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# A New Extention of the Odd Inverse Weibull-G Family of Distributions: Bayesian and Non-Bayesian Estimation with Engineering Applications

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**Abstract:** In this work, we propose a novel generator called the "extended odd inverse Weibull-generator" to obtain better distribution flexibility. This generator is considered as a generalization of the three well-known families. In comparison to the baseline model, the newly formed family may offer more efficient continuous symmetric and asymmetric models. The statistical features of the proposed family are analyzed, including quantile function, moments, incomplete moments, mean deviation, density function expansion, order statistics and entropy measures. The generated family is used to give a number of well-known models as special cases. We use the Weibull distribution as a baseline model and fully study a five-parameter special member of the extended odd inverse Weibull-G family. In order to evaluate the behavior of the parameter estimates, the point and interval estimation parameters are examined using both Bayesian and non-Bayesian methods. In both symmetric and asymmetric loss functions, Bayes estimates are computed using the Markov Chain-Monte-Carlo method. The effectiveness of the suggested estimators is evaluated using Monte Carlo simulations and certain criterion metrics. Two time between failure data sets are considered in order to apply the extended odd inverse Weibull Weibull distribution. We have demonstrated that, utilizing specific criteria for model selection and goodness of fit test statistics, the suggested model performs better than six other existing models.

**Keywords:** Inverse Weibull-G family; Bayesian estimation; entropy; moments; maximum likelihood method.
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## 1. Introduction

Statistical evaluation of lifetime data is unpredictable in applied sciences such as engineering, biology, medicine, environmental science, finance, and actuarial science, among others, and statistical modelling is the best and most efficient method to look into the ambiguity of any occurrence. Life time data play a crucial role in industries like insurance and finance because of the complicated nature and unique qualities. Therefore, it would seem that the classic statistical distributions that are now in use might benefit from enlargement and change. In fact, several efforts have been undertaken to create new classes of lifetime distributions in order to broaden existing families of distributions and provide the innovative model with greater flexibility. Over the past several years, numerous researchers have contributed additional classes of life time distributions, which are now available in the statistical literature. To do this, we advise the readers to look at these families, for example beta-G [1], the exponentiated generalized-G [2], the transformed-transformer (T-X) [3], the Weibull-G [4], Kumaraswamy Weibull-G [5], type II half logistic-G [6], odd Lindley-G [7], Topp-Leone odd log-logistic-G [8], odd Fréchet-G [9], extended odd Fréchet-G [10], Type II Fréchet-G [11], inverse Weibull-G [12], modified Weibull-G [13], odd Lomax-G [14], power Lindley- G [15], transmuted odd Fréchet-G [16], modified odd Weibull-G [17], new Kumaraswamy-G [18], new Marshall-Olkin extended-G [19], Type I half logistic Burr X-G [20], Gompertz-G [21], Teissier-G [22], odd inverted Topp-Leone-G [23], odd Chen-G [24], transmuted modified power-G [25], X-Gamma Lomax [26], new modified sine-G [27], Topp-Leone Weibull G family [28], and mixture exponentialted-G [29] among others.

There are a variety of lifetime distributions used in model construction and real-world data analysis, for which Ref. [30] and Ref. [31] illustrated the fundamental principles and concepts. The inverse Weibull (IW) distribution is one of well-known models, and it has many uses in reliability engineering, where Ref. [32], and Ref. [33] studied the various inference techniques in the distribution under discussion. The probability density function (PDF) of the IW distribution, for $y > 0$, is as below:

$$F(y) = e^{-\theta^{\alpha} y^{-\alpha}}, \tag{1.1}$$

where, $\theta > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter. The cumulative distribution function (CDF) related to (1.1), for $y > 0$, is as follows:

$$f(y) = \alpha\theta^{\alpha}\, y^{-\alpha-1} e^{-\theta^{\alpha} y^{-\alpha}}. \tag{1.2}$$

Refrences [34] and [35] provided the theoretical analyses of the IW distribution and order statistics using inference, respectively. Two genuine data sets were studied by Ref. [36], along with applications to the generalized modified Weibull distribution. Reference [37] applied the IW distribution to model the wind speed data. Inferences from the IW distribution in information theory were discussed by Ref. [38]. For recent studies see Refs. [39, 40].

Following the T-X family and utilising the PDF of the IW as the baseline distribution, Ref. [12] suggested the IW-G (IW-G) family with the following CDF and PDF:

$$F(x) = \int_0^{\frac{G(x;\eta)}{1-(G(x;\eta)}} \alpha\theta^{\alpha}\, y^{-\alpha-1} e^{-\theta^{\alpha} y^{-\alpha}}\, dy = e^{-\left[\theta^{\alpha}\left\{\frac{G(x;\eta)}{1-(G(x;\eta))}\right\}^{-\alpha}\right]}; x \in R, \tag{1.3}$$

and,

$$f(x) = \frac{\alpha\theta^{\alpha} g(x;\eta)\,[G(x;\eta)]^{-\alpha-1}}{[1-(G(x;\eta)]^{1-\alpha}} e^{-\left[\theta^{\alpha}\left\{\frac{G(x;\eta)}{1-(G(x;\eta)}\right\}^{-\alpha}\right]}; \quad x \in R, \tag{1.4}$$

where $\theta > 0$ is the scale parameter, $\alpha > 0$ is the shape parameter, $G\,(x;\,\eta)$ and $g\,(x;\eta)$ are the CDF and PDF of a baseline continuous distribution with $\eta$ as parameter vector, respectively.

The paper's major goals of the present work can be summarized as follows:

1. Provide a new and more flexible version of the IW-G family given by Ref. [12] with an additional shape parameter. The new family is called the extended odd IW-G (EOIW-G).
2. The EOIW-G family comprises two new families and generalizes three existing families from References [9], [10], and [12].
3. The PDF of the recommended family's submodels can have the right skewed, unimodal, U-shaped, reversed J-shaped, and symmetrical in shape. Also, possible are uni-modal, U-shaped, J-shaped, rising, decreasing, and constant hazard rate functions (HFs).
4. Explore a few of its statistical characteristics, including the quantile function, moments, incomplete moments, and order statistics.
5. Examine the EOIW-G family's statistical inference using Bayesian and maximum likehood (ML) techniques.
6. Carry out a simulation study to illustrate how the parameters of the model behave.
7. Owing to several beneficial attributes and tenable physical interpretations of the Weibull distribution, the EOIW-Weibull (EOIWW) model is contrasted with a few popular models through the use of goodness of fit tests.
8. To give better fits for the EOIW-Weibull (EOIWW) model than certain well-known models with favourable outcomes.

The EOIW-G family is created by inserting (1.1) into (1.3), where the upper limit is taken as $\frac{G(x;\eta)^{\beta}}{1-G(x;\eta)^{\beta}}$, with the following CDF and PDF

$$F\,(x) = \int_0^{\frac{G(x;\eta)^{\beta}}{1-G(x;\eta)^{\beta}}} \alpha\theta^{\alpha}\, y^{-\alpha-1} e^{-\theta^{\alpha} y^{-\alpha}}\, dy \, = e^{-\left[\theta^{\alpha}\left\{\frac{G(x;\eta)^{\beta}}{1-G(x;\eta)^{\beta}}\right\}^{-\alpha}\right]}; \qquad x \in R, \tag{1.5}$$

and

$$f\,(x) = \frac{\alpha\,\beta\,\theta^{\alpha}\, g\,(x;\;\eta)\left[1 - G(x;\;\eta)^{\beta}\right]^{\alpha-1}}{G\,(x;\;\eta)^{\beta\,\alpha+1}}\; e^{-\left[\theta^{\alpha}\left\{\frac{G(x;\eta)^{\beta}}{1-G(x;\eta)^{\beta}}\right\}^{-\alpha}\right]}; \quad x \in R, \tag{1.6}$$

where $\theta$ is the scale parameter and $\alpha$, $\beta$ are the shape parameters. For the random variable to have PDF (1.6), we will write $X$~EOIW-G$(\theta, \alpha, \beta, \eta)$.

The associated HF for the EOIW-G family is

$$h\,(x) = \frac{\alpha\,\beta\,\theta^{\alpha}\, g(x;\eta)\left(1 - G(x;\eta)^{\beta}\right)^{\alpha-1}}{G(x;\eta)^{\beta\alpha+1}\left(1 - \exp-\left[\theta^{\alpha}\left\{\frac{G(x;\eta)^{\beta}}{1-G(x;\eta)^{\beta}}\right\}^{-\alpha}\right]\right)}\; \exp-\left[\theta^{\alpha}\left\{\frac{G(x;\eta)^{\beta}}{1-G(x;\eta)^{\beta}}\right\}^{-\alpha}\right].$$

Furthermore, the EOIW-G family of distributions can be simulated by using a quantile function; let $q$ be a random variable having a uniform distribution on the interval [0,1]. The quantile function of the family is given by:

$$Q\,(q) = \; G^{-1}\left[\left(1 + (-\theta^{-\alpha}\ln q)^{\frac{1}{\alpha}}\right)^{\frac{-1}{\beta}}\right], \; q \in [0\,,\,1]\,,$$

where $G^{-1}(.)$ is the inverse function of the CDF of the baseline distribution. For $q = 0.5$, the median of the EOIW-G family is produced. Also, the quantile function is used in obtaining the skewness and kurtosis measures when the moments of the distribution do not exit.

In this paper we discussed two real data of engineearing applications: Time between failures (TBF) is a key metric in reliability engineering, and it is particularly relevant for repairable systems. TBF refers to the time elapsed between consecutive failures of a system or component. In the context of repairable items, this metric helps assess the reliability and maintainability of a system over time & "Vinyl chloride (VC) is a chemical compound that is of particular concern in environmental engineering due to its potential for groundwater contamination. When dealing with data obtained from cleanup gradient groundwater monitoring wells, engineers and environmental scientists use this information for various applications."

The rest of this article is divided into the following sections. Section 2 studies four specific examples of the EOIW-G family, along with plots of their PDFs and HFs. In Section 3, certain statistical properties of the EOIW-G family are discovered, such as quantiles, linear representations of the PDF and CDF, moments, entropy measure, mean deviation, and order statistics. In Section 4, parameter estimation is covered using the ML and Bayesian techniques. We provide a simulation analysis in Section 5 that compares the aforementioned estimation techniques. In order to highlight the significance of the new distribution, two real-world data sets are taken into consideration for the EOIWW distribution in Section 6. In Section 7, some last observations are presented.

## 2. Special Models of the EOIW-G Family

This section provides a list of notable new and existing families. A lot of new distributions can be deduced as special models from the EOIW-G family of distributions.

Following are a list of several notable new and existing families based on (1.6):

1. For $\theta = 1$, the PDF provides the extended odd Fréchet-G family presented by Ref. [10].
2. For $\theta = 1$, and $\beta = 1$ the PDF provides the odd Fréchet-G family presented by Ref. [9].
3. For $\beta = 1$, the PDF provides the IW-G family presented by Ref. [12].
4. For $\alpha = 1$, the PDF provides the extended odd inverse exponential-G family (new).
5. For $\alpha = 2$, the PDF provides the extended odd inverse Rayleigh-G family (new).

A lot of new distributions can be deduced as special models from the EOIW-G family of distributions. Consequently, we will introduce some of these special models as follows:

### 2.1. The EOIW-Uniform Distribution

Suppose the parent distribution is uniform on the interval $(0, a)$, where $a > 0$. The CDF of the EOIW-uniform (EOIW-U) distribution depending on the four parameters $(\theta, a, \alpha, \beta) > 0$ is given by:

$$F_{EOIWU}(x) = e^{-\left[\theta^{\alpha}\left[(x/a)^{-\beta}-1\right]^{\alpha}\right]};\quad 0 < x < a < \infty.$$

The corresponding PDF is:

$$f_{EOIWU}(x) = \frac{\alpha\beta\theta^{\alpha}}{a}\left[1-\left(\frac{x}{a}\right)^{-\beta}\right]^{\alpha-1}\left(\frac{x}{a}\right)^{\beta} e^{-\left[\theta^{\alpha}\left[(x/a)^{-\beta}-1\right]^{\alpha}\right]},\ 0 < x < a < \infty.$$

The hazard rate function and qth quantile function are, respectively, given by

$$h_{EOIWU}(x) = \frac{\alpha\,\beta\,\theta^{\alpha}}{a}\left[1-\left(\frac{x}{a}\right)^{\beta}\right]^{\alpha-1}\left(\frac{x}{a}\right)^{-(\beta\alpha+1)}\left[e^{\left[\theta^{\alpha}\left[(x/a)^{-\beta}-1\right]^{\alpha}\right]}-1\right]^{-1},$$

and

$$Q(q) = a\left[1+(-\theta^{\alpha}\ln q)^{\frac{1}{\alpha}}\right]^{-\beta},\ q\in[0,\ 1]\ .$$

*2.2. EOIW-Weibull Distribution*

Considering the baseline distribution is a Weibull with PDF and CDF given by $g(x) = \mu\,\lambda^{\mu}\,x^{\mu-1}e^{-(\lambda x)^{\mu}}$, and $G(x) = 1-e^{-(\lambda x)^{\mu}}$, where $\mu > 0$, $\lambda > 0$, respectively. The CDF of the EOIWW distribution depending on the five parameters $(\theta,\ \alpha,\ \beta,\ \mu,\ \lambda) > 0$, is:

$$F_{EOIWW}(x) = e^{-\theta^{\alpha}\left[\left(1-e^{-(\lambda x)^{\mu}}\right)^{-\beta}-1\right]^{\alpha}};\quad x>0.$$

The PDF of the EOIWW distribution is given by:

$$f_{EOIWW}(x) = \frac{\alpha\,\beta\,\mu\,\lambda^{\mu}\,\theta^{\alpha}\,x^{\mu-1}\,e^{-(\lambda x)^{\mu}}}{(1-e^{-(\lambda x)^{\mu}})^{\beta\,\alpha+1}}\left[1-\left(1-e^{-(\lambda x)^{\mu}}\right)^{\beta}\right]^{\alpha-1}e^{-\theta^{\alpha}\left[\left(1-e^{-(\lambda x)^{\mu}}\right)^{-\beta}-1\right]^{\alpha}};x>0\,.$$

The EOIW-W model is a highly adaptable model that contains several additional models. The sub-models of the EOIWW model are listed in Table 1. Twenty four special models are listed in Table 1 for the EOIWW model, including well-known distributions discussed and studied in the literature.

