(tX)^l}{l!}\right) \\ &= \sum_{l=0}^{\infty}\mathbb{E}\left(\frac{(tX)^l}{l!}\right) \\ &= \sum_{l=0}^{\infty}\frac{t^l}{l!}\mathbb{E}\left(X^l\right). \end{aligned}$$

Using equation (4.8), we have

$$\mathbb{E}\left(X^l\right) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{l+n}\left(\frac{1}{i}\right)^{\frac{l+n}{a}}\Gamma\left(\frac{l+n}{a}\right)\left[-\left(\frac{l}{a}\right)\right].$$

Therefore,

$$M_X(t) = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{l=0}^{\infty} S_{n,i}\frac{t^l}{l!}k^{l+n}\left(\frac{1}{i}\right)^{\frac{l+n}{a}}\left[-\left(\frac{l}{a}\right)\right]\Gamma\left(\frac{l+n}{a}\right).$$

4.6. Incomplete moments

Incomplete moments are an extension of ordinary moments, which allow us to determine the moments of a distribution by taking into account part of the distribution rather than the whole distribution. They are very useful when we are interested in particular subsets of the data or when we are partially observing the distribution. The incomplete moment is defined by $M_s(y) = \int_{-\infty}^{y} x^s f(x;\theta)dx$. For SHL-W distribution, we have:

$$M_s(y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}} R_\gamma(x), \tag{4.9}$$

with

$$R_\gamma(x) = \left(-\frac{s}{a}\right)\gamma\left(\frac{s+n}{a}, i\left(\frac{y}{k}\right)^a\right) + i^{\frac{s+n}{a}}\left(\frac{y}{k}\right)^{s+n} e^{-i\left(\frac{y}{k}\right)^a}.$$

Proof:

$$M_s(y) = \int_0^y x^s f(x;\theta)dx$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}\int_0^y e^{-i\left(\frac{x}{k}\right)^a} x^{s+n-1}\left[n-\frac{ia}{k^a}x^a\right]dx.$$

Making substitution $i\left(\frac{x}{k}\right)^a = t$, $0 < t < i\left(\frac{y}{k}\right)^a$.

$$M_s(y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}}\left[\frac{n}{a}\int_0^{i\left(\frac{y}{k}\right)^a} e^{-t}t^{\frac{s+n}{a}-1}dt - \int_0^{i\left(\frac{y}{k}\right)^a} e^{-t}t^{\frac{s+n}{a}}dt\right]$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n} k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}} S_{n,i}\left[\frac{n}{a}\gamma\left(\frac{s+n}{a}, i\left(\frac{y}{k}\right)^a\right) - \gamma\left(\frac{s+n}{a}+1, i\left(\frac{y}{k}\right)^a\right)\right],$$

because $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ where $\gamma(.)$ represents the incomplete gamma function. Knowing that $\gamma(s+1,x) = s\gamma(s,x) - x^s e^{-x}$, this leads to:

$$M_s(y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{s+n}\left(\frac{1}{i}\right)^{\frac{s+n}{a}}\left[\left(-\frac{s}{a}\right)\gamma\left(\frac{s+n}{a}, i\left(\frac{y}{k}\right)^a\right) + i^{\frac{s+n}{a}}\left(\frac{y}{k}\right)^{s+n} e^{-i\left(\frac{y}{k}\right)^a}\right].$$

4.7. Average waiting time

The term ”average waiting time” describes the typical amount of time you have to wait for anything to happen. It is a central tendency metric that expresses the average or normal waiting time. The definition of it is as follows:

$$m_t(s) = x - \frac{1}{F(x)}\int_{-\infty}^{x} tf(t)dt.$$

In the case of the SHL-W distribution, the average waiting time is:

$$m_t(x) = x - \frac{1}{F(x)}\sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{n+1}\left(\frac{1}{i}\right)^{\frac{n+1}{a}}\left[\left(-\frac{1}{a}\right)\gamma\left(\frac{n+1}{a}, i\left(\frac{x}{k}\right)^a\right) + i^{\frac{n+1}{a}}\left(\frac{x}{k}\right)^{n+1} e^{-i\left(\frac{x}{k}\right)^a}\right].$$

Proof:

$$m_t(s) = x - \frac{1}{F(x)}\int_0^x tf(t)dt.$$

Knowing that from equation (4.9), we have:

$$\int_0^x tf(t)dt = M_1(x)$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{n+1}\left(\frac{1}{i}\right)^{\frac{n+1}{a}}\left[\left(-\frac{1}{a}\right)\gamma\left(\frac{n+1}{a}, i\left(\frac{x}{k}\right)^a\right) + i^{\frac{n+1}{a}}\left(\frac{x}{k}\right)^{n+1} e^{-i\left(\frac{x}{k}\right)^a}\right].$$

Therefore,

$$m_t(x) = x - \frac{1}{F(x)} \sum_{n=0}^{\infty} \sum_{i=0}^{n} S_{n,i} k^{n+1} \left(\frac{1}{i}\right)^{\frac{n+1}{a}} \left[\left(-\frac{1}{a}\right) \gamma \left(\frac{n+1}{a}, i\left(\frac{x}{k}\right)^{a}\right) + i^{\frac{n+1}{a}} \left(\frac{x}{k}\right)^{n+1} e^{-i\left(\frac{x}{k}\right)^{a}} \right].$$

4.8. Mean Residual Life

A crucial idea in survival analysis and reliability theory is mean residual life (MRL). It gives the anticipated amount of time that a person or component will live if it survives till a specific time, t.

