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Research article

A Skew Product Distribution with Applications

Mohammad Shakil[1]*, Tassaddaq Hussain[2], Zahid Ur Rehman[2], Aneeqa Khadim[3], Musaddiq Sirajo[4], J. N. Singh[5], Mohammad Ahsanullah[6], B. M. Golam Kibria[7]

[1] Department of Mathematics, Miami Dade College, Hialeah, FL, USA
[2] Department of Statistics, Mirpur University of Science and Technology (MUST), Mirpur, Pakistan
[3] Department of Mathematics, Mirpur University of Science and Technology (MUST), Mirpur, Pakistan
[4] Department of Statistics, Ahmadu Bello University, Zaria, Nigeria
[5] Barry University, Dept. of Math & Comp. Sc., Miami Shores, FL, USA
[6] Department of Management Sciences, Professor Emeritus, Rider University, NJ, USA
[7] Department of Mathematics & Statistics, Florida International University, Miami, FL, USA

* **Correspondence:** mshakil@mdc.edu

Abstract: The Rayleigh and error function distributions arise in many problems of applied and physical sciences. In this paper, we derive a skew type product Rayleigh-Error Function distribution by taking the product of the probability density function of Rayleigh distribution and the cumulative distribution function of the error function distribution. Several characteristics of the new distribution are presented by providing its distributional properties, namely, the expression for its probability density function, cumulative distribution function, hazard function, moments, and Shannon entropy. To discuss the behavior of the proposed distribution, the corresponding graphs of its probability density function, cumulative distribution function, and hazard function are provided. The percentage points are also computed. Some characterizations, estimation of parameters, and simulation are presented. Also, d two real-life environmental data applications are provided. We observed that our proposed skew type product Rayleigh-Error Function distribution fits reasonably well to the datasets considered. We believe that our proposed skew product model would be useful to the practitioners in various research fields of applied, physical, environmental and other sciences.

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1. Introduction and Motivation

Skewed distributions are a crucial concept in many fields of research and play very important roles in modeling and analyzing of real lifetime in various fields of science and social sciences. For example, income distribution in a population often follows a skewed distribution, with a few individuals earning significantly more than the majority. Understanding this skewness helps analysts identify income inequality and inform economic policies. In economics the wealth distribution among individuals in a country often exhibits a skewed distribution, with a small percentage of the population holding a significant portion of the wealth. These understanding guides economic decisions and policies aimed at reducing wealth disparities. In finance Stock prices and returns often follow a skewed distribution, with extreme events like market crashes or sudden gains. Recognising this skewness helps investors and analysts manage risk, optimize portfolios, and make informed investment decisions. In Biology and Medicine, the spread of diseases like COVID-19 follows a skewed distribution, with a small percentage of individuals accounting for a significant portion of transmissions. Understanding this skewness guides public health decisions and contact tracing strategies. In Environmental Science Air pollution levels in a city can follow a skewed distribution, with extreme events like peak pollution days. Recognising this skewness helps policymakers develop targeted strategies to reduce pollution and protect public health. In each of these fields, skewed distributions play a vital role in:

1. Identifying outliers and extreme events
2. Understanding asymmetric data
3. Informing decision-making and policy development
4. Optimizing processes and resource allocation
5. Developing effective solutions to real-world problems

However, despite the potential applications of skewed distributions in various fields of research, it would be worth mentioning some of its limitations. Skewed distributions can be challenging to model accurately, especially when data is limited. They are sensitive to outliers, which can significantly impact analysis and modelling. They may not be representative of the entire population, limiting generalizability. Moreover, they can make hypothesis testing challenging due to non-normality, and can be difficult to interpret, especially for non-technical audiences. Nevertheless, skewed distributions can help model extreme weather events and climate change impact, complex ecosystem dynamics, income inequality, economic risks and uncertainties, quality control and defect detection, predict disease spread, data compression, image processing, machine learning, etc. By acknowledging and working with skewed distributions, professionals can develop more realistic models, make more informed decisions, and derive meaningful insights in their respective fields by studying various systems of non-normal or skew data and processes governing real-life phenomena. For some notable contributions and recently developed distributions, one is referred to [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], and [13]. We define a skew distribution as follows:

DEFINITION – 1.1. Let be a random variable with pdf (probability density function) $f_X()$. and cdf (cumulative distribution function) $F_X()$.. Then Y is called a skew symmetric when

$$f_Y(y) = f_X(y).w\left[F_X(y)\right], y \in \Re, \tag{1.1}$$

where $w\,[F_X(y)] \in (0,1)$ is a probability density function. Let $w\,(x) = 2\,F_X\left[\lambda\,F_X{}^{-1}(x)\right]$. Then,

$$f_Y\,(y) = 2 f_X(y).F_X\,(\lambda y)\,, \tag{1.2}$$

where λ is a *shape parameter* ([12]). Let $f_X(.) = \varphi\,(.)$ and $F_X(.) = \Phi\,(.)$, where $\phi\,(.)$ and $\Phi\,(.)$ denotes the pdf and cdf of the standard normal distribution, $N\,(0,\ 1)$, respectively. Then, from (1.2), we have

$$f_X\,(x\,;\ \lambda) = 2\,\phi\,(x)\;\Phi\,(\lambda\,x)\,,\ -\infty < x < \infty, \tag{1.3}$$

(see [14]). Many well-known families of skew distributions developed by various authors are special cases of the equations (1.1) and (1.2) ([14], [15], [16], [17], [18], [19],[20], [21], and [22]). Let X be a non-negative continuous random variable. In this paper, we derive a skew product distribution (which we call PREF distribution), $p_X\,(x) = C.g_X(x).G_X\,(x)$, where C is the normalizing constant, and $g_X(x)$ and $G_X\,(x)$ denote the pdf and cdf of the Rayleigh and error function distributions, respectively.

The organization of this paper is as follows: In Section 2, we provide the derivations of our proposed PREF distribution with several characteristics. Section 3 contains the estimation of parameters, applications to two environmental datasets and simulation. In Section 4, characterizations are given. Conclusions are outlined in Section 5. Some special functions and lemmas, used in the paper, are provided in Appendix A.

2. Derivation of the PREF Distribution

2.1. The Rayleigh and error function distribution are defined as follows:

DEFINITION – 2.1. Let X be a non-negative continuous random variable. Then X is said to have a Rayleigh distribution if its pdf and cdf are, respectively, given as

$$f_X(x) = \left(\frac{x}{\sigma^2}\right)\,e^{-\,x^2/2\sigma^2},\qquad x \geq 0,\ \sigma > 0, \tag{2.1}$$

and

$$F_X(x) = 1 - e^{-\,x^2/2\sigma^2}. \tag{2.2}$$

DEFINITION – 2.2. Let X be a non-negative continuous random variable. Then X is said to have an error function distribution if it has its pdf and cdf, respectively, as follows:

$$f_X(x) = \left(\frac{h}{\sqrt{\pi}}\right)\,e^{-\,h^2x^2},\qquad x \in \mathfrak{R},\ h > 0, \tag{2.3}$$

and

$$F_X(x) = \frac{1}{2}\,[1 + \Phi\,(h\,x)]\,, \tag{2.4}$$

where $\Phi\,(.)$ denotes the error function.

2.2. Expressions for the Normalizing Constant and PDF

Theorem 1: The equation $p_X(x) = C \,.\, g_X(x) \,.\, G_X(x)$, $x \geq 0$, $C > 0$, defines the pdf of the PREF distribution if it has its normalizing constant given as follows:

$$C = \frac{2\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}\,h\,\sigma}, \quad h \geq 0,\ \sigma > 0. \tag{2.5}$$

Without loss of generality, we assume that $h \geq 0$ in equation (2.5).

Proof: Clearly, $p_X(x) \geq 0$, $\forall\, x \in [0, \infty)$, and $C > 0$. Hence for $p_X(x)$ to be a pdf, we must have

$$\int_0^\infty p_X(x)\ dx = 1, \tag{2.6}$$

where $p_X(x) = C \,.\, g_X(x) \,.\, G_X(x)$. Substituting (2.1) and (2.4) in (2.6), we have

$$\int_0^\infty C.\left(\frac{x}{\sigma^2}\right) e^{-x^2/2\sigma^2}.\ \frac{1}{2}\,[1 + \Phi(h\,x)]\ dx = 1,$$

or,

$$\int_0^\infty \left(\frac{1}{2\,\sigma^2}\right) x\,e^{-x^2/2\sigma^2} dx + \int_0^\infty \left(\frac{1}{2\,\sigma^2}\right) x\,e^{-x^2/2\sigma^2}.\ \Phi(h\,x)\ dx = \frac{1}{C}. \tag{2.7}$$

Let $\frac{x^2}{2\,\sigma^2} = t$ in (2.7). Then, we easily have

$$\int_0^\infty \left(\frac{1}{2\,\sigma^2}\right) x\,e^{-x^2/2\sigma^2} dx = \frac{1}{2}, \tag{2.8}$$

and

$$\int_0^\infty \left(\frac{1}{2\,\sigma^2}\right) x\,e^{-x^2/2\sigma^2}.\ \Phi(h\,x)\ dx = \frac{h\,\sigma}{\sqrt{2}\,\sqrt{1 + 2h^2\sigma^2}}. \tag{2.9}$$

Using (2.8) and (2.9) in (2.7), the proof follows.