The associated HF is obtained as follows:

$$h_{EOIWW}(x) = \frac{\alpha\,\beta\,\mu\,\lambda^{\mu}\,\theta^{\alpha}\,x^{\mu-1}e^{-(\lambda x)^{\mu}}\left[1-\left(1-e^{-(\lambda x)^{\mu}}\right)^{\beta}\right]^{\alpha-1}}{(1-e^{-(\lambda x)^{\mu}})^{\beta\,\alpha+1}\left[\exp\left\{\theta^{\alpha}\left[(\,1-e^{-(\lambda x)^{\mu}})^{-\beta}-1\right]^{\alpha}\right\}-1\right]}\ .$$

Furtheremore, the quantile function of the EOIWW distrbution is as follows:

$$Q(q) = \frac{1}{\lambda}\left[-\ln\left[1-\left\{1+(-\theta^{-\alpha}\ln q)^{\frac{1}{\alpha}}\right\}^{\frac{-1}{\beta}}\right]\right]^{\frac{1}{\nu}}\ ,\ q\in[0,\ 1]\ . \tag{2.1}$$

Equation (2.1) used to generate random numbers of the EOIWW distribution.

*2.3. EOIW-Fréchet Distribution*

Suppose that the Fréchet is the baseline distribution with PDF $g(x) = a\,b^{a}\,x^{-(a+1)}e^{-(b/x)^{a}}$, and CDF $G(x) = e^{-(b/x)^{a}}$. The EOIW-Fréchet (EOIWF) distribution depending on the five parameters $(\theta,\ \alpha,\ \beta, a, b) > 0$ has CDF given by:

$$F_{EOIWF}(x) = e^{-\left\{\theta^{\alpha}\left[\left(e^{-(b/x)^{a}}\right)^{-\beta}-1\right]^{\alpha}\right\}};\quad x>0.$$

The corresponding PDF is:

$$f_{EOIWF}(x) = \frac{\alpha\,\beta\,a\,b^{a}\,\theta^{\alpha}\,x^{-(a+1)}\,e^{-(b/x)^{a}}}{(e^{-(b/x)^{a}})^{\beta\,\alpha+1}}\left[1-\left(e^{-(b/\,x)^{a}}\right)^{\beta}\right]^{\alpha-1}e^{-\left\{\theta^{\alpha}\left[\left(e^{-(b/x)^{a}}\right)^{-\beta}-1\right]^{\alpha}\right\}};x>0\,.$$

**Table 1.** Sub-models of the EOIWW model

| $\theta$ | $\alpha$ | $\beta$ | $\mu$ | $\lambda$ | Reduced Model |
|---|---|---|---|---|---|
| 1 | - | - | | - | The Extended Odd Fréchet-Weibull |
| 1 | - | - | 1 | - | The Extended Odd Fréchet-Exponential |
| 1 | - | - | 2 | - | The Extended Odd Fréchet-Rayleigh |
| 1 | - | 1 | | - | The Odd Fréchet-Weibull |
| 1 | - | 1 | 1 | - | The Odd Fréchet-Exponential |
| 1 | - | 1 | 2 | - | The Odd Fréchet-Rayleigh |
| - | 1 | - | | - | The Extended Inverse Exponential-Weibull |
| - | 1 | - | 1 | - | The Extended Inverse Exponential-Exponential |
| - | 1 | - | 2 | - | The Extended Inverse Exponential-Rayleigh |
| - | 1 | 1 | - | - | The Inverse Exponential-Weibull |
| - | 1 | 1 | 1 | - | The Inverse Exponential-Exponential |
| - | 1 | 1 | 2 | - | The Inverse Exponential-Rayleigh |
| - | 2 | - | - | - | The Extended Inverse Rayleigh-Weibull |
| - | 2 | | 1 | - | The Extended Inverse Rayleigh-Exponential |
| - | 2 | | 2 | - | The Extended Inverse Rayleigh-Rayleigh |
| - | 2 | 1 | - | - | The Inverse Rayleigh-Weibull |
| - | 2 | 1 | 1 | - | The Inverse Rayleigh-Exponential |
| - | 2 | 1 | 2 | - | The Inverse Rayleigh-Rayleigh |
| - | - | 1 | - | - | Inverse Weibul-Weibull |
| - | - | 1 | 1 | - | Inverse Weibul-Exponential |
| - | - | 1 | 2 | - | Inverse Weibul-Rayleigh |
| 1 | 1 | 1 | - | - | Weibull |
| 1 | 1 | 1 | 1 | - | Exponential |
| 1 | 1 | 1 | 2 | - | Rayleigh |

The HF of the EOIWF distribution is given by:

$$h_{EOIWF}(x) = \frac{\alpha \beta\, a\, b^{a}\, \theta^{\alpha}\, x^{-(a+1)} e^{-(b/x)^{a}} \left[1 - \left[e^{-(b/x)^{a}}\right]^{\beta}\right]^{\alpha-1}}{\left[e^{-(b/x)^{a}}\right]^{\beta\alpha+1} \left[\exp\left\{\theta^{\alpha}\left[\left(e^{-(b/x)^{a}}\right)^{-\beta} - 1\right]^{\alpha}\right\} - 1\right]} .$$

The quantile function of the EOIWF distribution is defined as:

$$Q(q) = b\left[-\ln\left[1 + \left(-\theta^{-\alpha} \ln q\right)^{\frac{1}{\alpha}}\right]^{\frac{-1}{\beta}}\right]^{\frac{-1}{a}} , \quad q \in [0,\ 1] .$$

### *2.4. EOIW-Lomax Distribution*

If the parent distribution is Lomax distribution with PDF and CDF given by $g(x) = \gamma\, \lambda^{\gamma}\ (\lambda + x)^{-\gamma-1}$, and $G(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\gamma}$, where $\lambda, \gamma > 0$, respectively. Then the CDF of the EOIW-Lomax (EOIWL) distribution depending on the five parameters ($\theta$, $\alpha$, $\beta, \lambda$, $\gamma > 0$) is given by:

$$F_{EOIWL}(x) = e^{-\theta^{\alpha}\left[\left(1-(1+(x/\lambda))^{-\gamma}\right)^{-\beta}-1\right]^{\alpha}};\ x > 0.$$

The PDF and HF of the EOIWL distribution are, respectively, given as follows:

$$f_{EOIWL}(x) = \frac{\alpha\beta\gamma\,\lambda^{\gamma}\,\theta^{\alpha}\,(\lambda+x)^{-(\gamma+1)}}{\left(1-\left(1+(x/\lambda)\right)^{-\gamma}\right)^{\beta\alpha+1}}\left[1-(1-(1+(x/\lambda))^{-\gamma})^{\beta}\right]^{\alpha-1} e^{-\theta^{\alpha}\left[\left(1-(1+(x/\lambda))^{-\gamma}\right)^{-\beta}-1\right]^{\alpha}};x > 0,$$

and,

$$h_{EOIWL}(x) = \frac{\alpha\beta\gamma\,\lambda^{\gamma}\,\theta^{\alpha}\,\left[1-(1-(1+(x/\lambda))^{-\gamma})^{\beta}\right]^{\alpha-1}}{(\lambda+x)^{(\gamma+1)}\,(1-(1+(x/\lambda))^{-\gamma})^{\beta\alpha+1}\left[\exp\theta^{\alpha}\left[(1-(1+(x/\lambda))^{-\gamma})^{-\beta}-1\right]^{\alpha}-1\right]}.$$

The quantile function of the EOIWL distribution can easily be done using the following equation:

$$Q(q) = \lambda\left[\left(1-\left[1+(-\theta^{-\alpha}\ln q)^{\frac{1}{\alpha}}\right]^{\frac{-1}{\beta}}\right)^{\frac{-1}{\gamma}}-1\right],\ q \in [0,\ 1].$$

The PDF and HF of EOIWU ($\theta$, $\alpha$, $\beta, a$), EOIWW ($\theta$, $\alpha$, $\beta, \mu$, $\lambda$), EOIWF ($\theta$, $\alpha$, $\beta, a, b$), and EOIWL ($\theta$, $\alpha$, $\beta, \lambda, \gamma$) for a range of parameter values are shown in Figures 1 and 2. The PDF of EOIW-G models can be left-skewed, symmetrical, reversed-J, J-shaped, unimodel, and U-shaped, as Figure 1 illustrates. Furthermore, Figure 2 illustrates the variety of shapes that the HF can assume, such as constant, decreasing, increasing, and upside-down bathtub shapes. Therefore, fitting data sets with several forms will be better suited for the EOIW-G family.

## 3. Statistical Properties

This section looks at several EOIW-G family member's mathematical characteristics. The overall results, at the very least, are applied, whenever feasible, to the EOIWW distribution as explained in Subsection 3.2.

### 3.1. Mixture Representation

Here, mixture representations of the EOIW-G family's PDF and CDF are deduced.

Using the exponential expansion in the PDF (1.6) provides

$$e^{-\theta^{\alpha}\left[\frac{G(x;\eta)^{\beta}}{1-G(x;\eta)^{\beta}}\right]^{-\alpha}} = \sum_{i_1=0}^{\infty}\frac{(-1)^{i_1}\,\theta^{i_1\,\alpha}}{i_1!}\left[\frac{G(x;\ \eta)^{\beta}}{1-G(x;\ \eta)^{\beta}}\right]^{-i_1\,\alpha}. \tag{3.1}$$

Inserting (3.1), in the PDF (1.6), then we have:

$$f(x) = \sum_{i_1=0}^{\infty}\frac{(-1)^{i_1}\,\theta^{\alpha\,(i_1+1)}\alpha\,\beta\,g(x;\ \eta)\left[1-G(x;\ \eta)^{\beta}\right]^{\alpha\,(i_1+1)-1}}{i_1!\ G(x;\ \eta)^{\alpha\beta\,(i_1+1)+1}}. \tag{3.2}$$

The generalized binomial expansion, for $s > 0$ is real non integer, is given by:

$$[1-Z]^{s-1} = \sum_{i_2=0}^{\infty}(-1)^{i_2}\binom{s-1}{i_2}Z^{i_2},\ |Z| < 1. \tag{3.3}$$

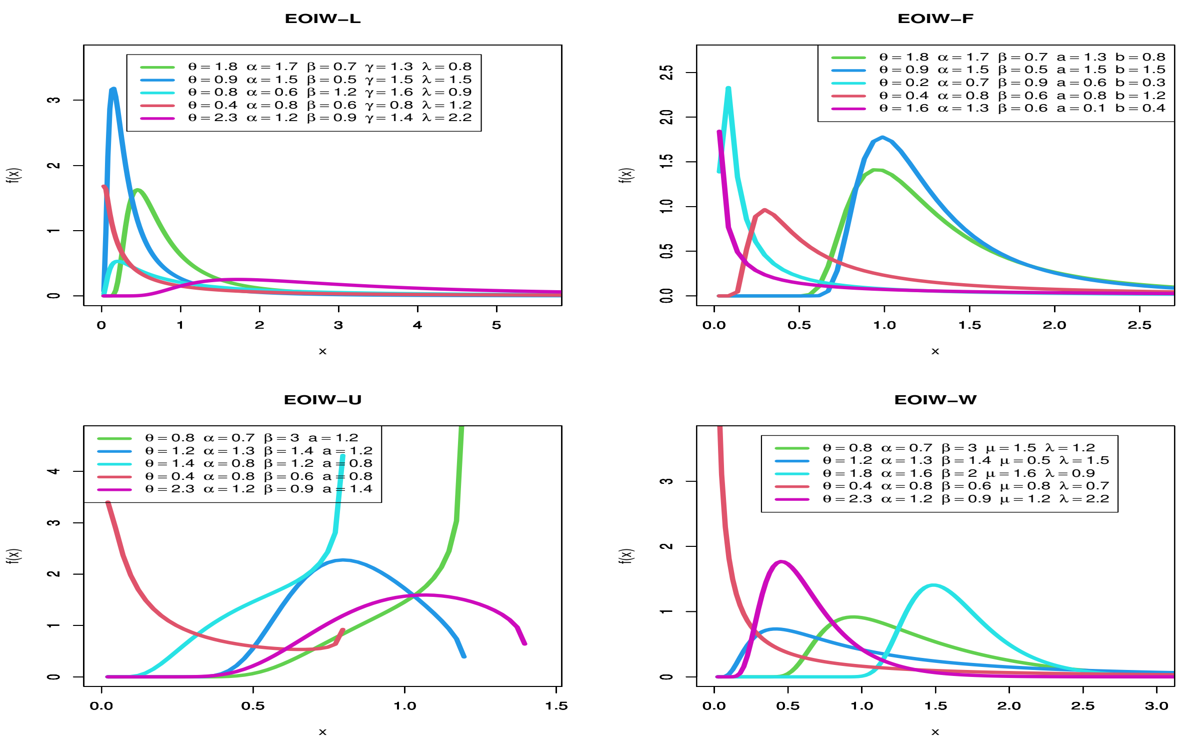


**Figure 1.** PDF plots for some special models

Using (3.3) in (3.2), then the PDF of the EIOW-G family is as follows

$$f(x) = \sum_{i_1,i_2=0}^{\infty} \psi_{i_1\, i_2}\ k\, g(x;\eta)\ G(x;\eta)^{k-1}\ , \tag{3.4}$$

where $\psi_{i_1 i_2} = \frac{(-1)^{i_1+i_2}\alpha\beta\theta^{\alpha\,(i_1+1)}}{i_1\,!\,k} \begin{pmatrix} \alpha\ (i_1+1)-1 \\ i_2 \end{pmatrix}$ , $k = \beta\ (i_2 - \alpha\ (i_1+1))$.