The average residual life function for the SHL-W distribution is defined below:

$$MRL = \frac{1}{1-F(x)} \sum_{n=0}^{\infty} \sum_{i=0}^{n} S_{n,i} k^{n+1} \left(\frac{1}{i}\right)^{\frac{n+1}{a}} \left[\left(-\frac{1}{a}\right) \gamma \left(\frac{n+1}{a}, i\left(\frac{x}{k}\right)^{a}\right) + i^{\frac{n+1}{a}} \left(\frac{x}{k}\right)^{n+1} e^{-i\left(\frac{x}{k}\right)^{a}} \right]$$

5. Actuarial measures

5.1. Risk measures

Financial risk quantification and management require the application of actuarial science's risk metrics, such as Value at Risk (VaR), Tail Value at Risk (TVaR), and Tail Value of the quantile (TVq).

5.1.1. VaR measure

Tail Value at Risk, or TVaR, is a risk metric primarily utilized by the insurance and banking sectors. It provides the ability to quantify the severe loss risk over and beyond a predetermined dependability level. TVaR assesses the average loss in the tail of the distribution once this threshold is exceeded, whereas VaR (Value at Risk) sets a maximum loss threshold for a certain confidence level. Put otherwise, it offers an illustration of the magnitude of damages that surpass the worst-case scenario that is protected by VaR. Thus, for the SHL-W distribution, the quantile function is defined as follows:

$$VaR_q = P^{-1}\left[\lambda\left((1-q)^{\frac{-1}{\alpha}} - 1\right)\right],$$

where $P^{-1}(x)$ is the inverse of the function $P(x) = x\left(1 - e^{-\left(\frac{x}{k}\right)^{a}} + m\right)$.

5.1.2. TVAR measure

Conditional Value at Risk (CVaR), also known as Tail Value-at-Risk, is a risk measure that evaluates losses in excess of Value at Risk (VaR). Since the probability of losses being less than or equal to VaR is p, TVaR calculates the expected value of losses in the worst-case scenario, i.e. when losses exceed VaR. The TVaR of SHL-W distribution is:

$$TVaR_q = \frac{1}{1-q} \sum_{n=0}^{\infty} \sum_{i=0}^{n} k^{n+1} \left(\frac{1}{i}\right)^{\frac{n+1}{a}} S_{n,i} R_{\Gamma}(x), \tag{5.1}$$

where $R_{\Gamma}(x) = \left[-\left(\frac{1}{a}\right)\right] \Gamma\left(\frac{n+1}{a}, i\left(\frac{VaR_q}{k}\right)\right)$.

Proof:

The TVAR is defined by:

$$TVaR_q = \frac{1}{1-q}\int_{VaR_q}^{\infty} xf(x)dx.$$

$$TVaR_q = \frac{1}{1-q}\sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}\int_{Var_q}^{\infty} e^{-i\left(\frac{x}{k}\right)^a} x^n\left[n-\frac{ia}{k^a}x^a\right]dx$$

$$TVaR_q = \frac{1}{1-q}\sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{n+1}\left(\frac{1}{i}\right)^{\frac{n+1}{a}}\left[-\left(\frac{1}{a}\right)\right]\Gamma\left(\frac{n+1}{a}, i\left(\frac{VaR_q}{k}\right)\right).$$

5.1.3. TVq measure

TVq is a measure that combines the concepts of VaR and TVaR, focusing on losses in the tail of the loss distribution. It is calculated by: $TV_q(X) = \mathbb{E}(X^2/X > VaR_q) - (TVaR_q)^2$. For SHL-W distribution this gives:

$$TV_q(X) = \frac{1}{1-q}\sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{n+2}\left(\frac{1}{i}\right)^{\frac{n+2}{a}}\left[-\left(\frac{2}{a}\right)\right]\Gamma\left(\frac{n+2}{a}\right) - (TVaR_q)^2. \quad (5.2)$$

Proof:

$$\begin{aligned}
TV_q(X) &= \mathbb{E}(X^2/X > VaR_q) - (TVaR_q)^2\\
&= \frac{1}{1-q}\int_{VaR_q}^{\infty} x^2 f(x)dx - (TVaR_q)^2\\
&= \frac{1}{1-q}\sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}\int_{VaR_q}^{\infty} e^{-i\left(\frac{x}{k}\right)^a} x^{n+1}\left[n-\frac{ia}{k^a}x^a\right]dx - (TVaR_q)^2\\
TV_q(X) &= \frac{1}{1-q}\sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{n+2}\left(\frac{1}{i}\right)^{\frac{n+2}{a}}\left[-\left(\frac{2}{a}\right)\right]\Gamma\left(\frac{n+2}{a}\right) - (TVaR_q)^2.
\end{aligned}$$

5.1.4. Expected Shortfall (ES)

Expected Shortfall at a confidence level $p\%$ measures the average loss that exceeds the VaR at the same confidence level. This means ES provides an average of the worst-case losses, giving a more comprehensive view of tail risk. It's defined by:

$$ES_q = \frac{1}{q}\int_0^q VaR_t dt$$

$$ES_q = \frac{1}{q}\int_0^q P^{-1}\left[\lambda\left((1-t)^{\frac{-1}{\alpha}} - 1\right)\right]dt.$$

5.2. Premium Principles

Premium principles are used to obtain insurance premiums for events, taking into account the level of risk associated with the event. Several premium principles have been developed over the decades.