Theorem 2: For some non-negative continuous random variable X, if $g_X(x)$ and $G_X(x)$ denote the pdf and the cdf of Rayleigh and the error function distribution, as defined in (2.1) and (2.4), respectively, and C denotes the normalizing constant given by (2.5), the following equation

$$\begin{aligned} p_X(x) &= C \,.\, g_X(x) \,.\, G_X(x) \\ &= \frac{\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}\,h\,\sigma}\,.\,\left(\frac{x}{\sigma^2}\right) e^{-x^2/2\sigma^2}\ .\ [1 + \Phi(h\,x)]\,, \quad h \geq 0,\ \sigma > 0, \end{aligned} \tag{2.10}$$

defines a pdf of the random variable X.

Proof: The proof of Theorem 2 easily follows, since from Theorem 1, we have

$$\int_0^\infty \left(\frac{x}{\sigma^2}\right) e^{-x^2/2\sigma^2}.\ \frac{1}{2}\,[1 + \Phi(h\,x)]\ dx = \frac{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}\,h\,\sigma}{2\sqrt{1 + 2h^2\sigma^2}}.$$

Special Case: For $h = 0$, the pdf in (2.10) reduces to the pdf of Rayleigh distribution given by (2.1).

2.3. Derivation of the CDF

This sub-section derives the associated cdf of the random variable X, when the normalizing constant C (> 0) satisfies the requirements for the product function $p_X(x)$ to be a probability density function, as shown in Section 2.2.

Theorem 3: The cumulative distribution function (cdf), $P_X(x)$, corresponding to our proposed product probability density function $p_X(x) = C \,.\, g_X(x) \,.\, G_X(x)$, $x \geq 0$, $C > 0$, where

$$C = \frac{2\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}\,h\,\sigma}, \quad h \geq 0,\ \sigma > 0,$$

is given by

$$P_X(x) = \frac{C}{2}\left[\gamma\left(1, \frac{x^2}{2\sigma^2}\right) + \frac{\sqrt{2}\,h\,\sigma}{\sqrt{2h^2\sigma^2 + 1}}\Phi\left(\frac{\sqrt{2h^2\sigma^2 + 1}}{\sqrt{2}\,\sigma}x\right) - e^{-\frac{x^2}{2\sigma^2}}\Phi(hx)\right], \tag{2.11}$$

$$P_X(x) = \frac{C}{2}\left[1 - e^{-x^2/2\sigma^2} + \frac{\sqrt{2}\,h\,\sigma}{\sqrt{2h^2\sigma^2 + 1}}\Phi\left(\frac{\sqrt{2h^2\sigma^2 + 1}}{\sqrt{2}\,\sigma}x\right) - e^{-\frac{x^2}{2\sigma^2}}\Phi(hx)\right], \tag{2.12}$$

where C is the normalizing constant given by (2.5).

Proof: We have

$$P_X(x) = \int_0^x C.\, g_X(t) \,.\, G_X(t)\, dt = \int_0^x C.\left(\frac{t}{\sigma^2}\right) e^{-t^2/2\sigma^2}.\, \frac{1}{2}\,[1 + \Phi(ht)]\, dt$$

$$= \frac{C}{2}\int_0^x \frac{1}{\sigma^2}.\, t.\, e^{-t^2/2\sigma^2} dt + \frac{C}{2\sigma^2}\int_0^x t.\, e^{-t^2/2\sigma^2}.\, \Phi(ht)\, dt. \tag{2.13}$$

Now, by substituting $\frac{t^2}{2\sigma^2} = z$ in the first integral of the Equation (2.13), simplifying, and then using Lemmas A.1.2 and A.1.3, it can easily be seen that

$$\frac{C}{2}\int_0^x \frac{1}{\sigma^2}.\, t.\, e^{-t^2/2\sigma^2} dt = \frac{C}{2}\gamma\left(1, \frac{x^2}{2\sigma^2}\right) \tag{2.14}$$

$$\frac{C}{2}\int_0^x \frac{1}{\sigma^2}.\, t.\, e^{-t^2/2\sigma^2} dt = \frac{C}{2}\left(1 - e^{-x^2/2\sigma^2}\right). \tag{2.15}$$

Furthermore, in the second integral of the Equation (2.13), using Lemma A.2.4, and simplifying, we have

$$\frac{C}{2\sigma^2}\int_0^x t.\, e^{-t^2/2\sigma^2}.\, \Phi(ht)\, dt = \frac{C}{2}\left[\frac{\sqrt{2}\,h\,\sigma}{\sqrt{2h^2\sigma^2 + 1}}\Phi\left(\frac{\sqrt{2h^2\sigma^2 + 1}}{\sqrt{2}\,\sigma}x\right) - e^{-\frac{x^2}{2\sigma^2}}\Phi(hx)\right]. \tag{2.16}$$

Using (2.14), (2.15) and (2.16) in Equation (2.13), the proof of Theorem 3 follows.

Remark: In view of the following derivatives $\frac{d\gamma(\alpha, z)}{dz} = z^{\alpha - 1}e^{-z}$, and $\frac{d\Phi(z)}{dz} = \frac{2}{\sqrt{\pi}}e^{-z^2}$, of the incomplete gamma and error functions, respectively, it can easily be seen, by direct differentiation of the expressions for the cdf in Equations (2.11) and (2.12), that $\frac{dP_X(x)}{dx} = p_X(x)$.

2.3.1. Some Plots of the PDF and CDF of the PREF Distribution

In this sub-section, we present some plots the PDF and CDF of the PREF distribution to discuss its behaviors for some selected values of the parameters. The possible shapes of the pdf (2.10) and cdf (2.12) for $h = 2$ and different values of σ, and for $\sigma = 2$ and different values of h, are provided in Figures 1, 2, 3, and 4, respectively.

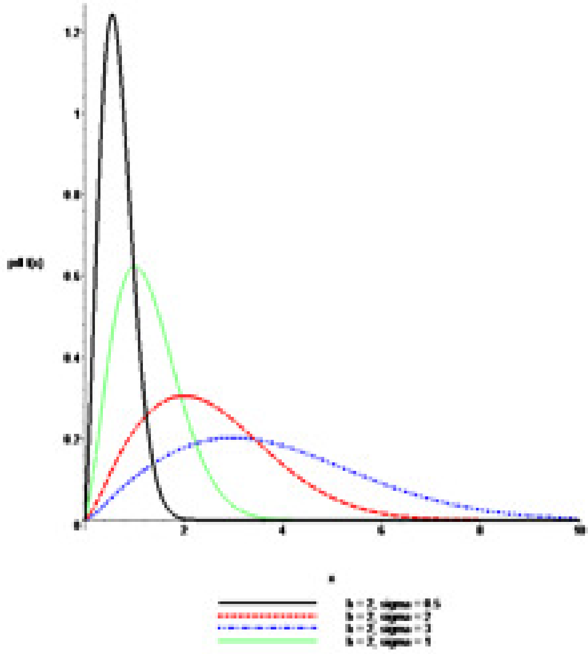

Figure 1. pdf, when $h = 2$ and $\sigma = 0.5,\ 1,\ 2,\ 3$

From the graphs in Figures 1, 2, 3, and 4, it is obvious that the proposed distribution is unimodal and right skewed.

2.3.2. Hazard Function and Reliability Analysis of the PREF Distribution:

The hazard function (hf) of the PREF distribution is given by

$$h(x) = \frac{p_X(x)}{1 - P_X(x)}$$

$$= \frac{\frac{\sqrt{1+2h^2\sigma^2}}{\sqrt{1+2h^2\sigma^2}+\sqrt{2}h\sigma}\cdot\left(\frac{x}{\sigma^2}\right)e^{-x^2/2\sigma^2}.\ [1+\Phi(hx)]}{1-\frac{\sqrt{1+2h^2\sigma^2}}{\sqrt{1+2h^2\sigma^2}+\sqrt{2}h\sigma}\left[1-e^{-x^2/2\sigma^2}+\frac{\sqrt{2}h\sigma}{\sqrt{2h^2\sigma^2+1}}\Phi\left(\frac{\sqrt{2h^2\sigma^2+1}}{\sqrt{2}\sigma}x\right)-e^{-\frac{x^2}{2\sigma^2}}\Phi(hx)\right]}, \tag{2.17}$$

where $h \geq 0$, $\sigma > 0$. For $h = 2$ and different values of σ, and for $\sigma = 2$ and different values of h, the possible shapes of the hf (2.17) of our new proposed skew product distribution are provided in Figures 5, and 6, respectively.