Hence, the PDF (3.4) is represented as a mixture form of the Exponentiated-G (Exp-G) PDFs. Another form of (3.4) can be written as:

$$f(x) = \sum_{i_1,i_2=0}^{\infty} \psi_{i_1 i_2}\, s_k\,(x;\,\eta)\ \ , \tag{3.5}$$

where $s_k\,(x;\,\eta) = k\,g\,(x;\,\eta)\ G\,(x;\,\eta)^{k-1}$ is the PDF of the exponentaiated (EXP)-G family with power parameter $k$. Additionally, the CDF of the EOIW-G family can be expressed as the mixture representation of Exp-G CDFs.

For $\delta$ is positive integer, $[F(x)]^{\delta}$ of the EOIW-G family can be expressed as follows:

$$[F(x)]^{\delta} = \sum_{i_3=0}^{\infty} \frac{(-1)^{i_3}\ \delta^{i_3}\theta^{i_3\,\alpha}}{i_3!} \left[\frac{G(x;\,\eta)^{\beta}}{1-G(x;\,\eta)^{\beta}}\right]^{-i_3\,\alpha}. \tag{3.6}$$

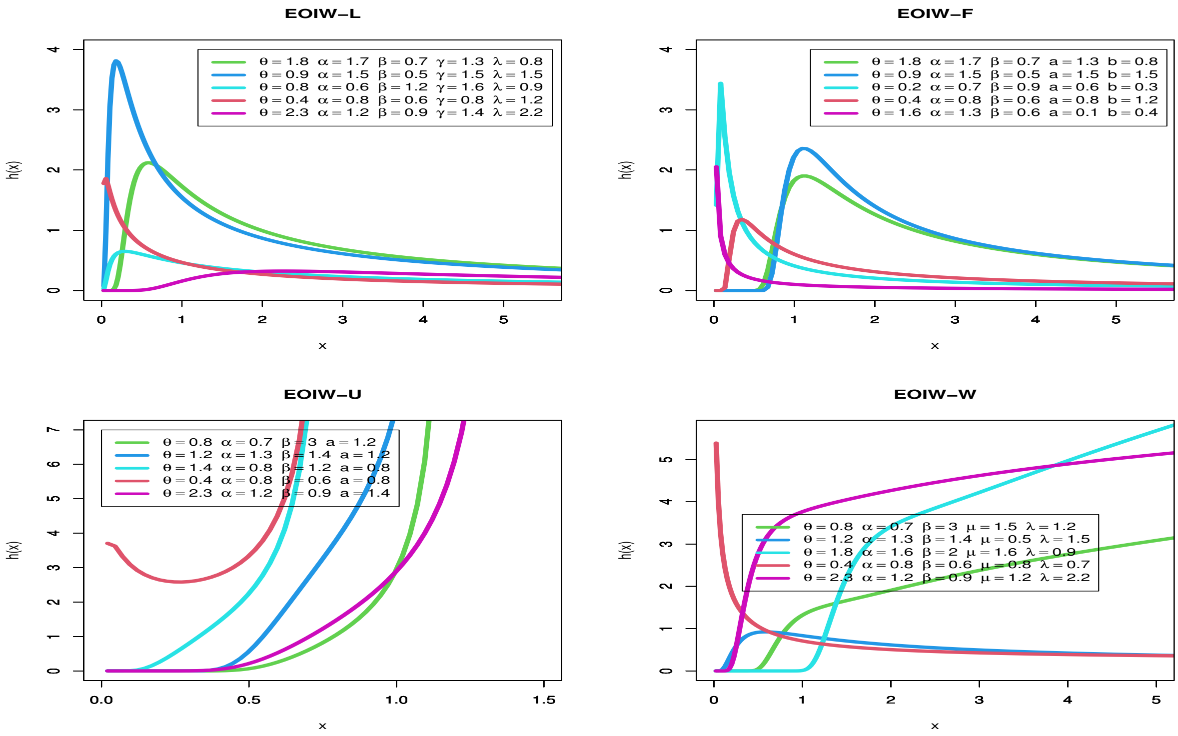


**Figure 2.** HF plots for some special models

Using binomial expansion, more than on time, in (3.6), we get

$$[F(x)]^{\delta} = \sum_{i_3,i_4=0}^{\infty} \zeta_{i_3,i_4}\, [G(x;\,\eta)]^{\beta(i_4-\alpha\, i_3)}\,, \tag{3.7}$$

where $\zeta_{i_3,i_4} = \frac{(-1)^{i_3+i_4}\delta^{i_3}\theta^{\alpha\, i_3}}{i_3\,!} \begin{pmatrix} \alpha\, i_3 \\ i_4 \end{pmatrix}$ .

### 3.2. *Moments and Associated Measures*

It is possible to derive the moments, moments generating function, and incomplete moments of the EOIW-G family of distributions. These moment measurements are specifically obtained for the EOIWW distribution.

The rth non-central moment of $X$ has the EOIWW distribution is derived from (3.4) as follows:

$$\mu_r' = \sum_{i_1,i_2=0}^{\infty} \psi_{i_1\, i_2} \int_{-\infty}^{\infty} x^r\, k\, g(x;\eta)\; G(x;\eta)^{k-1}\, dx\,. \tag{3.8}$$

Using the Weibull distribution as the baseline in (3.8), then rth-moment of the EOIWW distribution is given by

$$\mu_r' = \sum_{i_1,\, i_2=0}^{\infty} \psi_{i_1\, i_2}\, k\, \mu\, \lambda^{\mu} \int_{0}^{\infty} x^{r+\mu-1} e^{-(\lambda\, x)^{\mu}} \left[1 - e^{-(\lambda\, x)^{\mu}}\right]^{k-1} dx\;. \tag{3.9}$$

Using the binomial expansion in (3.9), the rth-moment of the EOIWW distribution is as follows:

$$\mu_r' = \sum_{i_1, i_2, i_3=0}^{\infty} \pi_{i_1 i_2 i_3} \Gamma\left(\frac{r}{\mu}+1\right) , \quad r = 1, 2, ...$$

where, $\pi_{i_1 i_2 i_3} = \frac{(-1)^{i_3} \psi_{i_1 i_2} k}{(i_3+1)^{(r/\mu)+1} \lambda^r} \left( \begin{array}{c} k-1 \\ i_3 \end{array} \right)$, $\psi_{i_1 i_2}$ and $k$ defined in (3.4), while $\Gamma(.)$ stands for gamma function. Furthermore, the moment generating function of the EOIWW distribution is

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i_1,i_2,i_3=0}^{\infty} \pi_{i_1 i_2 i_3} \Gamma\left(\frac{r}{\nu}+1\right) .$$

The rth incomplete moment of $X$ having EOIWW distribution is obtained as follows:

$$\rho_r(t) = \sum_{i_1, i_2=0}^{\infty} \psi_{i_1 i_2} k \mu \lambda^{\mu} \int_0^t x^{r+\mu-1} e^{-(\lambda x)^{\mu}} \left[1 - e^{-(\lambda x)^{\mu}}\right]^{k-1} dx = \sum_{i_1, i_2, i_3=0}^{\infty} \pi_{i_1 i_2 i_3} \bar{\Gamma}\left(\frac{r}{\mu}+1, (i_3+1)(\lambda t)^{\mu}\right) ,$$

where $\bar{\Gamma}(.)$ is the lower incomplete gamma function. Economics, reliability, demography, and medicine all benefit greatly from the usage of the Bonferroni and Lorenz curves. The incomplete moments allow for the computation of these curves. The Lorenz curve and the Bonferroni curve, are respectively, defined by $L(x) = \rho_1(x) \big/ \mu_1'$, $B(x) = L(x)/F(x)$. Hence these curves are easily obtained, by using $\rho_1(x)$, $\mu_1'$, and $F(x)$, as follows:

$$L(x) = (\mu_1')^{-1} \sum_{i_1, i_2, i_3=0}^{\infty} \pi_{i_1 i_2 i_3} \bar{\Gamma}\left(\frac{\mu+1}{\mu}, (i_3+1)(\lambda x)^{\mu}\right),$$

and

$$B(x) = (\mu_1')^{-1} \sum_{i_1, i_2, i_3=0}^{\infty} \pi_{i_1 i_2 i_3} \bar{\Gamma}\left(\frac{\mu+1}{\mu}, (i_3+1)(\lambda x)^{\mu}\right) \exp - \left\{\theta^{\alpha}\left[\left(1 - e^{-(\lambda x)^{\mu}}\right)^{-\beta} - 1\right]^{\alpha}\right\} .$$

### 3.3. *Mean Deviation*

The statistical measures of the mean deviations about mean and median can be computed using the incomplete moments. For a random variable $X$ distributed as EOIWW, the mean deviation about mean $\varepsilon_1$ is as follows:

$$\varepsilon_1 = 2\left[\mu_1' F\left(\mu_1'\right) - \rho_1(\mu_1')\right],$$

which is given by:

$$\varepsilon_1 = 2\mu_1' \left[\exp\left\{-\theta^{\alpha}\left[\left(1 - e^{-(\lambda \mu_1')^{\mu}}\right)^{-\beta} - 1\right]^{\alpha}\right\}\right] - 2 \sum_{i_1, i_2, i_3=0}^{\infty} \pi_{i_1 i_2 i_3} \bar{\Gamma}\left(\frac{\mu+1}{\mu}, (i_3+1)\left(\lambda \mu_1'\right)^{\mu}\right) ,$$

where $\mu_1'$ is the mean of the EOIWW distribution. Also the mean deviation about median $\varepsilon_2$ of the EOIWW distribution, with median $(M)$ is

$$\varepsilon_2 = \mu_1' - 2\rho_1(M),$$

which is given by:

$$\varepsilon_2 = \mu_1' - 2 \sum_{i_1, i_2, i_3=0}^{\infty} \pi_{i_1 i_2 i_3} \bar{\Gamma}\left(\frac{\mu+1}{\mu}, (i_3+1)(\lambda M)^{\mu}\right) .$$

### 3.4. Rényi Entropy

Information theory and probability distribution use entropy to measure unpredictability. It describes how much uncertainty there is in relation to random variables. The EOIWW distribution's Rényi entropy, for $r > 0,\ r \neq 1$, is given by:

$$R_r(x) = \frac{1}{(1-r)} \log \int_0^{\infty} f^r_{EOIWW}(x)\, dx\,. \tag{3.10}$$

Using the exponential expansion in $f^r_{EOIWW}(x)$ gives

$$f^r_{EOIWW}(x) = \frac{(\alpha\beta\mu\, \lambda^{\mu}\, \theta^{\alpha})^r\ x^{r(\mu-1)}\, e^{-r\,(\lambda\, x)^{\mu}}}{i\,!} \left[1-\left(1-e^{-(\lambda\, x)^{\mu}}\right)^{\beta}\right]^{\alpha\, i+r} \left[1-e^{-(\lambda\, x)^{\mu}}\right]^{-\beta\,\alpha\,(r+i)-r}. \tag{3.11}$$

Applying binomial expansion twice times in (3.11), we find

$$f^r_{EOIWW}(x) = \sum_{i,\,j,\,k=0}^{\infty} \Lambda_{i,\,j,\,k}\, (\alpha\beta\mu\,\lambda^{\mu}\,\theta^{\alpha})^r\ x^{r\,(\mu-1)} e^{-(k+r)\,(\lambda\, x)^{\mu}}, \tag{3.12}$$

where $\Lambda_{i,j,k} = \frac{(-1)^{i+j+k}(r\,\theta^{\alpha})^i}{i\,!} \begin{pmatrix} \alpha\, i + r \\ j \end{pmatrix} \begin{pmatrix} \beta\,(j - \alpha\,(i+r)) - r \\ k \end{pmatrix}$.

Hence, inserting (3.12) in (3.10), then the Rényi entropy of the EOIWW distribution is

$$R_r(x) = \frac{1}{(1-r)} \log\left[\frac{(\alpha\beta\,\theta^{\alpha})^r}{\mu^{1-r}\,(r+k)^{r-\left(\frac{r-1}{\mu}\right)}} \sum_{i,\,j,\,k=0}^{\infty} \Lambda_{i,\,j,\,k}\Gamma\left(r - \left(\frac{r-1}{\mu}\right)\right)\right],\ r > 1,$$

where $\Gamma\,(.)$ is the gamma function.