5.2.1. Expected Value Principle

The principle of expected value is defined as:

$$EVP = (1-\rho)\mathbb{E}(X),$$

where $0 < \rho < 1$.
The EVP of the SHL-W distribution is defined by:

$$EVP = (1-\rho)\sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{n+1}\left(\frac{1}{i}\right)^{\frac{n+1}{a}}\left[-\left(\frac{1}{a}\right)\right]\Gamma\left(\frac{n+1}{a}\right).$$

5.2.2. Tail Variance Premium Principle

TVP makes it possible to assess risk by taking into account the heavy tails characteristic of these distributions. The SHL-W's TVP is calculated as follows:

$$TVP = TVaR + \rho TV \tag{5.3}$$

The TVP of the SHL-W follows by substituting the expressions (5.1) and (5.2) in Eq. (5.3).

5.3. Income inequality metrics

5.3.1. Gini index

A population's inequality in the distribution of wealth or income is gauged by the Gini index. It is widely employed in economic research to evaluate the level of social inequality. For the SHL-W distriibution, it's define by:

$$GI = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{\left[\binom{\alpha+n-1}{n}-\binom{2\alpha+n-1}{n}\right]\binom{n}{i}I_G}{\mu_0}$$

with

$$I_G = (-1)^{n+i}(m+1)^{n-i}\frac{\lambda^{-n}}{a}\left(\frac{k}{i^a}\right)^{n+1}\Gamma\left(\frac{n+1}{a}\right)$$

and the mean (1st-order moment derived from equation(4.8)) is denoted by μ_0.

Proof:
The Gini index's described as follows:

$$GI = \frac{1}{\mu_0}\int_0^{\infty} F(x)\,(1-F(x))\,dx$$

$$\begin{aligned} F(x)\,(1-F(x)) &= F(x) - F^2(x) \\ &= 1-(1+z)^{-\alpha} - [1-(1+z)^{-\alpha}]^2 \end{aligned}$$

$$= (1+z)^{-\alpha} - (1+z)^{-2\alpha}$$

then,

$$(1+z)^{-\alpha} = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{\alpha+n-1}{n}\binom{n}{i}(-1)^{n+i}\lambda^{-n}(m+1)^{n-i}x^{n}e^{-i\left(\frac{x}{k}\right)^{a}},$$

$$(1+z)^{-2\alpha} = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{2\alpha+n-1}{n}\binom{n}{i}(-1)^{n+i}\lambda^{-n}(m+1)^{n-i}x^{n}e^{-i\left(\frac{x}{k}\right)^{a}}$$

and

$$\int_0^{\infty} x^{n}e^{-i\left(\frac{x}{k}\right)^{a}}dx = \frac{1}{a}\left(\frac{k}{i^{a}}\right)^{n+1}\Gamma\left(\frac{n+1}{a}\right).$$

So:

$$GI = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{\left[\binom{\alpha+n-1}{n} - \binom{2\alpha+n-1}{n}\right]\binom{n}{i}I_G}{\mu_0}$$

5.3.2. Lorenz curve

The distribution of wealth or income within a population is graphically represented by the Lorenz curve. One purpose for it is to illustrate distributional disparity. Economic fairness is often studied using this curve in conjunction with the Gini index. We have:

$$C(x) = \frac{1}{\mu_0}\int_0^{x} tf(t)dt$$

$$C(x) = \frac{1}{\mu_0}M_1(x)$$

where $M_1(x)$ is the incomplete moment of order 1.
Therefore

$$C(x) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{n+1}\left(\frac{1}{i}\right)^{\frac{n+1}{a}}\frac{\left[\left(-\frac{1}{a}\right)\gamma\left(\frac{n+1}{a}, i\left(\frac{x}{k}\right)^{a}\right) + i^{\frac{n+1}{a}}\left(\frac{x}{k}\right)^{n+1}e^{-i\left(\frac{x}{k}\right)^{a}}\right]}{\mu_0}$$