From the Figures 5, and 6, it is evident that the failure rate function, $h(x)$, is increasing and concave up shaped. Also, differentiating Equation (2.17) with respect to x, we have

$$h^{/}(x) = \frac{p^{/}(x)}{p(x)}h(x) + [h(x)]^2, \tag{2.18}$$

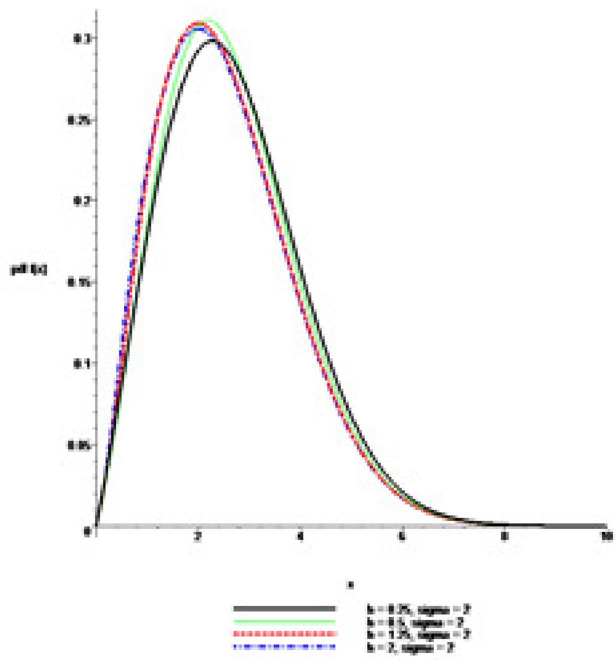

Figure 2. pdf, when $h = 0.25, 0.5, 1.25, 2$ and $\sigma = 2$

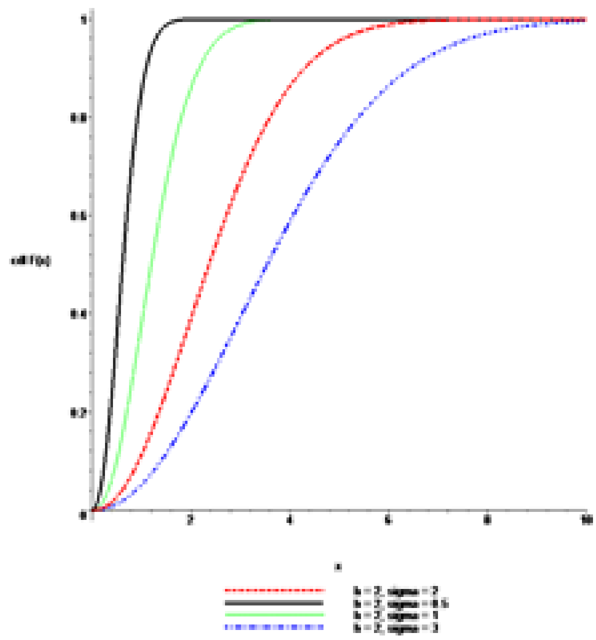

Figure 3. cdf, when $h = 2$ and $\sigma = 0.5, 1, 2, 3$

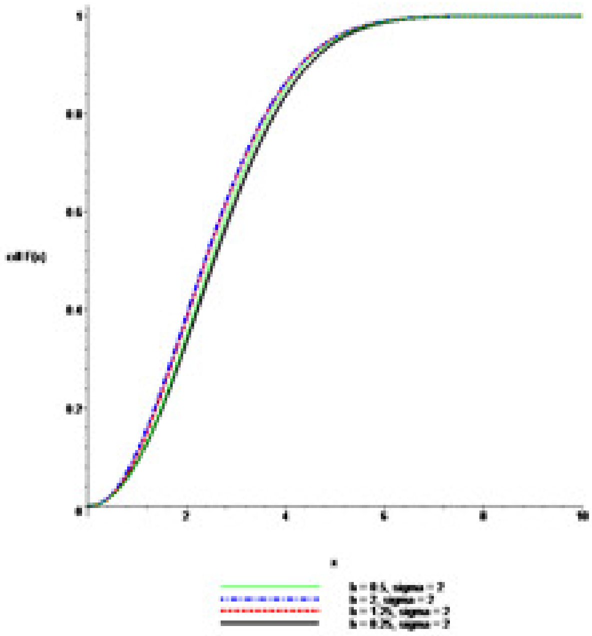

Figure 4. cdf, when $h = 0.25,\ 0.5,\ 1.25,\ 2$ and $\sigma = 2$.

for $x \geq 0$, where $p(x)$ and $h(x)$ are given by Equations (2.10) and (2.17), respectively, and $p^{/}(x)$ could be obtained by differentiating Equation (2.10) with respect to x, that is,

$$p^{/}(x) = \left(\frac{1}{\sigma^2}\right) e^{-x^2/2\sigma^2} \, . \, [1 + \Phi(h x)] \left(1 - \frac{x^2}{\sigma^2}\right) + \left(\frac{2h}{\sqrt{\pi}\,\sigma^2}\right) x \, e^{-h^2 x^2} e^{-x^2/2\sigma^2}. \tag{2.19}$$

In order to discuss the behavior of the failure rate function, $h(x)$, let $h^{/}(x) = 0$. We observe that the nonlinear equation $h^{/}(x) = 0$ does not have a closed form solution, but could be solved numerically by using some computer software. It is obvious from the Equations (2.18) and (2.19) that $h^{/}(x)$ is positive, provided $x < \sigma$, irrespective of the values of the parameters, $h \geq 0$, $\sigma > 0$. This shows that our new proposed PREF distribution has the increasing failure rate (IFR) property.

2.4. Some Properties of the PREF Distribution:

This sub-section discusses some characteristics of the PREF distribution.

2.4.1. Moments

Theorem 4: For some integer $k > 0$, the kth moment of the random variable X having the pdf (2.10) is given by

$$E\left(X^k\right) = \left(\frac{2\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}\,h\,\sigma}\right)$$
$$\times \left[2^{\frac{k}{2}-1}\sigma^k \Gamma\left(\frac{k}{2} + 1\right) + \left(\frac{h}{2\sqrt{\pi}\left(\frac{1}{2\sigma^2}\right)^{\frac{(k+1)}{2}}}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{\left(j + \frac{1}{2}\right)(j!)} \Gamma\left(\frac{2j + k + 3}{2}\right) \left(\sqrt{2}\,h\,\sigma\right)^{2j}\right]. \tag{2.20}$$

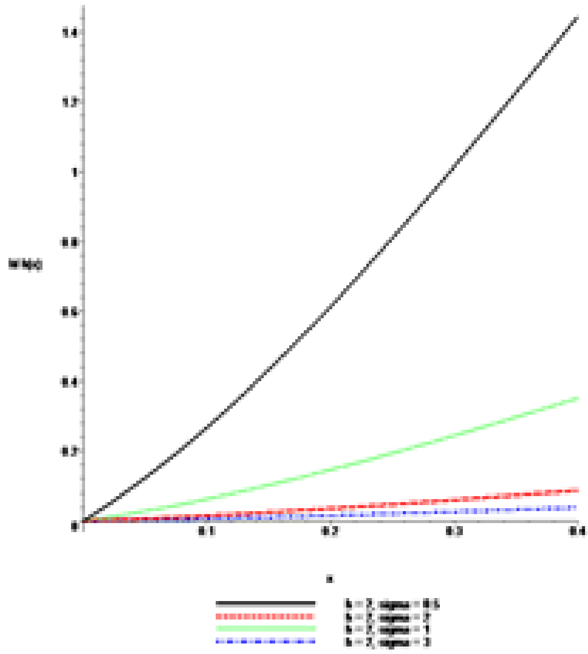

Figure 5. hf, when $h = 2$ and $\sigma = 0.5,\ 1,\ 2,\ 3$

Proof: Using the expression for the pdf (2.10), we have

$$E\left(X^k\right) = \left(\frac{2\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}h\sigma}\right) . \int_0^{\infty} x^k \left\{\left(\frac{x}{\sigma^2}\right) e^{-x^2/2\sigma^2} . \frac{1}{2}[1 + \Phi(hx)]\right\} dx$$
$$= \left(\frac{2\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}h\sigma}\right) . \left[\int_0^{\infty} x^k \left(\frac{1}{2\sigma^2}\right) x e^{-x^2/2\sigma^2} dx + \int_0^{\infty} x^k \left(\frac{1}{2\sigma^2}\right) x e^{-x^2/2\sigma^2} . \Phi(hx)\, dx\right]. \tag{2.21}$$