### 3.5. Order Statistics

Suppose $x_{(1)}, x_{(2)}, ..., x_{(n)}$ be an ordered random sample of size $n$ drawn from EOIW-G with CDF and PDF given by (1.5) and (1.6), respectively. The PDF of $X_{(r)}$ is given by:

$$f_{X_{(r)}}(x) = \sum_{d=0}^{n-r} \frac{(-1)^d}{B\,(r,\ n-r+1)} \begin{pmatrix} n-r \\ d \end{pmatrix} [F(x)]^{r+d-1}\ f(x). \tag{3.13}$$

Using (3.5) and (3.7) with $\delta = (r+d-1)$ in (3.13), we get the PDF of r$^{th}$ order statistic of the EOIW-G family as follows:

$$f_{X_{(r)}}(x) = w \sum_{d=0}^{n-r} \sum_{i,j,m,l=0}^{\infty} \rho_{d,i,j,m,l}\, g(x;\,\eta)\, G\,(x;\,\eta)^{w-1},$$

where

$\rho_{d,i,j,m,l} = \frac{\beta(j-\alpha(i+1))}{B(r,n-r+1)\,w} (-1)^d \begin{pmatrix} n-r \\ d \end{pmatrix} \psi_{i,j}\zeta_{m,l}$, $w = \beta\,[(j+l) - \alpha\,(i+m+l)]$ and B(., .) is the beta function. The density function of the EOIW-G order statistics can be written as a linear combination of the EXP-G density function with power parameter $w$, as follows:

$$f_{X_{(r)}}(x) = \sum_{d=0}^{n-r} \sum_{i,j,m,l=0}^{\infty} \rho_{d,i,j,m,l}\, D_w(x;\,\eta), \tag{3.14}$$

where $D_w(x;\,\eta) = w\, g(x;\,\eta)\, G(x;\,\eta)^{w-1}$is the PDF of EXP-G family with power parameter $w$. We may derive numerous statistical features for the PDF of the EOIW-G order statistics, including moments and L-moments, based on (3.14).

In particular: The density function of r$^{th}$ order statistics of the EOIWW distribution can be written as

$$f_{X_{(r)}}(x) = \sum_{d=0}^{n-r} \sum_{i,\,j,\,m,\,l=0}^{\infty} \rho_{d,\,i,\,j,\,m,\,l}\, w\, \mu\, \lambda^{\mu} x^{(\mu-1)} e^{-(\lambda x)^{\mu}} \left(1 - e^{-(\lambda x)^{\mu}}\right)^{w-1}. \tag{3.15}$$

For $r$ =1, and $r = n$, the smallest order statistics $X_{(1)}$ and the largest order statistics $X_{(n)}$ of the EOIWW distribution are produced.

## 4. Parameters Estimation

This section is devoted to estimate the parameters $\Xi = (\theta,\, \alpha,\, \beta, \eta)^T$ for the EOIW-G family of distributions using ML and Bayesian estimation methods.

### 4.1. Maximum Likelihood Estimates

Using complete samples, we derive the ML estimates (MLEs) of the unknown parameters for the EOIW-G family. Assume $x_1, x_2, ..., x_n$ be a random sample from the EOIW-G ($\Xi$) family with PDF (1.2) and $\Xi = (\theta,\, \alpha,\, \beta, \eta)^T$ be a parameter vector. The log-likelihood function is given by:

$$\begin{aligned} \log L(\Xi) = {} & n\, \log(\alpha\beta) + n\alpha \log(\theta) + \textstyle\sum_{i=1}^{n} \log g(x_i;\eta) + (\alpha-1)\, \sum_{i=1}^{n} \log\left[1 - G(x_i;\eta)^{\beta}\right] \\ & - (\beta\,\alpha + 1) \textstyle\sum_{i=1}^{n} \log G(x_i;\,\eta) - \theta^{\alpha} \sum_{i=1}^{n} \left[G(x_i;\eta)^{-\beta} - 1\right]^{\alpha}. \end{aligned} \tag{4.1}$$

The elements of the score function $U(\Xi) = \left(U_{\theta}, U_{\alpha}, U_{\beta}, U_{\eta}\right)^T$ are

$$U_{\theta} = \frac{n\,\alpha}{\theta} - \alpha\,\theta^{\alpha-1} \sum_{i=1}^{n} \left[G(x_i;\,\eta)^{-\beta} - 1\right]^{\alpha},$$

$$\begin{aligned} U_{\alpha} = {} & \tfrac{n}{\alpha} + n\, \log(\theta) + \textstyle\sum_{i=1}^{n} \log\left[1 - G(x_i;\,\eta)^{\beta}\right] - \beta\, \sum_{i=1}^{n} \ln G(x_i;\,\eta) - \theta^{\alpha} \log\theta \sum_{i=1}^{n} \left[G(x_i;\,\eta)^{-\beta} - 1\right]^{\alpha} \\ & -\theta^{\alpha} \textstyle\sum_{i=1}^{n} \left[G(x_i;\,\eta)^{-\beta} - 1\right]^{\alpha} \log\left[G(x_i;\,\eta)^{-\beta} - 1\right], \end{aligned}$$

$$\begin{aligned} U_{\beta} = {} & \tfrac{n}{\beta} - \textstyle\sum_{i=1}^{n} \frac{(\alpha-1)\, \log G(x_i;\eta)}{G(x_i;\eta)^{-\beta}-1} + \alpha\,\theta^{\alpha} \sum_{i=1}^{n} \left[1 - G(x_i;\,\eta)^{\beta}\right]^{\alpha-1} G(x;\,\eta)^{-\beta} \log\left[G(x_i;\eta)\right] \\ & -\alpha\, \textstyle\sum_{i=1}^{n} \log G(x_i;\,\eta), \end{aligned}$$

and

$$\begin{aligned} U_{\eta} = {} & \textstyle\sum_{i=1}^{n} \left[\frac{\partial g(x_i;\eta)/\partial\eta}{g(x_i;\eta)}\right] - (\alpha-1)\, \sum_{i=1}^{n} \frac{\beta G(x_i;\eta)^{\beta-1}\left[\partial G(x_i;\eta)/\partial\eta\right]}{\left[1-G(x_i;\eta)^{\beta}\right]} \\ & - (\beta\,\alpha - 1) \textstyle\sum_{i=1}^{n} \left[\frac{\partial G(x_i;\eta)/\partial\eta}{G(x_i;\eta)}\right] + \alpha\,\beta\,\theta^{\alpha} \sum_{i=1}^{n} \left[\frac{\left[1-G(x_i;\eta)^{\beta}\right]^{\alpha-1}\left[\partial G(x_i;\eta)/\partial\eta\right]}{G(x_i;\eta)^{1-\beta\alpha}}\right]. \end{aligned}$$

The MLEs $\hat{\Xi} = \left(\hat{\theta},\, \hat{\alpha},\, \hat{\beta}, \hat{\eta}\right)^T$ of $\Xi = (\theta,\, \alpha,\, \beta, \eta)^T$ are derived by setting the above score function to zero and numerically solving the resulting system of non-linear equations simultaneously using any statistical software.

Under standard regularity conditions, when $n \to \infty$, the distribution of $\Xi$ can be approximated by a multivariate normal $N_p\left(0,\ Var\left(\hat{\Xi}\right)\right)$, where variances of estimated parameters denotes the diagonal elements of inverting the information matrix evaluated at MLEs $\hat{\Xi}$ of the model parameters. The $100\,(1-\omega)\%$ approximate two-sided confidence intervals for the parameters $\Xi = (\theta,\ \alpha,\ \beta, \eta)^T$ of the EOIW-G family are respectively given by:

$\hat{\theta} \pm z_{\omega/2}\sqrt{var\left(\hat{\theta}\right)}$, $\hat{\beta} \pm z_{\omega/2}\sqrt{var\left(\hat{\beta}\right)}$, $\hat{\alpha} \pm z_{\omega/2}\sqrt{var\left(\hat{\alpha}\right)}$, and $\hat{\eta} \pm z_{\omega/2}\sqrt{var\left(\hat{\eta}\right)}$,

where the upper $(\omega/2)^{th}$ percentile of the standard normal distribution is $z_{(\omega/2)}$ .

### 4.2. Bayesian Estimation

This section utilises a Bayesian approach to estimate and examine the parameters of the EOIW-G family of distributions. In this approach, two different loss functions the asymmetric: linear exponential (LINEX) loss function and the symmetric: squared error loss function (SELF) are used to compute Bayes estimates (BEs). In this Bayesian approach, the gamma distribution is used as a prior distribution. When considered individually as prior joint density functions, the parameters of $\theta$, $\alpha$, $\beta$, and $\eta$ take the following form:

$$\pi\left(\theta,\ \alpha,\ \beta,\ \eta\right) \propto \theta^{h_1-1}e^{-q_1\theta}\alpha^{h_2-1}e^{-q_2\alpha}\beta^{h_3-1}e^{-q_3\beta}\eta^{h_4-1}e^{-q_4\eta};\ h_i,\ q_i > 0,\ i = 1,2,3,4. \tag{4.2}$$

To establish the hyper-parameters, we employ informative priors (IFs). These IFs priors are derived by equating the mean and variance of likelihood estimators for $\hat{\Xi} = \left(\hat{\theta},\ \hat{\alpha},\ \hat{\beta}, \hat{\eta}\right)^T$ to the mean and variance of the specified gamma priors for $h_i$ and $q_i$. Therefore, by setting the mean and variance of likelihood estimators for $\hat{\Xi} = \left(\hat{\theta},\ \hat{\alpha},\ \hat{\beta}, \hat{\eta}\right)^T$ equal to those of the gamma priors, we obtain the outcome as described in Ref. [38].

$$\frac{1}{I}\sum_{j=1}^{I}\hat{\Xi}_i^{\ j} = \frac{h_i}{q_i}\quad,\quad \frac{1}{I-1}\sum_{j=1}^{I}\left(\hat{\Xi}_i^{\ j} - \frac{1}{I}\sum_{j=1}^{I}\hat{\Xi}_i^{\ j}\right)^2 = \frac{h_i}{q_i^{\ 2}}\ ,\ i = 1,\ 2,\ 3,\ 4,$$

where $I$ is the number of samples iteration. Now on solving the above two equations, the estimated hyper-parameters can be written as:

$$h_i = \frac{\left(I^{-1}\sum_{j=1}^{I}\hat{\Xi}_i^{\ j}\right)^2}{(I-1)^{-1}\ \sum_{j=1}^{I}\left(\hat{\Xi}_i^{\ j} - \frac{1}{I}\sum_{j=1}^{I}\hat{\Xi}_i^{\ j}\right)}, q_i = \frac{\left(I^{-1}\sum_{j=1}^{I}\hat{\Xi}_i^{\ j}\right)}{(I-1)^{-1}\ \sum_{j=1}^{I}\left(\hat{\Xi}_i^{\ j} - \frac{1}{I}\sum_{j=1}^{I}\hat{\Xi}_i^{\ j}\right)^2}.$$

The joint posterior density function of $\Xi = (\theta,\ \alpha,\ \beta, \eta)^T$ is obtained as follows using the likelihood function of the EOIW-G family of distributions and joint prior density (4.2)

$$\Pi\left(\theta,\ \alpha,\ \beta,\ \eta|\ x\right) = \frac{\pi\left(\theta,\ \alpha,\ \beta,\ \eta\right)\ L\left(\theta,\ \alpha,\ \beta,\ \eta\right)}{\int\ldots\int\pi\left(\theta,\ \alpha,\ \beta,\ \eta\right)\ L\left(\theta,\ \alpha,\ \beta,\ \eta\right)\ d\theta\, d\alpha\, d\beta\, d\eta}.$$

The posterior of the EOIW-G family of distributions is as follows:

$$\Pi\left(\theta,\ \alpha,\ \beta,\ \eta|\ x\right) \propto \alpha^{n+h_2-1}\beta^{n+h_3-1}\theta^{n\,\alpha+h_i-1}\eta^{h_4-1}e^{-\theta^{\alpha}\sum_{i=1}^{n}\left\{\frac{G(x_i;\eta)^{\beta}}{1-G(x_i;\eta)^{\beta}}\right\}^{-\alpha}}e^{-q_1\theta-q_2\alpha-q_3\beta-q_4\eta}$$
$$\prod_{i=1}^{n}\frac{g(x;\eta)}{G(x_i;\eta)^{\beta\,\alpha+1}}\left[1-G\left(x_i;\ \eta\right)^{\beta}\right]^{\alpha-1}.$$

Most Bayesian inference methods rely on symmetric loss functions, with one notable example being SELF. Using SELF, we derive BEs for $\theta, \alpha, \beta$, and $\eta$, denoted as $\overset{\leftrightarrow}{\Xi}= \left(\overset{\leftrightarrow}{\theta}, \overset{\leftrightarrow}{\alpha}, \overset{\leftrightarrow}{\beta}, \overset{\leftrightarrow}{\eta}\right)^{T}$ that defined by:

$$L\left(\overset{\leftrightarrow}{\Xi}_i, \Xi_i\right) = \left(\overset{\leftrightarrow}{\Xi}_i - \Xi_i\right)^2 \; ; \; \Xi = (\theta, \alpha, \beta, \eta).$$

Under the SELF, the BEs can be defined by:

$$\overset{\leftrightarrow}{\Xi}_i = E\left(\Xi_i \mid x\right) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Xi_i \prod \left(\Xi_i \mid x\right) \, d\theta \, d\alpha \, d\beta \, d\eta. \tag{4.3}$$

The LINEX loss function is a type of asymmetric loss function used in statistical decision theory and Bayesian estimation. It is designed to balance between the squared-error loss and the absolute error loss. The LINEX loss function is defined as:

$$L\left(\overset{\leftrightarrow}{\Xi}_i, \Xi_i\right) = c\, e^{\overset{\leftrightarrow}{\Xi}_i - \Xi_i} - \overset{\leftrightarrow}{\Xi}_i + \Xi_i \; ; \; \Xi = (\theta, \alpha, \beta, \eta).$$

The BEs for $\Xi$ using the LINEX loss function is derived in the following manner:

$$\overset{\leftrightarrow}{\Xi}_i = \frac{-1}{c} \ln\left[E\left(e^{-c\,\Xi_i} \mid x\right)\right] = \frac{-1}{c} \ln\left[\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{c\,\Xi_i} \prod \left(\Xi_i \mid x\right) \, d\theta \, d\alpha \, d\beta \, d\eta\right]. \tag{4.4}$$

It's important to highlight that the integrals described in Equations (4.3) and (4.4) are not readily derivable. Therefore, we employ the Markov Chain-Monte-Carlo (MCMC) method to estimate the expected values in Equations (4.3) and (4.4).