5.3.3. Bonferroni Index

Comparable to the Gini, the Bonferroni inequality indicator is more sensitive to low earnings. This indicator concentrates more on inequality at the bottom of the distribution, whereas the Gini index is more evenly distributed. The Bonferroni index emphasizes inequality in this segment of the income distribution by concentrating on the percentage of the people with lower earnings. It is obtained by:

$$B_I(x) = \frac{C(x)}{F(x)}$$

$$B_I(x) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} S_{n,i}k^{n+1}\left(\frac{1}{i}\right)^{\frac{n+1}{a}}\frac{\left[\left(-\frac{1}{a}\right)\gamma\left(\frac{n+1}{a}, i\left(\frac{x}{k}\right)^{a}\right) + i^{\frac{n+1}{a}}\left(\frac{x}{k}\right)^{n+1}e^{-i\left(\frac{x}{k}\right)^{a}}\right]}{F(x)\mu_0}$$

6. Estimation of parameters

Let $x_1, x_2, ..., x_n$ represent a size n random sample of the variable X. The likelihood function can be obtained by utilizing the SHL-W distribution's pdf, which is provided by:

$$\begin{aligned} L(x;\alpha,\lambda,m,a,k) &= \prod_{i=1}^{n} f(x_i;\theta) \\ &= \alpha^n \lambda^{n\alpha} \prod_{i=1}^{n} \left[1 - e^{-\left(\frac{x_i}{k}\right)^a} + m + \frac{a}{k^a} x_i^a e^{-\left(\frac{x_i}{k}\right)^a}\right] \\ &\times \left[\lambda + x_i\left(1 - e^{-\left(\frac{x_i}{k}\right)^a} + m\right)\right]^{-\alpha-1}. \end{aligned}$$

By setting $v(x_i) = 1 - e^{-\left(\frac{x_i}{k}\right)^a} + m + \frac{a}{k^a} x_i^a e^{-\left(\frac{x_i}{k}\right)^a}$ and $u(x_i) = \lambda + x_i\left(1 - e^{-\left(\frac{x_i}{k}\right)^a} + m\right)$,
we obtain

$$L(x;\alpha,\lambda,m,a,k) = \alpha^n \lambda^{n\alpha} \prod_{i=1}^{n} v(x_i) u(x_i)^{-\alpha-1}.$$

The log-likelihood is defined by:

$$lnL(x;\alpha,\lambda,m,a,k) = nln\alpha + n\alpha ln\lambda + \sum_{i=1}^{n} ln(v(x_i)) - (\alpha+1)\sum_{i=1}^{n} ln(u(x_i)).$$

To obtain estimates of the parameters of the SHL-W distribution, we calculate the partial derivatives of the log-likelihood with respect to the various parameters, and derive the estimates by cancelling the first partial derivatives.
The first partial derivatives of $lnL(\hat{\alpha}, \hat{\lambda}, \hat{m}, \hat{a}, \hat{k})$ which must be equal to zero are:

$$\frac{\partial lnL}{\partial \alpha} = \frac{n}{\alpha} + nln\lambda - \sum_{i=1}^{n} ln(u(x_i));$$

$$\frac{\partial lnL}{\partial \lambda} = \frac{n\alpha}{\lambda} - (\alpha+1)\sum_{i=1}^{n} \frac{1}{u(x_i)};$$

$$\frac{\partial lnL}{\partial m} = \sum_{i=1}^{n} \frac{1}{v(x_i)} - (\alpha+1)\sum_{i=1}^{n} \frac{x_i}{u(x_i)};$$

$$\begin{aligned} \frac{\partial lnL}{\partial a} &= \sum_{i=1}^{n} \left(\frac{x_i}{k}\right)^a e^{-\left(\frac{x_i}{k}\right)^a} \frac{\left[2ln\left(\frac{x_i}{k}\right) - a\left(\frac{x_i}{k}\right)^a ln\left(\frac{x_i}{k}\right) + 1\right]}{v(x_i)} \\ &- (\alpha+1)\sum_{i=1}^{n} \frac{x_i ln\left(\frac{x_i}{k}\right)\left(\frac{x_i}{k}\right)^a e^{-\left(\frac{x_i}{k}\right)^a}}{u(x_i)}; \end{aligned}$$

$$\frac{\partial lnL}{\partial k} = \sum_{i=1}^{n} \frac{\frac{1}{k}\left(\frac{x_i}{k}\right)^a e^{-\left(\frac{x_i}{k}\right)^a}\left(-1 + a\left(\frac{x_i}{k}\right)^a + a^2\right)}{v(x_i)} +$$

$$\frac{a}{k^{(a+1)}}(\alpha+1)\sum_{i=1}^{n}\frac{x_i^{a+1}e^{-\left(\frac{x_i}{k}\right)^a}}{u(x_i)}.$$

The complexity of the equations, and the interdependence of the variables, make it difficult to obtain exact solutions in closed form. In this context, the use of numerical methods becomes unavoidable. Numerical methods provide us with a pragmatic and efficient approach to solve these complex equations. They allow us to navigate through the subtleties of the system and approximate solutions iteratively. Thanks to these techniques, we can manage the complexity of the model and obtain precise parameter estimates, which would otherwise be impossible with traditional analytical methods.