Now, by substituting $\frac{x^2}{2\sigma^2} = t$ in the first integral of the Equation (2.21), simplifying, and then using the definition of the complete gamma function, we have

$$\int_0^{\infty} x^k \left(\frac{1}{2\sigma^2}\right) x e^{-x^2/2\sigma^2} dx = 2^{\frac{k}{2} - 1} \sigma^k \Gamma\left(\frac{k}{2} + 1\right). \tag{2.22}$$

Furthermore, using Lemma A.1.5 in the second integral of the Equation (2.21) and simplifying, we have

$$\int_0^{\infty} x^k \left(\frac{1}{2\sigma^2}\right) x e^{-x^2/2\sigma^2} . \Phi(hx)\, dx = \left(\frac{1}{2\sigma^2}\right) \int_0^{\infty} x^{k+2-1} e^{-x^2/2\sigma^2} . \Phi(hx)\, dx$$
$$= \left[\frac{h}{2\sqrt{\pi}\left(\frac{1}{2\sigma^2}\right)^{\frac{(k+1)}{2}}}\right] \sum_{j=0}^{\infty} \frac{(-1)^j}{\left(j + \frac{1}{2}\right)(j!)} \Gamma\left(\frac{2j + k + 3}{2}\right) \left(\sqrt{2}h\sigma\right)^{2j}. \tag{2.23}$$

Using (2.22) and (2.23) in Equation (2.21), the proof of Theorem 4 easily follows.

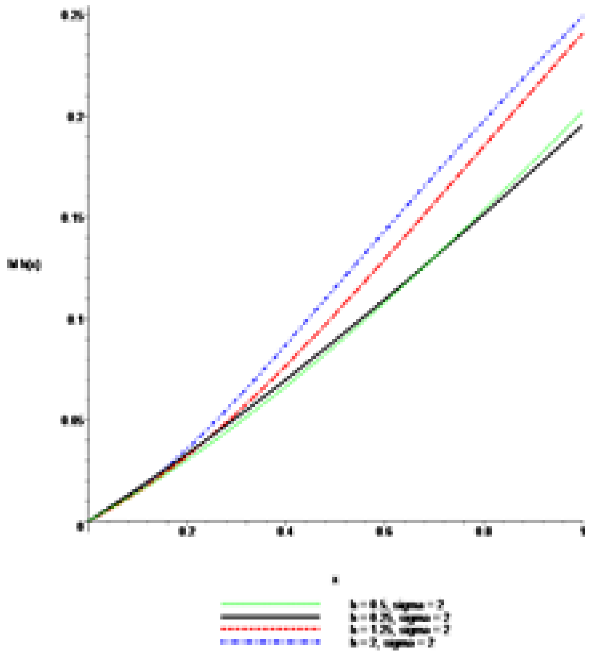

Figure 6. hf, when $h = 0.25, 0.5, 1.25, 2$ and $\sigma = 2$.

By taking $k = 1$ in Equation (2.20) and simplifying, the first moment (or the mean), α_1, of the PREF distribution is obtained as follows:

$$\alpha_1 = E(X) = \left(\frac{2\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}h\sigma} \right) \times \left[2^{-\frac{1}{2}} \sigma \Gamma\left(\frac{3}{2}\right) + \left(\frac{h}{2\sqrt{\pi}\left(\frac{1}{2\sigma^2}\right)} \right) \sum_{j=0}^{\infty} \frac{(-1)^j}{\left(j + \frac{1}{2}\right)(j!)} \Gamma(j + 2) \left(\sqrt{2}h\sigma\right)^{2j} \right]. \tag{2.24}$$

Similarly, using (2.20), it is easy to compute the *jth* (central) moment, variance, and coefficients of skewness and kurtosis.

2.4.2. *kth* Incomplete Moment:

Theorem 5: For some integer $k > 0$, the *kth* incomplete moment of the random variable X having the pdf (2.10) is given by

$$I_k(x) = \left(\frac{\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}h\sigma} \right) \cdot \left[2^{\frac{k}{2}} \sigma^k \gamma\left(\frac{k+2}{2}, \frac{x^2}{2\sigma^2} \right) - x^k e^{-x^2/2\sigma^2} \Phi(hx) + \frac{h}{\sqrt{\pi}} \left(h^2 + \frac{1}{2\sigma^2} \right)^{-\left(\frac{k+1}{2}\right)} \gamma\left(\frac{k+1}{2}, \left(h^2 + \frac{1}{2\sigma^2} \right) x^2 \right) \right]. \tag{2.25}$$

Proof: Using the expression for the pdf (2.10), we have

$$\begin{aligned} I_k(x) &= \left(\frac{2\sqrt{1+2h^2\sigma^2}}{\sqrt{1+2h^2\sigma^2}+\sqrt{2}h\sigma}\right) . \int_0^x t^k\left\{\left(\frac{t}{\sigma^2}\right)e^{-t^2/2\sigma^2} . \frac{1}{2}[1+\Phi(ht)]\right\}dt \\ &= \left(\frac{\sqrt{1+2h^2\sigma^2}}{\sqrt{1+2h^2\sigma^2}+\sqrt{2}h\sigma}\right) . \left[\int_0^x t^k\left(\frac{t}{\sigma^2}\right)e^{-t^2/2\sigma^2}dt + \int_0^x t^k\left(\frac{t}{\sigma^2}\right)e^{-t^2/2\sigma^2} . \Phi(ht)\,dt\right]. \end{aligned} \tag{2.26}$$

Now, by substituting $\frac{t^2}{2\sigma^2} = u$ in the first integral of the above Equation (2.26), simplifying, and then using Lemma A.1.2, we have

$$\int_0^x t^k\left(\frac{t}{\sigma^2}\right)e^{-t^2/2\sigma^2}dt = 2^{\frac{k}{2}}\sigma^k\gamma\left(\frac{k+2}{2}, \frac{x^2}{2\sigma^2}\right). \tag{2.27}$$

where γ (.) denotes the incomplete gamma function. Furthermore, using Lemma A.1.6 in the second integral of the above Equation (2.26) and simplifying, we have

$$\begin{aligned} \int_0^x t^k\left(\frac{t}{\sigma^2}\right)e^{-t^2/2\sigma^2} . \Phi(ht)\,dt &= \left(\frac{1}{\sigma^2}\right)\int_0^x t^{k+1}e^{-t^2/2\sigma^2} . \Phi(ht)\,dt \\ &= -x^k e^{-x^2/2\sigma^2}\Phi(hx) + \frac{2h}{\sqrt{\pi}}\int_0^x t^k e^{-\left(h^2+\frac{1}{2\sigma^2}\right)t^2}dt. \end{aligned} \tag{2.28}$$

Now, by substituting $t^2 = u$ in the integral of the above Equation (2.28), simplifying, and then using Lemma A.1.2, we have

$$\begin{aligned} \int_0^x t^k\left(\frac{t}{\sigma^2}\right)e^{-t^2/2\sigma^2} . \Phi(ht)\,dt = -x^k e^{-x^2/2\sigma^2}\Phi(hx) + \frac{h}{\sqrt{\pi}}\left(h^2+\frac{1}{2\sigma^2}\right)^{-\left(\frac{k+1}{2}\right)} \\ \gamma\left(\frac{k+1}{2}, \left(h^2+\frac{1}{2\sigma^2}\right)x^2\right). \end{aligned} \tag{2.29}$$

where Φ (.) denotes the error function, and γ (.) denotes the incomplete gamma function. Using (2.27) and (2.29) in Equation (2.26), the proof of Theorem 5 easily follows.

2.4.3. Entropy

For a continuous real random variable X, we define it as follows:

$$H[X] = E[-\ln(f_X(X)] = -\int_S f_X(x)\ \ln\ [f_X(x)]\,dx,$$

where $S = \{x :\ f_X(x) > 0\}$, [23]. Using this we obtain the following result:
Theorem 6: The entropy of the random variable X having the pdf (2.10) is given by

$$H[X] = -\ln\left(\frac{C}{2\sigma^2}\right) - E(\ln(X)) + \frac{1}{2\sigma^2}E\left(X^2\right) - E[\ln(1+\Phi(hx))],$$

where C is the normalizing constant given by (2.5), andΦ (.) denotes the error function.