It was noted that the integrals described in Equations (4.3) and (4.4) cannot be explicitly calculated. Therefore, we utilize the MCMC method to estimate the integral values in Equations (4.3) and (4.4). Numerous studies have employed the MCMC technique, as demonstrated by [39], [42], [43], [44] and [45].

Gibbs samplers are a subset of MCMC algorithms, and within this category, more comprehensive Metropolis algorithms play a crucial role. Two commonly used MCMC approaches are the Metropolis-Hastings (MH) and Gibbs sampling methods. The MH method, akin to acceptance-rejection sampling, postulates that each iteration in the algorithm can produce a candidate value from a proposal distribution. We incorporate the MH method during the Gibbs sampling steps to generate random samples of conditional posterior densities from the EOIW-G distributions:

$$\Pi\left(\theta \mid \alpha, \beta, \eta, x\right) \propto \theta^{n\alpha + h_1 - 1} e^{-\theta^{\alpha} \sum_{i=1}^{n} \left\{\frac{G(x_i;\eta)^{\beta}}{1-G(x_i;\eta)^{\beta}}\right\}^{-\alpha}} e^{-q_1 \theta},$$

$$\Pi\left(\alpha \mid \theta, \beta, \eta, x\right) \propto \alpha^{n+h_2-1} e^{-\theta^{\alpha} \sum_{i=1}^{n} \left\{\frac{G(x_i;\eta)^{\beta}}{1-G(x_i;\eta)^{\beta}}\right\}^{-\alpha}} e^{-q_2 \alpha} \prod_{i=1}^{n} \frac{\left[1 - G(x;\eta)^{\beta}\right]^{\alpha-1}}{G\left(x_i;\eta\right)^{\beta\alpha+1}},$$

$$\Pi\left(\beta \mid \theta, \alpha, \eta, x\right) \propto \beta^{n+h_3-1} e^{-\theta^{\alpha} \sum_{i=1}^{n} \left\{\frac{G(x_i;\eta)^{\beta}}{1-G(x_i;\eta)^{\beta}}\right\}^{-\alpha}} e^{-q_3 \beta} \prod_{i=1}^{n} \frac{\left[1 - G(x;\eta)\right]^{\alpha-1}}{G\left(x_i;\eta\right)^{\beta\alpha+1}},$$

and

$$\Pi\left(\eta \mid \theta, \alpha, \beta, x\right) \propto \eta^{h_4-1} e^{-\theta^{\alpha} \sum_{i=1}^{n} \left\{\frac{G(x_i;\eta)^{\beta}}{1-G(x_i;\eta)^{\beta}}\right\}} e^{-q_4 \eta} \prod_{i=1}^{n} \frac{g(x;\eta)}{G\left(x_i;\eta\right)^{\beta\alpha+1}} \left[1 - G\left(x_i;\eta\right)^{\beta}\right]^{\alpha-1}.$$

## 5. Numerical Study

In this section, we employ a Monte-Carlo simulation approach to compare the ML estimation method with the Bayesian estimation method. We use the R language to estimate the parameters of the EOIW-W model using ML and Bayesian approaches based on MCMC with both the SELF and LINEX loss function. The Monte-Carlo experiments are conducted with 10,000 randomly generated samples from the EOIW-G family of distributions, where 'x' represents the lifetime values of the EOIW-W model for various actual parameter values and sample sizes (35, 60, 100, and 200). The best estimator methods are evaluated based on their ability to minimize estimator relative absoulute bias (RAB), mean squared error (MSE), length of confidance intervals (LCI) and covarge probability (CP). The LCI for MLE is LACI, while for Bayesian is LCCI. The level of confidance intervals is 95%. The actual parameters of the EOIW-W model are known in advance as:

Case 1: $\theta = 0.5$ , $\alpha = 0.5$, $\beta = 0.5$, $\mu = 0.5$, $\lambda = 0.5$.
Case 2: $\theta = 0.5$ , $\alpha = 0.5$, $\beta = 0.5$, $\mu = 2$, $\lambda = 2$.
Case 3: $\theta = 0.5$ , $\alpha = 0.5$, $\beta = 2$, $\mu = 2$, $\lambda = 2$.
Case 4: $\theta = 2$ , $\alpha = 2$, $\beta = 2$, $\mu = 2$, $\lambda = 2$.
Case 5: $\theta = 0.9$ , $\alpha = 0.6$, $\beta = 2$, $\mu = 3$, $\lambda = 2$.
Case 6: $\theta = 0.9$ , $\alpha = 0.6$, $\beta = 2$, $\mu = 1.3$, $\lambda = 1.2$.

Tables 2, 3, and 4 provide an overview of the simulation outcomes for the methodologies introduced in this paper concerning point estimation. To facilitate a meaningful comparison among different point estimation methods, we evaluate the RAB, MSE, and LCI values. Consequently, the following insights and findings were derived:

1. Consistency of point estimates tends to improve with larger sample sizes. This means that as $n$ increases, the likelihood of point estimates converging to the true population parameter values also increases.
2. For parameters of the EOIW-W model distribution, the RAB and MSE decrease as sample size $n$ grows.
3. Larger samples typically result in narrower confidence intervals (LCI). This reflects a higher level of confidence in the precision of estimates. It also indicates that as the sample size increases, the range within which the true parameter value likely falls becomes more compact.
4. The best method of estimation is the Bayesian estimation.
5. While larger sample sizes yield more accurate estimates, they can also increase computational demands. Some estimation methods, especially those involving MCMC, may require more computational resources and time with larger samples.
6. The BEs under LINEX loss function have smallest RAB, and MSE.

## 6. Analysis to Real Data

For special model of EOIW-G family of distributions, called the EOIWW distribution, we present two real data sets to illustrate the performance and flexibility of the EOIW-W distribution with other competitive distributions namely: Additive Weibull(AW) [46], new modified Weibull (NMW) [47], Weibull Weibull (WW) [4], IW Weibull (IWW) [12], modified WeibullWeibull (MWW) [13] and Weibull (W) for data modeling.

The goodness of fit of the fitted distributions are measured by using some analytical measures called -2 log-likelihood function, say ($H_1$), Akaike information criterion, say ($H_2$), corrected Akaike information criteria, say ($H_3$), Bayesian information criterion, say ($H_4$), and Kolmogorov-Smirnov, say ($H_5$) statistic. The smallest value of criteria, is the best fit distribution corresponds to data.

**Data Set 1:**

The first data set is provided in Ref. [48] about time between failures for repairable item. The data are listed as the following:

1.43, 0.11, 0.63, 0.71, 0.77, 1.23, 2.63, 1.49, 1.24, 3.46, 2.46, 1.97, 0.59, 0.74, 1.86, 1.23, 0.94, 1.17, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37.

Table 5 discussed MLE and BE for parameters of EOIWW distribution. The analytical measures are given in Table 6 shows that the EOIWW distribution has the smallest measure values compared with those values of the other distributions. So, the EOIWW distribution is the best fit than the other fitted distributions for this data.

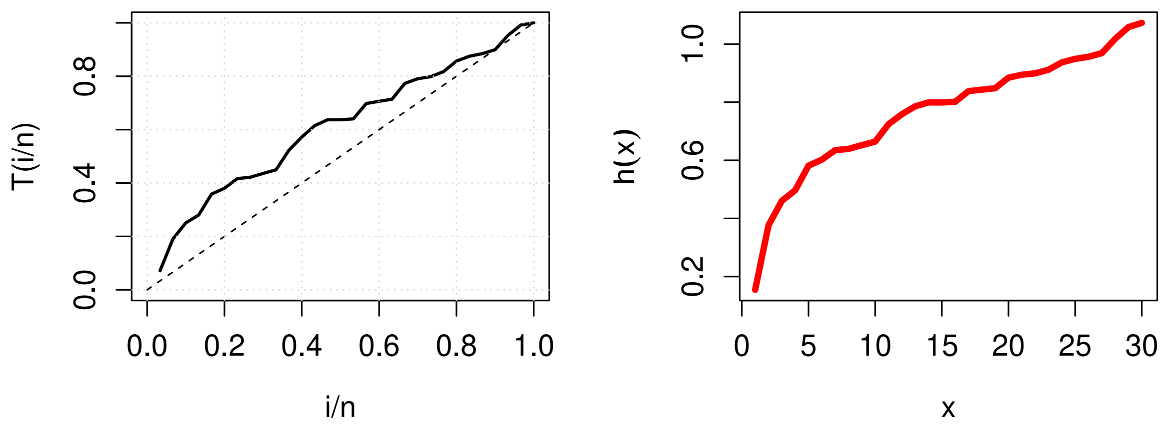


**Figure 3.** TTT plot and HF of EOIWW distribution

Figure 3 gives TTT (Time-to-Failure) plot and HF of the EOIWW distribution, which a TTT plot, also known as a ”survival plot” or ”reliability plot,” is a graphical representation used to visualize the time to failure or time to an event for a items of data set. By Figure 3, we can note the TTT line and the HF line are increases. When both the TTT line and the hazard rate line of EOIWW distribution are increasing, it suggests that the model and data set under analysis are experiencing an increasing rate of events or failures as time goes. Figure 4 discussed non-parameter plots distribution for data set 1 as Box-plot, Violine, and Kernal density. As well as, Figure 5 illustrates the PDF, empirical CDFs and probability plots, respectively, of the EOIWW distribution. Figures 4 and 5 confirm that the EOIWW distribution is fitting for data set 1. Therefore, we should be confirm that the EOIWW distribution is fit of this data set 1.

Table 6 presents a comparison of MLE and BEs using standard error (SE) as the metric. It’s worth noting that BEs demonstrates smaller SE values compared to MLEs.

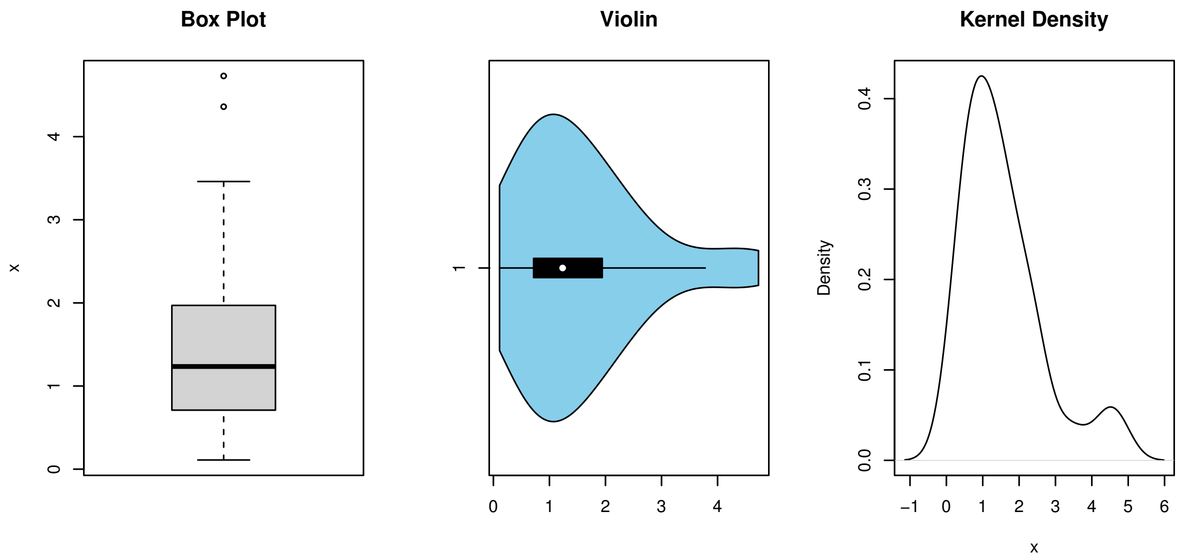


**Figure 4.** Non-parameter plots distribution for data set 1

Figure 6 displays trace plots and convergence plots for the parameters obtained using the MCMC technique for the EOIWW distribution. In the figure, the posterior density of MCMC results for each parameter is depicted in the center, showing a symmetric normal distribution that closely resembles the proposed distribution. Figure 7 discussed profile likelihood for parameters of EOIW-W distribution for data set 1.

**Data Set 2:**

The second data set represents 34 observations of vinyl chloride data obtained from clean up gradient ground-water monitoring wells in mg/L. the data are obtained from [49] and recorded as follows:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

Results in Table 7, indicate that the EOIWW distribution is the better fit than the other competitive distributions for this data set based on the selected analytical measures.