7. Monte Carlo Simulation

In order to assess the consistency of the maximum likelihood estimator of the SHL-Weibull distribution, we will study the behavior of the maximum estimator as the sample size increases, using a full simulation analysis performed with R software. We have generated 300 independent samples, with variable sizes of $n = 100, 250, 400, 500$ and 1000 from the SHL-Weibull distribution. For each replication, after determining the MLEs, we appraise the root mean square error (RMSE) and mean bias (Bias) to determine the precision and coherence of the estimated parameters. The table 2 shows the results obtained from these measurements with the R software.

8. Application

This section demonstrates the adaptability of the SHL-W distribution by applying it to three different data sets. The suggested model's maximum likelihood estimates (MLEs) and goodness-of-fit standards are contrasted with those of rival models.

- Lomax-Weibull of Lomax-G family [11]

$$f_{LW}(x) = \alpha m^{\alpha} a \frac{x^{a-1}}{k^a\left[m + (x/k)^a\right]^{\alpha+1}}.$$

- Weibull-Weibull [31]

$$f_{WW}(x) = \alpha a k \lambda x^{k-1}\left[e^{\lambda x^k} - 1\right]^{a-1} e^{-\alpha\left(e^{\lambda x^k}-1\right)^a + \lambda x^k}.$$

- Odd burr XII Exponentiated Weibull [32]

$$\begin{aligned} f_{OBXIIEW}(x) = m\lambda\alpha a k x^{a-1}\left(1 - e^{-kx^a}\right)^{2\alpha-1}\left(2 - \left(1 - e^{-kx^a}\right)^{\alpha}\right) \\ \times\left(1 - \left(1 - e^{-kx^a}\right)^{\alpha}\right)^{-2} j(x;\theta)^{\lambda-1}\left[1 + j(x;\theta)^{\lambda}\right]^{-(m+1)}, \end{aligned}$$

where $j(x;\theta) = \left[\frac{\left(1-e^{-kx^a}\right)^{2\alpha}}{1-\left(1-e^{-kx^a}\right)^{\alpha}}\right]$.

Table 2. Findings from Monte Carlo simulations of the SHL-W distribution: Mean, RMSE, and mean bias computations

	$(\alpha, \lambda, m, a, k) = (1.5, 2, 0.004, 1.8, 0.8)$ —			$(\alpha, \lambda, m, a, k)$ =(0.5, 1, 0.005, 2, 0.5) —		
n	Mean	RMSE	Bias	Mean	RMSE	Bias
100	1.7103	0.8411	0.2103	0.6970	0.2181	0.1970
	2.5163	1.9797	0.5163	1.9146	1.1841	0.9146
	0.0317	0.0888	0.0277	0.0541	0.1479	0.0491
	2.2104	1.0391	0.4104	3.1227	3.6452	1.1227
	0.8860	0.4340	0.0860	0.4822	0.1445	-0.0178
250	1.5960	0.3377	0.0960	0.6885	0.1972	0.1885
	2.2471	0.7704	0.2471	1.8147	0.9255	0.8147
	0.0179	0.0395	0.0139	0.0314	0.0818	0.0264
	1.9442	0.3847	0.1442	2.4668	1.0845	0.4668
	0.8100	0.1387	0.0100	0.4639	0.0768	-0.0361
400	1.5676	0.2428	0.0676	0.6862	0.1914	0.1862
	2.1663	0.5688	0.1663	1.7827	0.8501	0.7827
	0.0141	0.0308	0.0101	0.0229	0.0541	0.0179
	1.9079	0.3107	0.1079	2.2861	0.6335	0.2861
	0.8032	0.0889	0.0032	0.4591	0.0656	-0.0409
500	1.5649	0.2116	0.0649	0.6852	0.1898	0.1852
	2.1493	0.4930	0.1493	1.7705	0.8271	0.7705
	0.0124	0.0263	0.0084	0.0241	0.0564	0.0191
	1.8745	0.2571	0.0745	2.2476	0.5295	0.2476
	0.8038	0.0774	0.0038	0.4613	0.0598	-0.0387
1000	1.5514	0.1483	0.0514	0.6820	0.1841	0.1820
	2.1132	0.3403	0.1132	1.7379	0.7660	0.7379
	0.0072	0.0137	0.0032	0.0154	0.0308	0.0104
	1.8399	0.1594	0.0399	2.1502	0.3174	0.1502
	0.7985	0.0540	-0.0015	0.4590	0.0518	-0.0410

- Beta exponentiated Weibull [33]

$$f_{BEW}(x) = \frac{a\alpha}{k^a B(\lambda, m)} x^{a-1} e^{-\left(\frac{x}{k}\right)^a} \left(1 - e^{-\left(\frac{x}{k}\right)^a}\right)^{\alpha\lambda-1} \left(1 - \left(1 - e^{-\left(\frac{x}{k}\right)^a}\right)^{\alpha}\right)^{-m-1},$$

where $B(\lambda, m) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

- Exponentiated Weibull Weibull [34]

$$f_{EWW}(x) = \frac{m\alpha a}{k\lambda^\alpha} \left(\frac{x}{k}\right)^{a\alpha-1} e^{\left[-\lambda^{-\alpha}\left(\frac{x}{k}\right)^{a\alpha}\right]} \times \left[1 - e^{\left[-\lambda^{-\alpha}\left(\frac{x}{k}\right)^{a\alpha}\right]}\right]^{m-1}$$