Proof: We have

$$H[X] = E[-\ln(p_X(X))] = -\int_0^\infty p_X(x)\ \ln\ [p_X(x)]\ dx$$
$$= -\int_0^\infty \left(\frac{C}{2\sigma^2}\right) x\, e^{-x^2/2\sigma^2}\,.\ [1 + \Phi\,(h\,x)] \times \ln\left\{\left(\frac{C}{2\sigma^2}\right) x\, e^{-x^2/2\sigma^2}\,.\ [1 + \Phi\,(h\,x)]\right\} dx, \tag{2.30}$$

which, using (2.10) and moments, easily reduces to

$$H[X] = -\ln\left(\frac{C}{2\sigma^2}\right) - E\,(\ln(X)) + \frac{1}{2\sigma^2} E\left(X^2\right) - E\,[\ln(1 + \Phi\,(h\,x))], \tag{2.31}$$

where C is the normalizing constant given by (2.5), $E\left(X^2\right)$ is given by (2.31) when $k = 2$, $E\,[\ln(1 + \Phi\,(h\,x))]$ cannot be evaluated analytically in closed forms and so requires some quadrature formulas for computations, and $E\,(\ln(X))$ is derived as follows:

$$E\,(\ln(X)) = \int_0^\infty \left(\frac{C}{2\sigma^2}\right) x\, e^{-x^2/2\sigma^2}\,.\ [1 + \Phi\,(h\,x)]\,.\ln(x)\ dx$$
$$= \left(\frac{C}{2\sigma^2}\right)\left[\int_0^\infty x\, e^{-x^2/2\sigma^2}.\ \ln(x)\ dx + \int_0^\infty x\, e^{-x^2/2\sigma^2}.\ \Phi\,(h\,x)\,.\ln(x)\ dx\right],$$

which, on substituting $x^2 = u$, simplifying, and then using Lemmas A1.7, A.1.8 and A.1.9, reduces to

$$E\,(\ln(X)) = \left(\frac{C}{2\sigma^2}\right)\left[\left(\frac{\sigma^2}{2}\right)\left\{\psi(1) + \ln\left(2\sigma^2\right)\right\} + \left(\frac{1}{2\sqrt{\pi}}\right)\sum_{k=0}^{\infty}\frac{2^k h^{2k+1}}{(2k+1)\,!!}\left(\frac{2\sigma^2}{1 + 2h^2\sigma^2}\right)^{\frac{2k+3}{2}}\right.$$
$$\left.\Gamma\left(\frac{2k+3}{2}\right)\left\{\psi\left(\frac{2k+3}{2}\right) - \ln\left(\frac{1 + 2h^2\sigma^2}{2\sigma^2}\right)\right\}\right],$$

where C is the normalizing constant given by (2.5), and $\Gamma(.)$ and $\psi(.)$ denote the gamma and psi functions, respectively, (cf. [[24], 8.31, p. 933, and 8.36, p. 943]). Also, see [35]. This completes the proof of Theorem 6.

2.4.4. Percentile Points

The percentage points x_p of our proposed PREF distribution are computed by numerically solving the equation $F(x_p) = \int_\lambda^{x_p} f_X(u)\,du = p$ (say), for any $0 < p < 1$, for different sets of values of the parameters, which are provided in Table 1 as follows:

3. Characterizations

A probability distribution can be characterized through various methods [25], [26], [27], [28], [29], [30], and [31]. In this section, we provide the characterization of our proposed product distribution by the left and right truncated moment methods in Theorems 3.1 and 3.2. For this, we will need the following assumption and lemmas.

Table 1. Percentile Points of PREF Distribution

Percentiles p		0.75	0.80	0.85	0.90	0.95	0.99
$h = 0.25, \sigma = 2$	x_p	2.93928	3.26727	3.70443	4.34327	5.48245	8.27443
$h = 0.5, \sigma = 2$	x_p	3.64562	4.28976	5.17756	6.51398	8.96187	15.10587
$h = 1.25, \sigma = 2$	x_p	4.06721	4.91374	6.09339	7.88452	11.18757	19.51993
$h = 2, \sigma = 2$	x_p	4.53702	5.61670	7.13374	9.45117	13.74370	24.60521
$\sigma = 0.5, h = 2$	x_p	1.43344	1.48370	1.54484	1.62580	1.75436	2.02480
$\sigma = 1, h = 2$	x_p	1.68344	1.73370	1.79484	1.87580	2.00436	2.27480
$\sigma = 2, h = 2$	x_p	1.93344	1.98370	2.04484	2.12580	2.25436	2.52480
$\sigma = 3, h = 2$	x_p	2.18343	2.23370	2.29484	2.37580	2.50436	2.77480

Assumptions 3.1. Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = \inf\{x \mid F(x) > 0\}$, and $\delta = \sup\{x \mid F(x) < 1\}$. We also assume that $f(x)$ is a differentiable for all x, and$E(X)$ exists.

Lemma 3.1. Under the Assumption 3.1, if $E(X|X \leq x) = g(x)\,\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_0^x \frac{u - g^{/}(u)}{g(u)} du$ is finite for $x > 0$, then $f(x) = c\, e^{\int_0^x \frac{u - g^{/}(u)}{g(u)} du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof. For proof, see [32].

Lemma 3.2. Under the Assumption 3.1, if $E(X \mid X \geq x) = \tilde{g}(x)\, r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and $\tilde{g}(x)$ is a continuous differentiable function of x with the condition that $\int_x^\infty \frac{u + \left[\tilde{g}(u)\right]^{/}}{\tilde{g}(u)} du$ is finite for $x > 0$, then $f(x) = c\, e^{-\int_0^x \frac{u + \left[\tilde{g}(u)\right]^{/}}{\tilde{g}(u)} du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof. For proof, see [32].

Theorem 3.1: Suppose the random variable X is absolutely continuous with the cumulative distribution function $P(x)$, given by (2.12), and the probability density function $p(x)$, given by (2.10). We assume that $\omega = \inf\{x \mid P(x) > 0\}$, and $\delta = \sup\{x \mid P(x) < 1\}$. We also assume that $p(x)$ is a differentiable for all x, and$E(X)$ exists. We assume that $\omega = \lambda$ and $\delta = \infty$, where $0 < \lambda < +\infty$. Then, $E(X|X \leq x) = g(x)\,\eta(x)$,where $\eta(x) = \frac{p(x)}{P(x)}$, and

$$g(x) = \frac{\left[\sqrt{2}\,\sigma\,\gamma\left(\frac{3}{2}, \frac{x^2}{2\sigma^2}\right) - x\,e^{-x^2/2\sigma^2}\,\Phi(h\,x) + \frac{h}{\sqrt{\pi}}\left(\frac{2\sigma^2}{2\sigma^2 h^2 + 1}\right)\gamma\left(1, \left(h^2 + \frac{1}{2\sigma^2}\right)x^2\right)\right]}{\left(\frac{x}{\sigma^2}\right)e^{-x^2/2\sigma^2}\,.\,[1 + \Phi(h\,x)]}, \tag{3.1}$$

if and only if X has the pdf

$$p(x) = \frac{\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}\,h\,\sigma}\cdot\left(\frac{x}{\sigma^2}\right)e^{-x^2/2\sigma^2}\,.\,[1 + \Phi(h\,x)], \quad h \geq 0, \sigma > 0. \tag{3.2}$$

Proof: Following [32], the proof easily follows.

Theorem 3.2: Suppose the random variable X is absolutely continuous with the cumulative distribution function $P(x)$, given by (2.12), and the probability density function $p(x)$, given by (2.10). We assume that $\omega = \inf\{x \mid P(x) > 0\}$, and $\delta = \sup\{x \mid P(x) < 1\}$. We also assume that $p(x)$ is a differentiable for all x, and$E(X)$ exists. We assume that $\omega = \lambda$ and $\delta = \infty$, where $0 < \lambda < +\infty$. Then $E(X|X \geq x) = r(x)\,\frac{p(x)}{1-P(x)}$, where

$$r(x) = \frac{(E(X) - g(x)\,p(x))}{p(x)}$$

, $g(x)$ being given by $E(X)$ being given by the Equation (2.24), if and only if X has the pdf

$$p(x) = \frac{\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}h\sigma} \cdot \left(\frac{x}{\sigma^2}\right) e^{-x^2/2\sigma^2} \cdot [1 + \Phi(hx)], \quad h \geq 0, \sigma > 0.$$

Proof. Following [32], the proof easily follows.

4. Estimation of Parameters, Applications and Simulation

In this section, we provide the estimation of the parameters of the PREF distribution.