Figure 8 displays a TTT plot and the HF for the EOIWW distribution. The TTT plot, provides a visual representation for assessing the time to failure or time to an event within a dataset 2. In Figure 8, an decreasing TTT line that falls below a central line suggests that the cumulative probability of an event (typically a failure) is decreasing over time, and it's doing so at a rate faster than expected or compared to a reference line (the central line). This indicates that the system or process is experiencing more failures at a higher rate than anticipated or compared to a standard performance. A J-shaped HF line implies a non-constant hazard rate. Initially, the hazard rate is relatively low, indicating a period of lower risk. However, as time progresses, it sharply increases, signifying an escalating risk of events or failures. This J-shaped pattern is characteristic of certain types of distributions, such as the Weibull distribution. In summary, when the TTT line is both increasing and situated below the central line, and the HF line has a J-shaped pattern, it suggests that the system or data under analysis is experiencing an accelerating rate of events or failures as time advances. This could indicate a system that is deteriorating or becoming less reliable over time, and the increasing hazard rate signifies

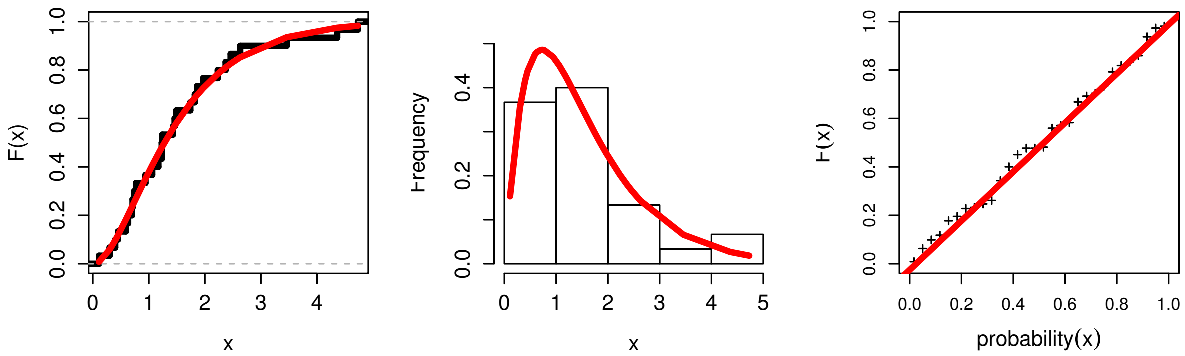


**Figure 5.** Fitting plots of EOIWW distribution: data set 1

a sudden surge in the risk of events or failures after an initial period of lower risk. This may call for closer monitoring and possible corrective actions to address the rising failure rate. Additionally, Figure 8 presents the PDF, empirical CDFs, and probability plots for the EOIWW distribution. By examining Figures 8, 9 and 10, we can confirm that the EOIWW distribution is a suitable fit for data set 2. Consequently, we can assert with confidence that the EOIWW distribution is appropriate for this specific dataset 2. Figure 9 discussed non-parameter plots distribution for data set 2 as Box-plot, Violine, and Kernal density.

Table 8 provides a side-by-side evaluation of MLE and BE using the SE as the measure. Notably, the BEs exhibit smaller SE values in comparison to the MLEs. Figure 11 exhibits profile for parameters of EOIWW distribution for data set 2. Figure 12 exhibits trace plots and convergence plots for the parameters derived through the MCMC method for the EOIWW distribution. Within the figure, the central representation illustrates the posterior density of MCMC results for each parameter, revealing a symmetric normal distribution closely resembling the proposed distribution.

**Table 2.** ML and Bayesian estimation based on symmetric and asymetic loss function: Case 1 and 2

| | | | MLE | | | | SELF | | | LINEX c=-0.5 | | | LINEX c=1.25 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Case | n | | RAB | MSE | LACI | CP | RAB | MSE | LCCI | RAB | MSE | LCCI | RAB | MSE | LCCI |
| 1 | 35 | $\theta$ | 1.6462 | 3.0418 | 4.4580 | 98.2% | 0.1459 | 0.0778 | 0.9403 | 0.1717 | 0.0856 | 0.9610 | 0.0827 | 0.0613 | 0.8761 |
| | | $\alpha$ | 0.1217 | 0.0164 | 0.4423 | 94.8% | 0.2108 | 0.0153 | 0.4079 | 0.2050 | 0.0255 | 0.4122 | 0.2248 | 0.0254 | 0.3887 |
| | | $\beta$ | 0.4523 | 0.1449 | 4.4580 | 98.2% | 0.1761 | 0.0287 | 0.5848 | 0.1703 | 0.0292 | 0.5941 | 0.1901 | 0.0276 | 0.5501 |
| | | $\mu$ | 0.1825 | 0.0363 | 0.4423 | 94.8% | 0.1485 | 0.0087 | 0.3257 | 0.1142 | 0.0089 | 0.3306 | 0.0631 | 0.0085 | 0.3174 |
| | | $\lambda$ | 0.3835 | 0.1444 | 4.4580 | 98.2% | 0.3077 | 0.1086 | 1.1006 | 0.3097 | 0.1207 | 1.1247 | 0.2304 | 0.0829 | 1.0093 |
| | 60 | $\theta$ | 1.3834 | 2.4995 | 2.7498 | 91.8% | 0.0288 | 0.0195 | 0.5227 | 0.0027 | 0.0198 | 0.5268 | 0.0168 | 0.0190 | 0.5196 |
| | | $\alpha$ | 0.0899 | 0.0099 | 0.3473 | 94.8% | 0.2029 | 0.0082 | 0.2317 | 0.2028 | 0.0177 | 0.2321 | 0.2134 | 0.0182 | 0.2316 |
| | | $\beta$ | 0.4463 | 0.1290 | 2.7498 | 91.8% | 0.1621 | 0.0158 | 0.2684 | 0.1621 | 0.0157 | 0.2681 | 0.1822 | 0.0161 | 0.2697 |
| | | $\mu$ | 0.0944 | 0.0153 | 0.3473 | 94.8% | 0.1224 | 0.0077 | 0.2310 | 0.1121 | 0.0076 | 0.2311 | 0.0613 | 0.0079 | 0.2304 |
| | | $\lambda$ | 0.3850 | 0.1001 | 2.7498 | 91.8% | 0.1191 | 0.0256 | 0.5891 | 0.1251 | 0.0265 | 0.5983 | 0.1039 | 0.0236 | 0.5722 |
| | 100 | $\theta$ | 1.2539 | 2.3829 | 1.6742 | 95.0% | 0.0237 | 0.0129 | 0.4319 | 0.0023 | 0.0129 | 0.4350 | 0.0146 | 0.0130 | 0.4302 |
| | | $\alpha$ | 0.0607 | 0.0063 | 0.2881 | 94.3% | 0.1924 | 0.0062 | 0.1489 | 0.2004 | 0.0161 | 0.1489 | 0.2045 | 0.0166 | 0.1488 |
| | | $\beta$ | 0.4269 | 0.1274 | 1.6742 | 95.0% | 0.1521 | 0.0131 | 0.1894 | 0.1521 | 0.0130 | 0.1887 | 0.1802 | 0.0133 | 0.1920 |
| | | $\mu$ | 0.0330 | 0.0053 | 0.2881 | 94.3% | 0.1140 | 0.0051 | 0.2051 | 0.1038 | 0.0075 | 0.2062 | 0.0601 | 0.0077 | 0.2011 |
| | | $\lambda$ | 0.3464 | 0.0910 | 1.6742 | 95.0% | 0.1025 | 0.0159 | 0.4065 | 0.1209 | 0.0164 | 0.4138 | 0.1014 | 0.0147 | 0.3970 |
| | 200 | $\theta$ | 1.2053 | 2.0928 | 1.5392 | 96.3% | 0.0236 | 0.0056 | 0.2730 | 0.0023 | 0.0056 | 0.2695 | 0.0139 | 0.0057 | 0.2770 |
| | | $\alpha$ | 0.0506 | 0.0032 | 0.1971 | 95.8% | 0.1825 | 0.0030 | 0.0769 | 0.1925 | 0.0156 | 0.0766 | 0.2005 | 0.0159 | 0.0785 |
| | | $\beta$ | 0.3973 | 0.1132 | 1.3917 | 96.3% | 0.1492 | 0.0102 | 0.1210 | 0.1491 | 0.0102 | 0.1201 | 0.1793 | 0.0103 | 0.1230 |
| | | $\mu$ | 0.0240 | 0.0019 | 0.1971 | 95.8% | 0.1035 | 0.0018 | 0.1309 | 0.1016 | 0.0075 | 0.1308 | 0.0596 | 0.0077 | 0.1312 |
| | | $\lambda$ | 0.3066 | 0.0910 | 1.3917 | 96.3% | 0.1022 | 0.0087 | 0.2727 | 0.1124 | 0.0089 | 0.2767 | 0.1002 | 0.0082 | 0.2670 |
| 2 | 35 | $\theta$ | 0.0381 | 0.2762 | 2.0618 | 94.5% | 0.1603 | 0.0692 | 0.9156 | 0.1868 | 0.0767 | 0.9455 | 0.0968 | 0.0543 | 0.8438 |
| | | $\alpha$ | 0.1202 | 0.0444 | 0.7927 | 95.0% | 0.0692 | 0.0223 | 0.5350 | 0.0790 | 0.0235 | 0.5441 | 0.0453 | 0.0195 | 0.5044 |
| | | $\beta$ | 0.3705 | 0.1467 | 2.0618 | 94.5% | 0.1222 | 0.0392 | 0.6266 | 0.1373 | 0.0423 | 0.6511 | 0.0861 | 0.0327 | 0.5892 |
| | | $\mu$ | 0.1762 | 0.3986 | 0.7927 | 95.0% | 0.0045 | 0.1356 | 1.4873 | 0.0059 | 0.1377 | 1.4932 | 0.0300 | 0.1356 | 1.4654 |
| | | $\lambda$ | 0.0305 | 0.1501 | 2.0618 | 94.5% | 0.0040 | 0.1492 | 1.5187 | 0.0154 | 0.1642 | 1.5256 | 0.0241 | 0.1540 | 1.4847 |
| | 60 | $\theta$ | 0.0379 | 0.2064 | 1.7810 | 94.3% | 0.0079 | 0.0148 | 0.4763 | 0.0129 | 0.0150 | 0.4825 | 0.0046 | 0.0143 | 0.4615 |
| | | $\alpha$ | 0.1074 | 0.0317 | 0.6655 | 96.8% | 0.0107 | 0.0053 | 0.2770 | 0.0131 | 0.0053 | 0.2799 | 0.0047 | 0.0051 | 0.2706 |
| | | $\beta$ | 0.2664 | 0.0989 | 1.7810 | 94.3% | 0.0229 | 0.0081 | 0.3397 | 0.0258 | 0.0082 | 0.3414 | 0.0155 | 0.0078 | 0.3393 |
| | | $\mu$ | 0.1407 | 0.2888 | 0.6655 | 96.8% | 0.0045 | 0.0253 | 0.6013 | 0.0057 | 0.0254 | 0.6005 | 0.0290 | 0.0253 | 0.5973 |
| | | $\lambda$ | 0.0301 | 0.1466 | 1.7810 | 94.3% | 0.0016 | 0.0225 | 0.5985 | 0.0030 | 0.0226 | 0.6013 | 0.0056 | 0.0224 | 0.5957 |
| | 100 | $\theta$ | 0.0347 | 0.1645 | 1.5888 | 95.5% | 0.0075 | 0.0113 | 0.4129 | 0.0042 | 0.0114 | 0.4139 | 0.0046 | 0.0111 | 0.4099 |
| | | $\alpha$ | 0.0512 | 0.0227 | 0.5819 | 96.2% | 0.0080 | 0.0040 | 0.2429 | 0.0097 | 0.0041 | 0.2438 | 0.0040 | 0.0040 | 0.2387 |
| | | $\beta$ | 0.1500 | 0.0582 | 1.5888 | 95.5% | 0.0210 | 0.0054 | 0.2869 | 0.0230 | 0.0055 | 0.2895 | 0.0146 | 0.0053 | 0.2841 |
| | | $\mu$ | 0.0838 | 0.1725 | 0.5819 | 96.2% | 0.0015 | 0.0158 | 0.4797 | 0.0046 | 0.0157 | 0.4812 | 0.0040 | 0.0159 | 0.4861 |
| | | $\lambda$ | 0.0235 | 0.1374 | 1.5888 | 95.5% | 0.0015 | 0.0168 | 0.5221 | 0.0017 | 0.0168 | 0.5207 | 0.0055 | 0.0170 | 0.5218 |
| | 200 | $\theta$ | 0.0030 | 0.0796 | 1.1068 | 95.0% | 0.0068 | 0.0047 | 0.2602 | 0.0042 | 0.0047 | 0.2620 | 0.0042 | 0.0047 | 0.2591 |
| | | $\alpha$ | 0.0291 | 0.0125 | 0.4340 | 94.3% | 0.0036 | 0.0017 | 0.1548 | 0.0043 | 0.0017 | 0.1549 | 0.0019 | 0.0017 | 0.1543 |
| | | $\beta$ | 0.0878 | 0.0262 | 1.1068 | 95.0% | 0.0055 | 0.0019 | 0.1657 | 0.0063 | 0.0019 | 0.1650 | 0.0036 | 0.0019 | 0.1659 |
| | | $\mu$ | 0.0418 | 0.0742 | 0.4340 | 94.3% | 0.0014 | 0.0066 | 0.3081 | 0.0023 | 0.0066 | 0.3086 | 0.0007 | 0.0065 | 0.3063 |
| | | $\lambda$ | 0.0077 | 0.0448 | 1.1068 | 95.0% | 0.0000 | 0.0067 | 0.3241 | 0.0005 | 0.0067 | 0.3236 | 0.0011 | 0.0067 | 0.3235 |