- McDonald Modified Weibull Distribution [35]

$$f_{McDW}(x) = \frac{\lambda}{B(a,b)}\left(\alpha + kmx^{m-1}\right)e^{-\alpha x - kx^m} \times\left(1 - e^{-\alpha x - kx^m}\right)\left(1 - e^{-\alpha x - kx^m}\right)^{a\lambda - 1}$$

The adoption of the Weibull distribution as a foundation allows for a comparison between various models and is the essential similarity between these distributions. These distributions have been compared and assessed using a number of well-known statistical metrics as well as others. It should be mentioned that the most ideal model is the one with the lowest criterion. We utilized the R programming language to compute each of these metrics.
The suggested models are assessed and contrasted using a variety of metrics. The criteria that are designated are the Akaike Information Criterion (AIC), the Hannan-Quinn Information Criterion (HQIC), and the Bayes Information Criterion (BIC). These criteria are defined as follows:

$$AIC = 2k - 2\log L,$$

$$BIC = k\log n$$

and

$$HQIC = 2k\log(\log n) - 2\log L,$$

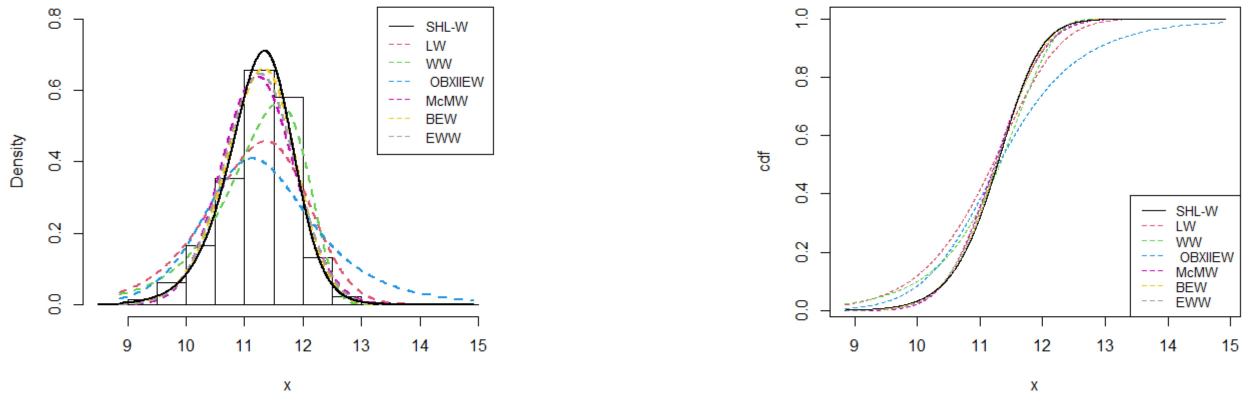
where n represents the sample size, k represents the number of parameters in the statistical model, and L represents the maximal value of the log-likelihood function in the model under consideration.

8.1. *Dataset of an actual life-risk insurance portfolio*

With the minimal technical requirements needed to compute life-risk insurance premiums on an annual and infra-annual basis, the first dataset describes anonymised data pertaining to policies in effect on December 31, 2009 in the life insurance portfolio of a Spanish insurance firm [36].

It is made up of 76102 observations with 15 variables, the data sheet for which is available at https://doi.org/10.3886/E178881V1. The economic amount that beneficiaries would receive in the event of death, given in euros, is referred to as the capital at risk (variable capital) in this set.
Capital at risk represents the amount that the insurer could potentially lose in unfavorable scenarios. Accurate modeling of risk capital is fundamental to guarantee solvency and optimize capital utilization.

Figure 6 provides a visualization of the empirical probability density and empirical distribution functions, which can be used to better understand the data distribution.



(a) pdfs representation

(b) CDFs representation

Figure 6. Visualization of empirical pdfs and CDFs for dataset I

Tables 3 and 4 present the parameter estimators and criteria for dataset I.

Table 3. Predicted values for various models

Models	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{k}$	$\hat{a}$	$\hat{m}$	$\hat{b}$
SHL-W	2.44	0.39	13.44	25.46	0.00	-
LW	5.78	-	11.78	15.36	3.58	-
WW	0.000059	0.359	1.82	0.318	-	-
OBXIIEW	5.8	2.95	0.07	1.48	0.75	-
BEW	14.447	0.57	0.098	6.04	3.38	-
EWW	5.11	3.7	5.13	1.84	5.126	-
McDW	0.092	102.38	3.87	275.11	0	317.09