4.1. The Method of Moments:

If $\{X_i\}_{i=1}^n$ be an *iid*sample from a distribution with a m-dimensional parameter vector ϕ, then, according to the method of moment (MOM), the estimator $\tilde{\phi}$ is the solution of the following system of equations:

$$E_{\tilde{\phi}}\left(X^j\right) = \frac{\sum_{i=1}^n X_i^j}{n}, \quad j = 1, 2, 3, \ldots, m. \tag{4.1}$$

Thus, using the above-mentioned definition (4.1) of the method of moments (MOM), we can obtain the respective moments from the Equation (2.20) of the *jth*moment, $E\left(X^j\right)$, of our proposed product distribution by taking the respective values of j, $j = 1, 2$, that is, $E(X)$, see Equation (2.24), and $E\left(X^2\right)$, and evaluating the respective expressions of the respective moments numerically. Then, the moment estimations of the respective parameters of our proposed product distribution can be determined by solving the system of respective equations thus obtained by Newton-Raphson's iteration method, and using some computer packages like Maple, or Mathematica, or R, or MathCAD, or other software.

4.2. The Method of Maximum Likelihood

The estimation of the parameters of our proposed PREF distribution is carried out by using the method of maximum likelihood (MLE). Given a sample $\{x_i\}$, $i = 1, 2, 3, \ldots, n$, the likelihood function of the PREF distribution pdf (2.10) is given by $L = \prod_{i=1}^n p(x_i)$, that is, .

$$L = \prod_{i=1}^{n} \frac{\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}h\sigma} \cdot \left(\frac{x_i}{\sigma^2}\right) e^{-x_i^2/2\sigma^2} \cdot [1 + \Phi(hx_i)]. \tag{4.2}$$

Table 2. Descriptive Summary of the Data Sets

Data Sets	Sample Size	Mean	Median	Standard Deviation	Skewness	Kurtosis	Skewness Kurtosis
I	50	12.3200	10.5000	6.0826	1.0511	4.0556	0.2592
II	102	2.3990	2.4000	0.7767	0.2749	3.5998	0.0763

The objective of the likelihood function approach is to determine those values of the parameters that maximize the function L given by (4.2). Suppose its log-likelihood function is given by

$$R = \ln(L) = \sum_{i=1}^{n} \ln\left[\frac{\sqrt{1 + 2h^2\sigma^2}}{\sqrt{1 + 2h^2\sigma^2} + \sqrt{2}h\sigma} \cdot \left(\frac{1}{\sigma^2}\right)\right] - \sum_{i=1}^{n} \frac{x_i^2}{2\sigma^2} + \sum_{i=1}^{n} \ln[(1 + \Phi(h x_i))] + \sum_{i=1}^{n} \ln(x_i) \tag{4.3}$$

Differentiating the Equation (4.3) partially with respect to the respective parameters, h and σ, respectively, the maximum likelihood system of equations will be given by

$$\left.\begin{array}{l} \frac{\partial R}{\partial h} = 0, \\ \frac{\partial R}{\partial \sigma} = 0 \end{array}\right]. \tag{4.4}$$

The maximum likelihood estimates (MLE) of the parameters $\{h, \sigma\}$ can be obtained by solving the maximum likelihood system of equations (4.4) numerically by Newton-Raphson's iteration method using some computer software like Maple, or Mathematica, or R, or MathCAD, or other software.

4.3. Applications

In this sub-section, by considering four real-life data sets, the goodness of fit tests of the PREF distribution is provided by comparing it with some well-known skew distributions.

Dataset I [[33], p. 31]: The following data represent the length of life, in seconds, of 50 fruit flies subject to a new spray in a controlled laboratory experiment with the observations: 17, 20, 10, 9, 23, 13, 12, 19, 18, 24, 12, 14, 6, 9, 13, 6, 7, 10, 13, 7, 16, 18, 8, 13, 3, 32, 9, 7, 10, 11, 13, 7, 18, 7, 10, 4, 27, 19, 16, 8, 7, 10, 5, 14, 15, 10, 9, 6, 7, 15.

Dataset II (Source: is https: //www.emsc − csem.org/Earthquake/Europe): The following data sets show the 102 Magnitudes of the latest Earthquakes in Euro-Mediterranean region. The magnitudes are 2.1, 1.9, 2.3, 1.8, 2., 1.6, 1.5, 2.8, 2.9, 1.5, 2.2, 3.4, 2.8, 3.1, 1.6, 0.6, 0.5, 1.7, 2.0, 0.8, 2.9, 4.7, 1.6, 1.6, 2.7, 1.7, 1.5, 4.3, 3.0, 2.6, 3.3, 2.6, 2.6, 3.1, 1.3, 2.8, 2.8, 0.7, 1.2, 1.9, 2.6, 2.0, 2.7, 2., 1.9, 3.0, 2.7, 2.8, 2.4, 2.4, 3.1, 3.1, 2.0, 1.8, 2.1, 1.8, 2.9, 3.2, 2.3, 2., 2., 2.9, 2.1, 2.5, 1.8, 1.8, 1.9, 3.1, 2.7, 3.7, 4.1, 2.4, 1.6, 2.9, 2.5, 2.2, 2.3, 2.1, 2.4, 2.7, 2.7, 2.3, 1.9, 2.3, 2.2, 4.2, 2.1, 2.1, 2.0, 2.6, 3.3, 4.0, 3.1, 2.8, 2.5, 3.1, 1.8, 2.9, 1.2, 4.0, 2.6, 2.5

The descriptive and theoretical statistics of datasets are portrayed in Table 2 and Table 3, respectively.

Analytical Discussion About Data Set I: Descriptive features like sample size, mean, median, standard deviation, skewness, kurtosis, and the ratio of Skewness/ Kurtosis, which reveals close coordination between descriptive and theoretical results as given in Tables 2 and 3, respectively. The

Table 3. Theoretical Summary of the Data Sets

Data Sets	SampleSize	Mean	Median	StandardDeviation	Skewness	Kurtosis	Skewness /Kurtosis
I	50	9.2342	8.4130	4.4156	0.1145	3.3659	0.0340
II	102	1.9526	1.0567	0.5433	0.1047	2.3547	0.0445

Table 4. Goodness-of-Fit measure based on listed MLEs for the Data Set I

Distribution	$\hat{h}$	$\hat{\sigma}$	X^2	A_0^*	W_0^*	KS	$P-Value$
Proposed	0.54412	9.9204	4.5308(5)	0.8023	0.1199	0.1355	0.8135
Rayleigh	9.6963	-	5.028(6)	1.1004	0.1641	0.1527	0.6833
Maxwell	7.9171	-	6.9284(6)	1.6024	0.1680	0.1610	0.6183
Weibull	2.1793	13.9628	4.693(5)	1.1234	0.1437	0.1486	0.7155
Lindley	0.15164	-	6.1583(5)	1.4152	0.2692	0.1745	0.5137
Frechet	1.9857	8.5541	6.4047(5)	1.4579	0.2354	0.19234	0.3897

proposed model has a minimum value of χ^2 goodness of-fit statistic compared to the rest of the model's values. The proposed model has also, the least value of A_0^* , W_0^* and , KS statistic and highest p-value (See Table 4) with minimum loss of information criterion (see Table 5) supports the suitability of the suggested model. In addition, to draw a valid conclusion for the Data Set I, we have grouped the observation by using R as [3, 7], (7, 8], (8, 10], (10, 13], (13, 15], (15, 18], (18, 32] and the frequencies are 13, 2, 10, 8, 4, 6, 7, respectively. Moreover, Tables 3 and 4 shows that the developed model is the most suitable one, with the least values for all statistics and the highest p-value for χ2.

The following histogram (Figure 7) also supports the suitability of the suggested model (PREF) for Data Set I.

Analytical Discussion about Data Set II: For Data Set II to draw a valid conclusion, we have grouped the observation by using R as [0.5, 1.6],(1.6, 1.8], (1.8, 2], (2, 2.1],(2.1, 2.4], (2.4, 2.6], (2.6, 2.8], (2.8, 2.9], (2.9, 3.19], (3.19, 4.7] and the frequencies are 15, 8 ,13, 6, 12, 10, 12, 6, 9, 11, respectively. Moreover, the results shown in Tables 6 and 7 confirm the authenticity of the proposed model (PREF).

The following histogram (Figure 8) also supports the suitability of the suggested model (PREF) for Data Set II.