**Table 3.** ML and Bayesian estimation based on symmetric and asymetic loss function: Case 3 and 4

| | | | MLE | | | | SELF | | | LINEX c=-0.5 | | | LINEX c=1.25 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Case | n | | RAB | MSE | LACI | CP | RAB | MSE | LCCI | RAB | MSE | LCCI | RAB | MSE | LCCI |
| 3 | 35 | $\theta$ | 0.0248 | 0.4001 | 2.4812 | 94.3% | 0.1180 | 0.0711 | 0.9571 | 0.1382 | 0.0739 | 0.9795 | 0.0568 | 0.0564 | 0.8787 |
| | | $\alpha$ | 0.0740 | 0.0711 | 1.0358 | 93.3% | 0.0970 | 0.0271 | 0.5551 | 0.1073 | 0.0291 | 0.5705 | 0.0722 | 0.0230 | 0.5291 |
| | | $\beta$ | 0.1141 | 0.5565 | 2.4812 | 94.3% | 0.0040 | 0.1977 | 1.7261 | 0.0114 | 0.2021 | 1.7409 | 0.0311 | 0.1923 | 1.6789 |
| | | $\mu$ | 0.2577 | 0.7406 | 1.0358 | 93.3% | 0.0020 | 0.1515 | 1.4414 | 0.0121 | 0.1551 | 1.4574 | 0.0231 | 0.1462 | 1.4003 |
| | | $\lambda$ | 0.0715 | 0.1344 | 2.4812 | 94.3% | 0.0140 | 0.0837 | 1.1842 | 0.0068 | 0.0840 | 1.1699 | 0.0317 | 0.0857 | 1.1678 |
| | 60 | $\theta$ | 0.0231 | 0.3198 | 2.1994 | 94.2% | 0.0292 | 0.0156 | 0.4956 | 0.0080 | 0.0158 | 0.4961 | 0.0096 | 0.0151 | 0.4893 |
| | | $\alpha$ | 0.0388 | 0.0500 | 0.8737 | 93.5% | 0.0174 | 0.0050 | 0.2483 | 0.0197 | 0.0051 | 0.2483 | 0.0116 | 0.0048 | 0.2457 |
| | | $\beta$ | 0.0612 | 0.4183 | 2.1994 | 94.2% | 0.0031 | 0.0255 | 0.5920 | 0.0048 | 0.0256 | 0.5919 | 0.0012 | 0.0254 | 0.5849 |
| | | $\mu$ | 0.1630 | 0.4306 | 0.8737 | 93.5% | 0.0017 | 0.0244 | 0.5763 | 0.0092 | 0.0244 | 0.5748 | 0.0058 | 0.0247 | 0.5809 |
| | | $\lambda$ | 0.0372 | 0.1090 | 2.1994 | 94.2% | 0.0017 | 0.0168 | 0.5076 | 0.0035 | 0.0168 | 0.5090 | 0.0050 | 0.0168 | 0.5071 |
| | 100 | $\theta$ | 0.0192 | 0.2191 | 1.8275 | 95.0% | 0.0209 | 0.0102 | 0.3986 | 0.0078 | 0.0102 | 0.3981 | 0.0093 | 0.0101 | 0.3999 |
| | | $\alpha$ | 0.0341 | 0.0343 | 0.7217 | 94.8% | 0.0163 | 0.0029 | 0.2124 | 0.0177 | 0.0030 | 0.2130 | 0.0103 | 0.0029 | 0.2114 |
| | | $\beta$ | 0.0526 | 0.3233 | 1.8275 | 95.0% | 0.0005 | 0.0162 | 0.4897 | 0.0006 | 0.0162 | 0.4873 | 0.0010 | 0.0163 | 0.4894 |
| | | $\mu$ | 0.1263 | 0.3205 | 0.7217 | 94.8% | 0.0015 | 0.0147 | 0.4945 | 0.0013 | 0.0148 | 0.4945 | 0.0048 | 0.0146 | 0.4909 |
| | | $\lambda$ | 0.0269 | 0.0858 | 1.8275 | 95.0% | 0.0015 | 0.0115 | 0.4180 | 0.0032 | 0.0115 | 0.4171 | 0.0041 | 0.0118 | 0.4219 |
| | 200 | $\theta$ | 0.0164 | 0.1397 | 1.4609 | 94.5% | 0.0048 | 0.0049 | 0.2798 | 0.0062 | 0.0049 | 0.2796 | 0.0011 | 0.0048 | 0.2778 |
| | | $\alpha$ | 0.0166 | 0.0196 | 0.5480 | 95.3% | 0.0011 | 0.0011 | 0.1288 | 0.0016 | 0.0011 | 0.1292 | 0.0004 | 0.0011 | 0.1280 |
| | | $\beta$ | 0.0130 | 0.2065 | 1.4609 | 94.5% | 0.0004 | 0.0078 | 0.3360 | 0.0009 | 0.0078 | 0.3370 | 0.0008 | 0.0078 | 0.3361 |
| | | $\mu$ | 0.0766 | 0.1793 | 0.5480 | 95.3% | 0.0004 | 0.0063 | 0.3083 | 0.0000 | 0.0063 | 0.3097 | 0.0015 | 0.0063 | 0.3101 |
| | | $\lambda$ | 0.0196 | 0.0700 | 1.4609 | 94.5% | 0.0015 | 0.0053 | 0.2896 | 0.0018 | 0.0053 | 0.2900 | 0.0005 | 0.0053 | 0.2904 |
| 4 | 35 | $\theta$ | 0.0444 | 0.1475 | 1.9506 | 94.0% | 0.0083 | 0.1439 | 1.7055 | 0.0217 | 0.2046 | 1.7145 | 0.0244 | 0.1945 | 1.6659 |
| | | $\alpha$ | 0.0132 | 0.2686 | 2.0307 | 95.2% | 0.0121 | 0.1288 | 1.3516 | 0.0220 | 0.1334 | 1.3652 | 0.0121 | 0.1216 | 1.2810 |
| | | $\beta$ | 0.0319 | 0.0422 | 1.8506 | 94.0% | 0.0073 | 0.0418 | 1.6283 | 0.0052 | 0.1815 | 1.6320 | 0.0374 | 0.1735 | 1.5558 |
| | | $\mu$ | 0.1084 | 0.3755 | 2.0307 | 95.2% | 0.0233 | 0.1080 | 1.2537 | 0.0322 | 0.1144 | 1.2671 | 0.0045 | 0.0966 | 1.1963 |
| | | $\lambda$ | 0.0129 | 0.0096 | 1.5061 | 94.0% | 0.0085 | 0.0094 | 0.7510 | 0.0056 | 0.0367 | 0.7483 | 0.0156 | 0.0366 | 0.7550 |
| | 60 | $\theta$ | 0.0221 | 0.1205 | 1.7659 | 94.8% | 0.0023 | 0.0255 | 0.6422 | 0.0040 | 0.0256 | 0.6431 | 0.0019 | 0.0256 | 0.6339 |
| | | $\alpha$ | 0.0022 | 0.2332 | 2.0026 | 96.0% | 0.0055 | 0.0226 | 0.5909 | 0.0040 | 0.0226 | 0.5907 | 0.0093 | 0.0228 | 0.5933 |
| | | $\beta$ | 0.0309 | 0.0417 | 1.7659 | 94.8% | 0.0062 | 0.0275 | 0.6348 | 0.0023 | 0.0276 | 0.6301 | 0.0036 | 0.0275 | 0.6472 |
| | | $\mu$ | 0.1000 | 0.3481 | 2.0026 | 96.0% | 0.0020 | 0.0202 | 0.5784 | 0.0012 | 0.0202 | 0.5775 | 0.0037 | 0.0204 | 0.5790 |
| | | $\lambda$ | 0.0116 | 0.0081 | 1.4766 | 94.8% | 0.0018 | 0.0058 | 0.3033 | 0.0023 | 0.0058 | 0.3035 | 0.0007 | 0.0058 | 0.3021 |
| | 100 | $\theta$ | 0.0016 | 0.0435 | 0.8181 | 94.5% | 0.0021 | 0.0182 | 0.5238 | 0.0014 | 0.0182 | 0.5228 | 0.0016 | 0.0184 | 0.5298 |
| | | $\alpha$ | 0.0021 | 0.0955 | 1.2104 | 94.5% | 0.0014 | 0.0150 | 0.4876 | 0.0004 | 0.0150 | 0.4901 | 0.0040 | 0.0151 | 0.4872 |
| | | $\beta$ | 0.0105 | 0.0127 | 0.8181 | 94.5% | 0.0056 | 0.0117 | 0.5110 | 0.0005 | 0.0167 | 0.5055 | 0.0033 | 0.0167 | 0.5113 |
| | | $\mu$ | 0.0386 | 0.1117 | 1.2104 | 94.5% | 0.0019 | 0.0114 | 0.4634 | 0.0012 | 0.0136 | 0.4631 | 0.0036 | 0.0136 | 0.4687 |
| | | $\lambda$ | 0.0049 | 0.0039 | 0.8181 | 94.5% | 0.0005 | 0.0031 | 0.2196 | 0.0008 | 0.0031 | 0.2197 | 0.0002 | 0.0031 | 0.2192 |
| | 200 | $\theta$ | 0.0015 | 0.0372 | 0.7560 | 94.3% | 0.0014 | 0.0063 | 0.2878 | 0.0013 | 0.0063 | 0.2874 | 0.0003 | 0.0063 | 0.2899 |
| | | $\alpha$ | 0.0020 | 0.0815 | 1.1194 | 94.5% | 0.0014 | 0.0059 | 0.3046 | 0.0003 | 0.0059 | 0.3038 | 0.0004 | 0.0059 | 0.3069 |
| | | $\beta$ | 0.0093 | 0.0119 | 0.7560 | 94.3% | 0.0017 | 0.0066 | 0.3171 | 0.0004 | 0.0066 | 0.3162 | 0.0028 | 0.0066 | 0.3175 |
| | | $\mu$ | 0.0259 | 0.0847 | 1.1194 | 94.5% | 0.0008 | 0.0055 | 0.2900 | 0.0004 | 0.0055 | 0.2894 | 0.0018 | 0.0055 | 0.2908 |
| | | $\lambda$ | 0.0029 | 0.0036 | 0.7560 | 94.3% | 0.0001 | 0.0013 | 0.1417 | 0.0002 | 0.0013 | 0.1420 | 0.0002 | 0.0013 | 0.1411 |