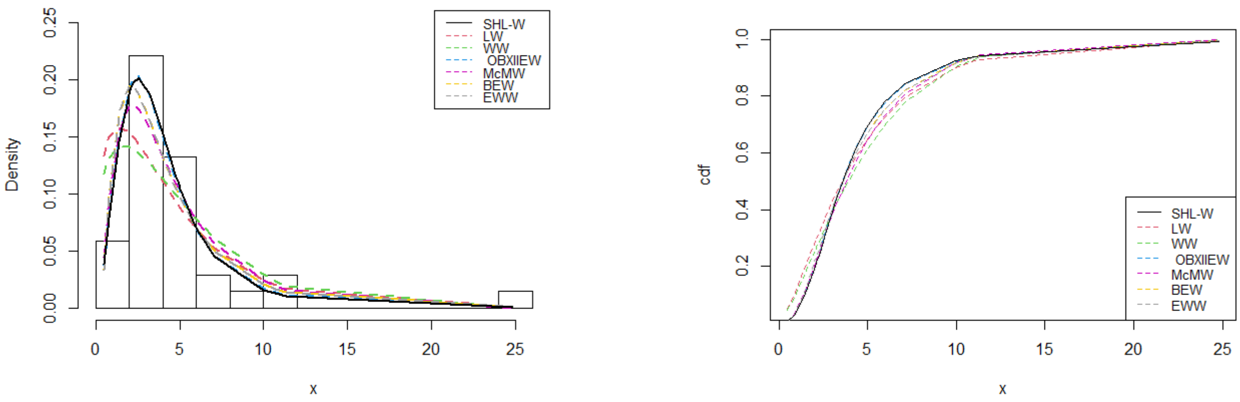
Table 4. Measures of selection for various models

Models	$-\hat{l}$	AIC	BIC	HQIC
SHL-W	70406.15	140822.3	140868.5	140836.5
LW	79524.7	159057.4	159094.34	159068.74
WW	79237	158482.49	158519.45	158493.85
OBXIIEW	86749.96	173509.9	173556.12	173524.12
BEW	70988.19	141986.37	142032.57	142000
EWW	71393.65	142797.31	142843.51	142811.5
McDW	72031.6	144075.3	144130.73	144092.33

8.2. Insurance and financial services dataset

The second dataset comes with annual statistics on insurance and financial services (% of services exports, BoP) broken down by nation and is sourced from the World Bank website. Insurance and financial services include the different kinds of insurance that resident insurance companies offer to

non-residents and vice versa, as well as the financial auxiliary and intermediary services that are traded between residents and non-residents (not including insurance company and pension fund services).
We will focus on Benin between 1988 and 2023, i.e. 34 observations: 4.95, 4.37, 5.95, 4.94, 7.07, 6.19, 5.04, 5.7, 2.61, 3. 25, 4.68, 3.56, 2.18, 2.11, 2.54, 0.48 , 1.41, 2.11, 2.24, 2.05, 2.12, 2.3, 2.07, 0.82, 10.72, 1.98, 3.74, 4.8, 3.74 , 3.91, 24.8, 11.38, 9.96, 5.61.
Figure 7 provides a visualization of the empirical probability density and empirical distribution functions, which can be used to better understand the data distribution.



(a) pdfs representation **(b)** CDFs representation

Figure 7. Visualization of empirical pdfs and CDFs for dataset II

The estimated values for the dataset studied are presented in Table 5. In order to evaluate the performance of the various models used for this data set, we performed information criteria calculations, the results of which are summarized in table 6.

Table 5. Predicted values for various models

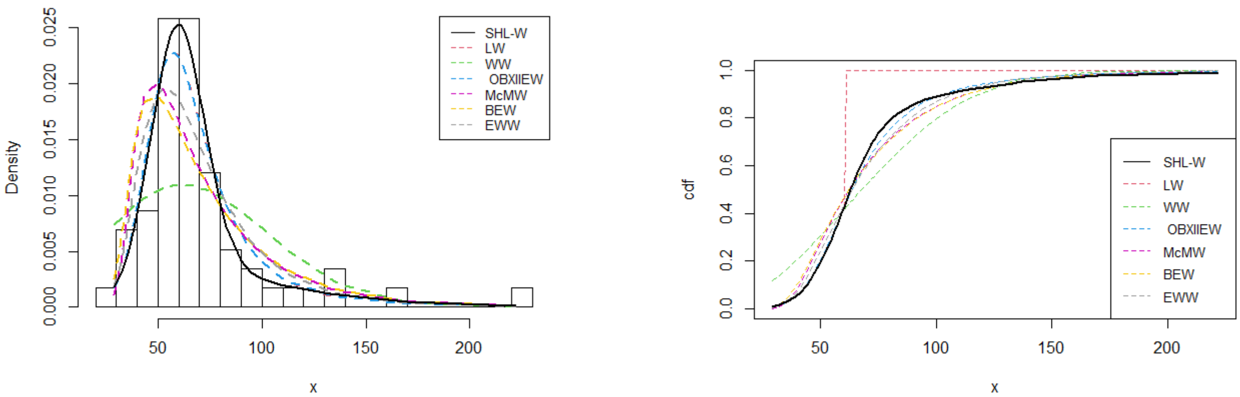
Models	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{k}$	$\hat{a}$	$\hat{m}$	$\hat{b}$
SHL-W	3.59	9.18	4.25	1.46	0.0	-
LW	5.0	-	1.93	1.34	15.34	-
WW	195.4	0.68	0.004	246.3	-	-
OBXIIEW	0.54	7.31	0.4	0.23	0.89	-
BEW	1.46	8.6	2	0.491	1.25	-
EWW	10.8	0.91	0.74	0.29	1.84	-
McDW	0.94	1.86	0.05	1.92	0.004	0.31