Table 5. Information Criterion for the Data Set I (Model's Information Adaptability in Data Set I)

Distribution	$-l$	AIC	AICC	BIC	HQIC	CAIC
Proposed	127.594	259.188	259.443	263.012	257.916	259.443
Rayleigh	157.425	316.85	316.934	318.762	317.578	316.934
Maxwell	157.142	316.285	316.368	318.197	317.013	316.368
Weibull	157.104	318.208	318.463	322.032	316.936	318.463
Lindley	164.498	330.995	331.079	332.907	331.723	331.079
Frechet	160.132	324.264	324.519	328.088	322.992	324.519

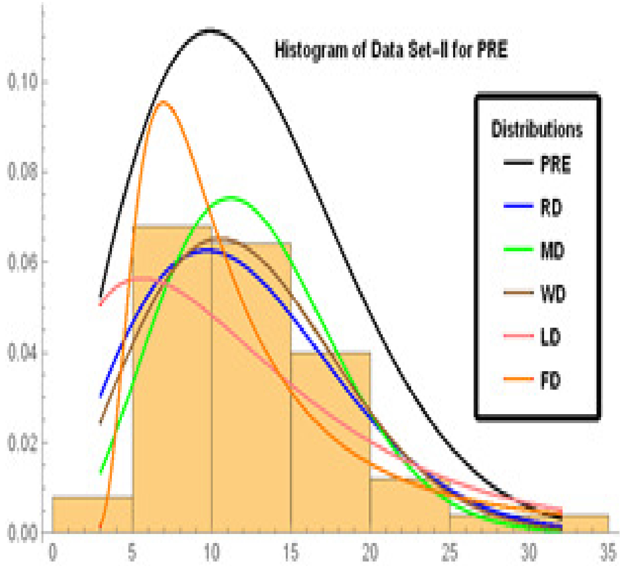

Figure 7. Histogram of Data Sets I for PREF

Table 6. Goodness-of-Fit measure based on listed MLEs for the Data Set II

Distribution	$\hat{h}$	$\hat{\sigma}$	X^2	A_0^*	W_0^*	KS	$P-Value$
Proposed	0.8315	1.9162	12.618(7)	0.2405	0.02897	0.06149	0.9997
Rayleigh	1.7822	-	31.910(7)	1.0661	0.2099	0.1276	0.6742
Maxwell	1.4551	-	17.132(7)	0.9788	0.1301	0.1108	0.8269
Weibull	3.3226	2.6668	12.420(7)	2.3865	0.1746	0.1361	0.5937
Lindley	0.6669	-	80.091(4)	2.7892	0.5074	0.2241	0.0805
Frechet	1.9731	1.8284	40.038(7)	2.5091	0.1362	0.1438	0.5226

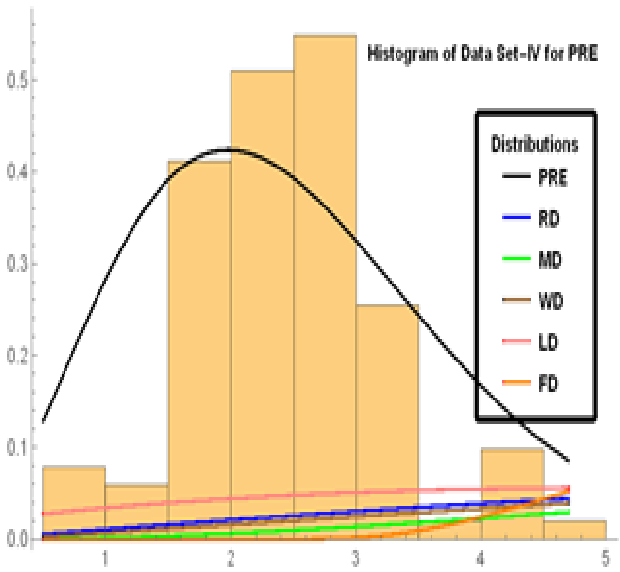

Figure 8. Histogram of Data Sets II for PREF

Table 7. Information Criterion for the Data Set II (Model's Information Adaptability in Data Set II)

Distribution	$-l$	AIC	AICC	BIC	HQIC	CAIC
Proposed	108.919	221.839	221.960	227.089	220.902	221.960
Rayleigh	136.900	275.799	275.839	278.424	276.862	275.839
Maxwell	124.852	251.704	251.744	254.329	252.767	251.744
Weibull	119.214	242.428	242.549	247.677	241.490	242.549
Lindley	175.945	353.889	353.929	356.514	354.952	353.929
Frechet	157.949	319.899	320.020	325.149	318.962	320.020

4.4. Simulation Study

Simulation studies with 1000 replications are carried out in this section. Stochastic variate X following PRE (h, σ) is generated using the following four sets of parameters. Set-I, h = 0.9856 and σ = 0.2178; Set-II, h = 0.2256 and σ = 2.4698, Set-III, h = 0.1156 and σ = 5.2658 and Set-IV, h = 1.2156 and σ = 4.1658. We use samples of sizes n = 15, 25, 50, 100, 150, and 250 and estimated the MLEs by using Mathematica 8.0. As shown in Table 8, both bias and MSE of the MLEs tend to zero as sample size increases.

5. Conclusion

In this paper, we have derived the exact distribution of the product of the probability density function of Rayleigh distribution and cumulative distribution of the error function for some

continuous random variable X, which we called as the PREF distribution. Various characteristics of the PREF distribution are presented. We have used two environmental data sets to illustrate the applications of proposed distribution, which model skewed distribution effectively. Thus, by acknowledging the limitations and leveraging the applications of the proposed PREF distribution, researchers and practitioners can better understand and analyse complex phenomena in various fields, deriving informed decision-making and innovation.

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APPENDIX A

A.1 Some Useful Lemmas

The following Lemmas have been used to complete the derivations.

Lemma A.1.1 [[24], 6.283.2, p. 649].

Table 8. Simulation Study

Parameter	n	Bias (h)	Bias (σ)	MSE (h)	MSE (σ)
Set-I	n = 15	0.000012500	0.075302000	0.000264000	0.002977000
	n = 25	0.000003120	0.056778100	0.000139000	0.001368000
	n = 50	0.000001950	0.031789300	0.000069000	0.000328000
	n = 100	0.000001610	0.001336800	0.000013000	0.000169000
	n = 150	0.000000930	0.000163100	0.000000000	0.000043000
	n = 250	0.000000100	-0.007214000	0.000000000	0.000023000
Set-II	n = 15	0.000005680	0.184670000	0.000000130	0.084300000
	n = 25	0.000004900	0.091150000	0.000000090	0.077160000
	n = 50	0.000002610	0.004560000	0.000000060	0.071212000
	n = 100	0.000001300	0.000315000	0.000000020	0.043046000
	n = 150	0.000000000	0.000011300	0.000000000	0.004741000
	n = 250	0.000000000	0.000010000	0.000000000	0.001121000
Set-III	n = 15	0.000000000	0.018467000	0.000000050	0.042668060
	n = 25	0.000000000	0.011150000	0.000000030	0.031651600
	n = 50	0.000000000	0.004560000	0.000000020	0.015902120
	n = 100	0.000000000	0.002315000	0.000000010	0.004210000
	n = 150	0.000000000	0.000113000	0.000000000	0.002136000
	n = 250	0.000000000	0.000010000	0.000000000	0.000101000
Set-IV	n = 15	0.000000000	0.000000760	0.000000000	0.077860000
	n = 25	0.000000000	0.000000680	0.000000000	0.063723000
	n = 50	0.000000000	0.000000490	0.000000000	0.051441000
	n = 100	0.000000000	0.000000340	0.000000000	0.001456000
	n = 150	0.000000000	0.000000100	0.000000000	0.002347000
	n = 250	0.000000000	0.000000000	0.000000000	0.004202000

For $Re\ (p) > 0,\ Re\ (q + p) > 0,\ \int_0^\infty \Phi\left(\sqrt{q\,t}\right)\ e^{-p\,t} dt = \frac{\sqrt{q}}{p}\frac{1}{\sqrt{p+q}}$, where Φ (.) denotes the error function.

Lemma A.1.2 [[24], 3.381.1, p. 317].

For $\nu > 0,\ \int_0^u x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu}\gamma\ (\nu,\ \mu u)$, where γ (.) denotes the incomplete gamma function.

Lemma A.1.3 [[24], 8.352.1, p. 940].For $n = 0, 1, \ldots,\ \gamma\ (1 + n,\ z) = (n!)\left[1 - e^{-z}\left(\sum_{m=0}^{n}\frac{z^m}{m!}\right)\right]$.

Lemma A.1.4 [[34], Vol. 2, 1.5.3.9, p. 32].

$$\int x\, e^{-b^2 x^2}\,\Phi\ (a\,x)\ dx = \frac{a}{2\,b^2\sqrt{a^2 + b^2}}\Phi\left(\sqrt{a^2 + b^2}\ x\right) - \frac{e^{-b^2 x^2}}{2\,b^2}\Phi\ (a\,x)\,,$$

where Φ (.) denotes the error function.

Lemma A.1.5 [[34], Vol. 2, 2.8.1.5, p. 93]. For $r > 0,\ Re\ (p) > 0,\ \left|\arg\ (c)\right| < \frac{\pi}{4},\ Re\ (\alpha) > -\frac{1}{2}$, $\int_0^\infty x^{\alpha-1} e^{-p\,x^r}\,\Phi\ (c\,x)\ dx = \left[\frac{c}{\sqrt{\pi}\,p^{\frac{(\alpha+1)}{r}}\,r}\right]\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(k+\frac{1}{2}\right)(k!)}\Gamma\left(\frac{2k+\alpha+1}{r}\right)\left(\frac{c}{p^{\frac{1}{r}}}\right)^{2k}$,

where Φ (.) denotes the error function.

Lemma A.1.6 [[34], Vol. 2, 1.5.3.1, p. 31].

$$\int x^{\lambda} e^{-b^2 x^2} \Phi(a x)\, dx = -\frac{1}{2 b^2} x^{\lambda-1} e^{-b^2 x^2} \Phi(a x)$$

$$+ \frac{a}{b^2 \sqrt{\pi}} \int x^{\lambda-1} e^{-(a^2+b^2) x^2} dx + \frac{\lambda - 1}{2 b^2} \int x^{\lambda-2} e^{-b^2 x^2} \Phi(a x)\, dx,$$

where $\Phi(.)$ denotes the error function.

Lemma A.1.7 [[24], 4.331.1, p. 573, and 8.367.1, p. 946]. $\int_0^{\infty} e^{-\mu x} \ln(x)\, dx = \frac{1}{\mu}[\psi(1) - \ln(\mu)]$, $[Re\, \mu > 0]$,where $\psi(1) \approx -0.577216$ denotes Euler's constant.

Lemma A.1.8 [[24], 8.253.1, p. 931]. Series representation of the error function $\Phi(.)$: $\Phi(x) = \left(\frac{2}{\sqrt{\pi}}\right) e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!!}$.

Lemma A.1.9 [[24], 4.352.1, p. 576]. $\int_0^{\infty} x^{\nu-1} e^{-\mu x} \ln(x)\, dx = \frac{1}{\mu^{\nu}} \Gamma(\nu)[\psi(\nu) - \ln(\mu)]$, $[Re\, \mu > 0,\ Re\, \nu > 0]$.

References

1. Patel, J.K., Kapadia, C. H., and Owen, D. B. (1976). Handbook of Statistical Distributions, Marcel Dekker, New York, USA.
2. Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). Continuous Univariate Distributions, Volumes 1 – 2, 6th Edition, John Wiley & Sons, Inc., New York, USA.
3. Balakrishnan, N., and Nevzorov, V. B. (2003). A Primer on Statistical Distributions, John Wiley & Sons, Inc., New York, USA.
4. Kotz, S., and Vicari, D. (2005). Survey of developments in the theory of continuous skewed distributions. METRON - International Journal of Statistics, vol. LXIII, n. 2, pp. 225 – 261.
5. Forbes, C., Evans, M., Hastings, N., and Peacock, B. (2010). Statistical Distributions, 4th Edition, John Wiley & Sons, Inc., New York, USA.
6. Azzalini, A., and Regoli, G. (2012). Some properties of skew-symmetric distributions. Annals of the Institute of Statistical Mathematics, 64, 857 – 879. https://doi.org/10.1007/s10463-011-0338-5.
7. Tahir, M. H., and Nadarajah, S. (2015). Parameter induction in continuous univariate distributions: Well-established G families. Anais da Academia Brasileira de Ciências, 87 (2), 539 - 568. DOI: 10.1590/0001-3765201520140299
8. Tahir, M., and Cordeiro, G. M. (2016). Compounding of distributions: a survey and new generalized classes. Journal of Statistical Distributions and Applications, 3 (13). DOI: 10.1186/s40488-016-0052-1.
9. Azzalini, A., and Capitanio, A. (2018). The Skew-Normal and Related Families, Cambridge University Press, Cambridge, U.K.
10. de Brito, C., Rego, L., de Oliveira, W., and Gomes-Silva, F. (2019). Method for Generating Distributions and Classes of Probability Distributions: The Univariate Case. Hacettepe University Bulletin of Natural Sciences and Engineering Series B: Mathematics and Statistics, 48(2), 897–930. DOI: 10.15672/HJMS.2018.619.

11. Ghosh, I., and Ng, H. K. T. (2019). A class of skewed distributions with applications in environmental data. Communications in Statistics: Case Studies, Data Analysis and Applications. 5 (4), 346 – 365. DOI: 10.1080/23737484.2019.1648194.
12. Ferreira, J. T. A. S., and Steel, M. F. J. (2006). A constructive representation of univariate skewed distributions. Journal of the American Statistical Association, 101, 823–829.
13. Azzalini, A. (2022). An overview on the progeny of the skew-normal family—A personal perspective, Journal of Multivariate Analysis, Volume 188, 104851, https://doi.org/10.1016/j.jmva.2021.104851.
14. Azzalini, A. (1985). A Class of Distributions Which Includes the Normal Ones. Scandinavian Journal of Statistics, 12 (2), 171–178.
15. Azzalini, A. (1986). Further results on a class of distributions which includes the normal ones. Statistica XLVI, 199–208.
16. Azzalini, A., and Dalla Valle, A. (1996). The multivariate skew-normal distribution. Biometrika, 83, 715–726.
17. Azzalini, A., and Capitanio, A. (1999). Statistical applications of the multivariate skew normal distribution. Journal of the Royal Statistical Society, Series B, 61, 579–602.
18. Nadarajah, S., and Kotz, S. (2006). Skew distributions generated from different families. Acta Applicandae Mathematica, 91 (1), 1–37.
19. Nadarajah, S., and Kotz, S. (2003). Skewed distributions generated by the normal kernel. Statistics and Probability Letters, 65, 269–277.
20. Sharafi, M., and Behboodian, J. (2006). A new skew-normal density. Journal of Statistical Research of Iran, 3, 47–61.
21. Nadarajah, S., Srivastava, H. M., and Gupta, A. K. (2007). Skewed Bessel function distributions with application to rainfall data, Statistics: A Journal of Theoretical and Applied Statistics, 41 (4), 333 – 344. DOI: 10.1080/02331880701270606.
22. Sharafi, M., and Behboodian, J. (2006). A new skew-normal density. Journal of Statistical Research of Iran, 3, 47–61.
23. Shannon, C. E. (1948). A mathematical theory of communication. Bell System Tech. J., Vol. 27, 379 – 423; 623 – 656.
24. Gradshteyn, I. S. and Ryzhik, I. M. (2000). Table of Integrals, Series, and Products (6th edition). Academic Press, San Diego, USA.
25. Galambos, J. and Kotz, S. (1978). Characterizations of probability distributions. A unified approach with an emphasis on exponential and related 18 models, Lecture Notes in Mathematics, 675. Berlin, Germany: Springer. doi: 19 10.1007/BFb0069530.
26. Kotz, S., and Shanbhag, D. N. (1980). Some new approaches to probability distributions. Advances in Applied Probability, 12(4), 903-921.
27. Glänzel, W., Telcs, A., and Schubert, A. (1984). Characterization by truncated moments and its application to Pearson-type distributions. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 66(2), 173-183. doi: 1 10.1007/BF00531527.

28. Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. In P. Bauer, F. 22 Konecny, & W. Wertz (Eds.), Mathematical Statistics and Probability Theory 23 (Vol. B) (75-84). Dordrecht, Netherlands: Reidel. doi: 10.1007/978-94-009-3965- 24 3_8.

29. Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments. Statistics, 21(4), 613-618. doi: 27 10.1080/02331889008802273.

30. Ahsanullah, M., Kibria, B. M. G., and Shakil, M. (2014). Normal and Student´s t Distributions and Their Applications, Atlantis Press, France.

31. Ahsanullah, M. (2017). Characterizations of Univariate Continuous Distributions, Atlantis Press, Paris, France.

32. Shakil, M., Ahsanullah, M., and Kibria, B. G. (2018). On the Characterizations of Chen's Two-Parameter Exponential Power Life-Testing Distribution, Journal of Statistical Theory and Applications, 17 (3), 393 - 407.

33. Walpole, R. E., Myers, R. H., Myers, S. L., and Ye, K. (2012). Probability & Statistics for Engineers & Scientists, 9th Edition, Page 31, Prentice Hall, New York, USA.

34. Prudnikov, A. P., Brychkov, Y. A., and Marichev, O. I. (1986). Integrals and Series, Volumes 1, 2, and 3, Gordon and Breach Science Publishers, Amsterdam, Netherlands.

35. Abramowitz, M., and Stegun, I. A. (1970). Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables. Dover, New York, USA.

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