Table 6. Measures of selection for various models

Models	$-\hat{l}$	AIC	BIC	HQIC
SHL-W	81.02	172.04	179.68	174.65
LW	83.8	175.6	181.71	177.69
WW	84.92	177.83	183.94	179.9
OBXIIEW	81.07	172.13	179.8	174.74
BEW	81.6	173.2	180.84	175.81
EWW	81.61	173.21	180.84	175.81
McDW	82.24	176.47	185.63	179.6

8.3. Unemployment insurance data set

This is a real sawtooth data set from the insurance industry. The Maryland Department of Labor reported that it represents monthly unemployment insurance measures from July 2008 to April 2013. The data set comprises 21 variables, of which variable number 12 (called "# of First UI Checks issued - Ex. Fed Employees") was the focus of our analysis.
You can access the data at: https://catalog.data.gov/dataset/unemployment-insurance-data-july-2008-to-april-2013.



(a) pdfs representation **(b)** CDFs representation

Figure 8. Visualization of empirical pdfs and CDFs for dataset III

Tables 7 and 8 present the parameter estimators and criteria for dataset III.

Table 7. Predicted values for various models

Models	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{k}$	$\hat{a}$	$\hat{m}$	$\hat{b}$
SHL-W	1.59e+04	7.21e+05	67.8	5	0	
LW	0.58		34.96	6.54	21.77	
WW	1.06e-06	1.26	0.13	6.37	-	
OBXIIEW	0.325	54.55	0.16	0.12	0.57	
BEW	2.26	41.26	0.1	1.12	0.2	
EWW	277.56	3.1	1.74	5.56	0.094	
McDW	0.145	2.8	0.032	92.01	0.007	0.22

Table 8. Measures of selection for various models

Models	$-\hat{l}$	AIC	BIC	HQIC
SHL-W	262.73	535.46	545.76	539.47
LW	264.18	536.35	544.6	539.56
WW	281.55	571.11	579.35	574.32
OBXIIEW	264.18	538.36	548.66	542.37
BEW	268.43	546.86	557.16	550.87
EWW	266.15	542.3	552.6	546.31
McDW	268.53	549.05	561.41	553.87

With a comparatively low AIC of 535.46 when compared to the other models, the SHL-Weibull (SHL-W) model appears to provide a decent balance between accuracy and complexity. We observe that a better fit to the data is provided by the SHL-W distribution. Among all the distributions examined, its AIC, BIC, and HQIC values are actually the lowest. As a result, we determine that among its models, the SHL-W distribution performs the best.
A thorough examination shows that the newly proposed model performs substantially better than the other models, as evidenced by PDF and CDF graphs for the SHL-W model and its rival models (Fig. 6–8), as well as the computation of information criteria like AIC, BIC, and HQIC as shown in tables 4, 6, 8. This outcome highlights the exceptional adaptability of our model in capturing the intricate features of insurance and financial data, demonstrating its capacity to offer more precise and dependable modifications for risk assessment and well-informed decision-making in intricate ever-changing settings.

9. Conclusion

The family of shifted Lomax-X distributions (SHL-X), which we propose as a new class of models, generates distributions with Lomax as the basic distribution. In this paper, we have developed a member of this family, the SHL-W distribution. Numerous mathematical features have been explored, including actuarial measures, Renyi entropy, ordinary and incomplete moments, and quantiles. The maximum likelihood method is used to estimate the model parameters.Three applications on real data, in particular in insurance and finance, show the importance and potential of SHL-W distribution in actuarial science.

SHL-W distributions have potential uses outside of actuarial science and finance because of their intrinsic modeling qualities. These adaptable statistical models will be especially helpful to the insurance sector because they offer better coverage.
However, this article does not address all of the family's properties and extensions. These include the following: quantile estimation, bootstrap estimation, uniform unbiased and minimum variance estimation, uniform residual entropy, Song measure, goodness-of-fit tests, tolerance intervals, multivariate generalizations, Bayesian and empirical Bayes estimation, uniform unbiased and minimum variance estimation, weighted least squares estimation, order statistics estimation, L-moment estimation, and order of magnitude estimation. We want to write on a few of these topics in an upcoming piece. We intend to investigate these facets in upcoming projects. For instance, we intend to create quantile estimation through the use of Monte Carlo methods or distribution function inversion techniques. Additionally, we will conduct quality tests by utilizing statistical tests like the Kolmogorov-Smirnov and Cramér-von Mises tests to compare theoretical distributions with empirical data. Furthermore, we will investigate Bayesian estimation with special application to medical and actuarial data, utilizing techniques like Variational Bayesian Approximation (VBA) and Markov Chain Sampling (MCMC).

As a result, our study clears the path for upcoming innovations that may improve our knowledge of and ability to use SHL-X models, solidifying their position in cutting-edge statistical modeling.

Conflict of Interest: The authors declare no competing interests.

Data Availability: Any data that supports the findings of this study is included in the article.

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