**Table 4.** ML and Bayesian estimation based on symmetric and asymetic loss function: Case 5 and 6

| | | | MLE | | | | SELF | | | LINEX c=-0.5 | | | LINEX c=1.25 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Case | n | | RAB | MSE | LACI | CP | RAB | MSE | LCCI | RAB | MSE | LCCI | RAB | MSE | LCCI |
| 5 | 35 | $\theta$ | 0.0881 | 0.6282 | 3.1079 | 94.3% | 0.0925 | 0.1180 | 1.2924 | 0.0248 | 0.1242 | 1.3086 | 0.0516 | 0.1074 | 1.2438 |
| | | $\alpha$ | 0.0863 | 0.0871 | 1.1399 | 93.2% | 0.0967 | 0.0270 | 0.5772 | 0.1063 | 0.0289 | 0.5876 | 0.0733 | 0.0231 | 0.5429 |
| | | $\beta$ | 0.1747 | 0.8526 | 3.1079 | 94.3% | 0.0188 | 0.2074 | 1.6936 | 0.0058 | 0.2113 | 1.7077 | 0.0503 | 0.2060 | 1.6287 |
| | | $\mu$ | 0.2606 | 1.5598 | 1.1399 | 93.2% | 0.0075 | 0.1985 | 1.6484 | 0.0093 | 0.2010 | 1.6359 | 0.0209 | 0.1994 | 1.6188 |
| | | $\lambda$ | 0.0446 | 0.0539 | 3.1079 | 94.3% | 0.0198 | 0.0394 | 0.7000 | 0.0156 | 0.0390 | 0.6999 | 0.0303 | 0.0416 | 0.7170 |
| | 60 | $\theta$ | 0.0870 | 0.6037 | 3.0117 | 95.5% | 0.0732 | 0.0242 | 0.6248 | 0.0098 | 0.0243 | 0.6205 | 0.0483 | 0.0241 | 0.6237 |
| | | $\alpha$ | 0.0508 | 0.0646 | 0.9898 | 93.7% | 0.0176 | 0.0055 | 0.2862 | 0.0198 | 0.0055 | 0.2887 | 0.0120 | 0.0053 | 0.2829 |
| | | $\beta$ | 0.1059 | 0.6013 | 3.1172 | 95.5% | 0.0049 | 0.0263 | 0.6336 | 0.0032 | 0.0263 | 0.6347 | 0.0093 | 0.0267 | 0.6293 |
| | | $\mu$ | 0.1626 | 0.9274 | 0.9898 | 93.7% | 0.0069 | 0.0280 | 0.6487 | 0.0057 | 0.0278 | 0.6488 | 0.0099 | 0.0286 | 0.6493 |
| | | $\lambda$ | 0.0276 | 0.0406 | 3.1172 | 95.5% | 0.0013 | 0.0076 | 0.3392 | 0.0021 | 0.0077 | 0.3397 | 0.0017 | 0.0076 | 0.3397 |
| | 100 | $\theta$ | 0.0734 | 0.3448 | 2.3007 | 95.0% | 0.0120 | 0.0169 | 0.5069 | 0.0097 | 0.0168 | 0.5065 | 0.0178 | 0.0171 | 0.5042 |
| | | $\alpha$ | 0.0412 | 0.0430 | 0.8077 | 95.2% | 0.0131 | 0.0032 | 0.2060 | 0.0144 | 0.0032 | 0.2066 | 0.0097 | 0.0031 | 0.2046 |
| | | $\beta$ | 0.0783 | 0.4242 | 2.3007 | 95.0% | 0.0013 | 0.0188 | 0.5311 | 0.0013 | 0.0188 | 0.5274 | 0.0041 | 0.0190 | 0.5314 |
| | | $\mu$ | 0.1272 | 0.6389 | 0.8077 | 95.2% | 0.0028 | 0.0160 | 0.4997 | 0.0021 | 0.0159 | 0.5007 | 0.0046 | 0.0163 | 0.4940 |
| | | $\lambda$ | 0.0270 | 0.0299 | 2.3007 | 95.0% | 0.0009 | 0.0046 | 0.2585 | 0.0004 | 0.0046 | 0.2584 | 0.0016 | 0.0046 | 0.2572 |
| | 200 | $\theta$ | 0.0616 | 0.2823 | 2.0730 | 95.8% | 0.0026 | 0.0072 | 0.3321 | 0.0016 | 0.0072 | 0.3309 | 0.0051 | 0.0072 | 0.3305 |
| | | $\alpha$ | 0.0088 | 0.0297 | 0.6761 | 96.0% | 0.0016 | 0.0016 | 0.1537 | 0.0022 | 0.0016 | 0.1541 | 0.0008 | 0.0016 | 0.1527 |
| | | $\beta$ | 0.0189 | 0.3373 | 2.0730 | 95.8% | 0.0012 | 0.0072 | 0.3343 | 0.0012 | 0.0072 | 0.3326 | 0.0009 | 0.0072 | 0.3358 |
| | | $\mu$ | 0.0586 | 0.3153 | 0.6761 | 96.0% | 0.0004 | 0.0074 | 0.3262 | 0.0007 | 0.0074 | 0.3262 | 0.0003 | 0.0074 | 0.3258 |
| | | $\lambda$ | 0.0151 | 0.0289 | 2.0730 | 95.8% | 0.0008 | 0.0021 | 0.1729 | 0.0003 | 0.0021 | 0.1726 | 0.0008 | 0.0021 | 0.1734 |
| 6 | 35 | $\theta$ | 0.1673 | 0.6551 | 3.1201 | 94.0% | 0.0100 | 0.1156 | 1.3036 | 0.0301 | 0.1233 | 1.3316 | 0.0388 | 0.1015 | 1.2138 |
| | | $\alpha$ | 0.1296 | 0.0847 | 1.1000 | 94.5% | 0.1048 | 0.0414 | 0.7040 | 0.1191 | 0.0453 | 0.7155 | 0.0704 | 0.0334 | 0.6800 |
| | | $\beta$ | 0.0778 | 0.5263 | 3.1201 | 94.0% | 0.0072 | 0.1856 | 1.6165 | 0.0050 | 0.1879 | 1.6123 | 0.0374 | 0.1856 | 1.6371 |
| | | $\mu$ | 0.4130 | 0.6590 | 1.1000 | 94.5% | 0.0158 | 0.0915 | 1.1637 | 0.0272 | 0.0954 | 1.1880 | 0.0119 | 0.0846 | 1.0940 |
| | | $\lambda$ | 0.1704 | 0.1220 | 3.1201 | 94.0% | 0.0099 | 0.0666 | 1.0183 | 0.0053 | 0.0682 | 1.0166 | 0.0339 | 0.0646 | 0.9895 |
| | 60 | $\theta$ | 0.1352 | 0.4878 | 2.6981 | 95.0% | 0.0090 | 0.0224 | 0.5799 | 0.0044 | 0.0225 | 0.5832 | 0.0074 | 0.0224 | 0.5743 |
| | | $\alpha$ | 0.0817 | 0.0743 | 1.0522 | 93.8% | 0.0245 | 0.0084 | 0.3537 | 0.0270 | 0.0085 | 0.3586 | 0.0181 | 0.0081 | 0.3490 |
| | | $\beta$ | 0.0635 | 0.3923 | 2.6981 | 95.0% | 0.0043 | 0.0301 | 0.6577 | 0.0046 | 0.0303 | 0.6536 | 0.0031 | 0.0298 | 0.6575 |
| | | $\mu$ | 0.2793 | 0.3814 | 1.0522 | 93.8% | 0.0056 | 0.0184 | 0.5073 | 0.0036 | 0.0184 | 0.5089 | 0.0106 | 0.0183 | 0.5149 |
| | | $\lambda$ | 0.1260 | 0.0901 | 2.6981 | 95.0% | 0.0033 | 0.0127 | 0.4466 | 0.0052 | 0.0128 | 0.4521 | 0.0014 | 0.0125 | 0.4430 |
| | 100 | $\theta$ | 0.0983 | 0.3748 | 2.3767 | 95.7% | 0.0085 | 0.0159 | 0.4947 | 0.0042 | 0.0158 | 0.4939 | 0.0061 | 0.0160 | 0.5050 |
| | | $\alpha$ | 0.0729 | 0.0456 | 0.8202 | 95.2% | 0.0087 | 0.0043 | 0.2515 | 0.0102 | 0.0044 | 0.2528 | 0.0048 | 0.0042 | 0.2489 |
| | | $\beta$ | 0.0388 | 0.3134 | 2.3767 | 95.7% | 0.0023 | 0.0166 | 0.5056 | 0.0012 | 0.0166 | 0.5015 | 0.0025 | 0.0167 | 0.5029 |
| | | $\mu$ | 0.2107 | 0.2520 | 0.8202 | 95.2% | 0.0046 | 0.0118 | 0.4301 | 0.0035 | 0.0119 | 0.4313 | 0.0012 | 0.0116 | 0.4278 |
| | | $\lambda$ | 0.0976 | 0.0720 | 2.3767 | 95.7% | 0.0032 | 0.0079 | 0.3362 | 0.0045 | 0.0079 | 0.3399 | 0.0011 | 0.0079 | 0.3350 |
| | 200 | $\theta$ | 0.0606 | 0.2675 | 2.0178 | 94.8% | 0.0048 | 0.0063 | 0.3095 | 0.0039 | 0.0063 | 0.3108 | 0.0072 | 0.0064 | 0.3062 |
| | | $\alpha$ | 0.0200 | 0.0281 | 0.6563 | 95.2% | 0.0038 | 0.0022 | 0.1728 | 0.0045 | 0.0022 | 0.1734 | 0.0021 | 0.0021 | 0.1711 |
| | | $\beta$ | 0.0027 | 0.2494 | 2.0178 | 94.8% | 0.0005 | 0.0069 | 0.3118 | 0.0001 | 0.0069 | 0.3117 | 0.0017 | 0.0069 | 0.3128 |
| | | $\mu$ | 0.1155 | 0.1163 | 0.6563 | 95.2% | 0.0025 | 0.0050 | 0.2839 | 0.0031 | 0.0050 | 0.2836 | 0.0010 | 0.0050 | 0.2845 |
| | | $\lambda$ | 0.0675 | 0.0612 | 2.0178 | 94.8% | 0.0013 | 0.0035 | 0.2263 | 0.0008 | 0.0035 | 0.2269 | 0.0010 | 0.0035 | 0.2262 |

**Table 5.** MLE and BE for parameters of EOIWW distribution

| | ML | | Bayesian | |
|---|---|---|---|---|
| | Estimates | SE | Estimates | SE |
| $\theta$ | 108.0009 | 793.7478 | 108.3383 | 37.3926 |
| $\alpha$ | 1.1787 | 2.5512 | 1.7648 | 1.1904 |
| $\beta$ | 0.0108 | 0.0142 | 0.0211 | 0.0156 |
| $\mu$ | 1.1847 | 2.4984 | 0.9064 | 0.5091 |
| $\lambda$ | 0.6257 | 3.4024 | 1.0145 | 0.6773 |

**Table 6.** Analytical results for the data set 1

| **Model** | $\mathbf{H_1}$ | $\mathbf{H_2}$ | $\mathbf{H_3}$ | $\mathbf{H_4}$ | $\mathbf{H_5}$ |
|---|---|---|---|---|---|
| **EOIWW** | **79.2238** | **89.2238** | **91.7238** | **96.2298** | **0.064** |
| **MWW** | 145.130 | 155.130 | 157.630 | 162.136 | 0.223 |
| **NMW** | 242.501 | 250.051 | 251.651 | 255.656 | 0.942 |
| **AW** | 159.642 | 167.642 | 169.242 | 173.246 | 0.283 |
| **W** | 92.751 | 96.751 | 97.196 | 99.554 | 0.134 |

**Table 7.** Analytical results for the data set 2

| **Model** | $\mathbf{H_1}$ | $\mathbf{H_2}$ | $\mathbf{H_3}$ | $\mathbf{H_4}$ | $\mathbf{H_5}$ |
|---|---|---|---|---|---|
| **EOIWW** | **107.694** | **117.694** | **119.837** | **125.326** | **0.082** |
| **IWW** | 108.470 | 116.47 | 117.850 | 122.576 | 0.087 |
| **MWW** | 190.306 | 200.306 | 202.306 | 207.938 | 0.598 |
| **NMW** | 314.807 | 322.807 | 324.186 | 328.912 | 1 |
| **AW** | 221.798 | 229.798 | 231.178 | 235.904 | 0.982 |
| **WW** | 111.160 | 119.160 | 120.539 | 125.265 | 0.094 |
| **W** | 117.253 | 121.253 | 121.640 | 124.306 | 0.113 |

**Table 8.** MLE and BE for parameters of EOIWW distribution: data set 2

| | ML | | Bayesian | |
|---|---|---|---|---|
| | Estimates | SE | Estimates | SE |
| $\theta$ | 0.4518 | 6.2905 | 0.2578 | 0.2600 |
| $\alpha$ | 1.1216 | 5.2383 | 0.8783 | 0.4424 |
| $\beta$ | 0.0696 | 0.6054 | 0.1357 | 0.0715 |
| $\mu$ | 7.0474 | 0.1665 | 6.9482 | 0.1586 |
| $\lambda$ | 0.1231 | 0.2232 | 0.1221 | 0.0380 |

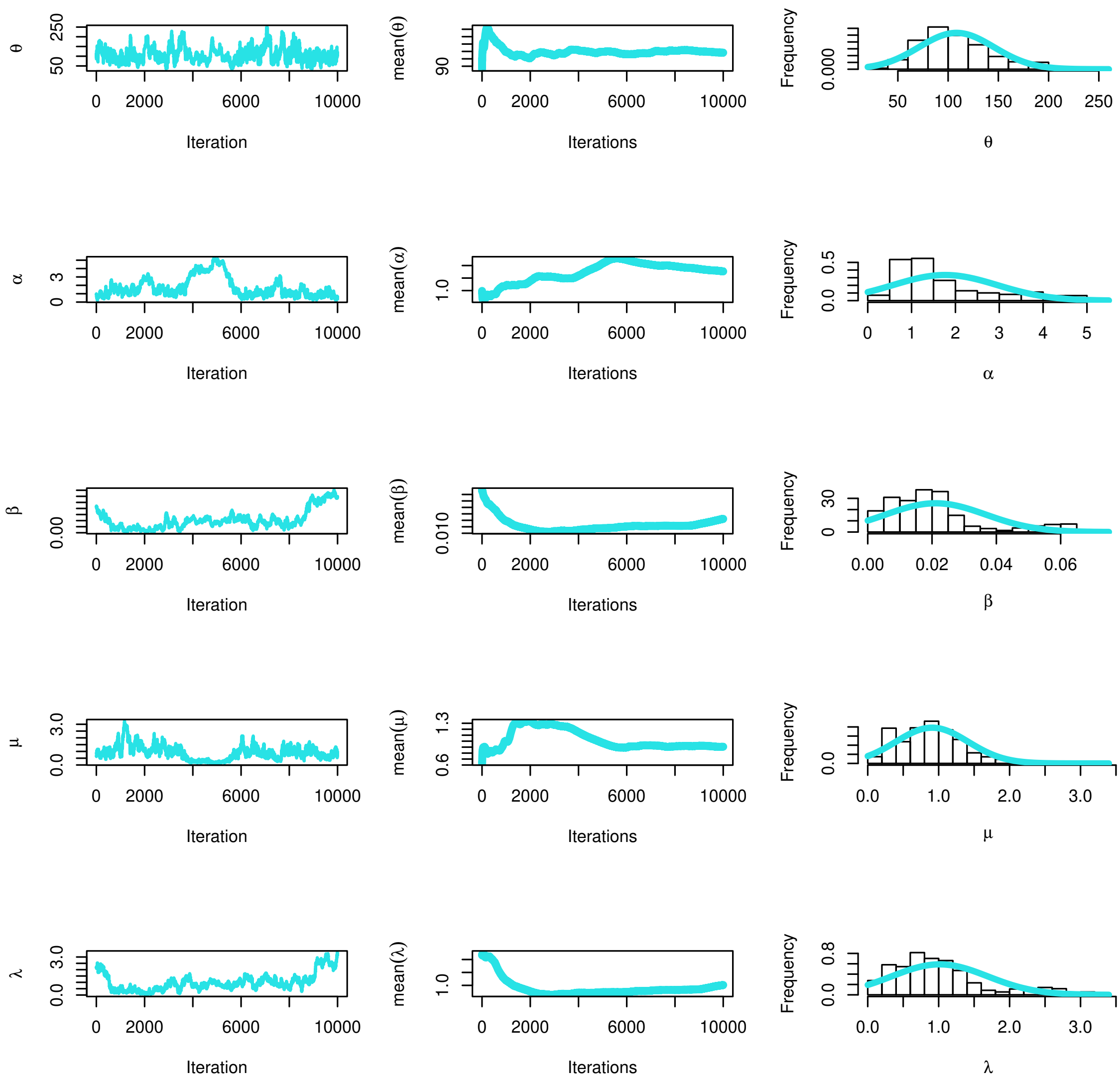

**Figure 6.** MCMC plots for parameters of EOIW-W distribution: data set 1

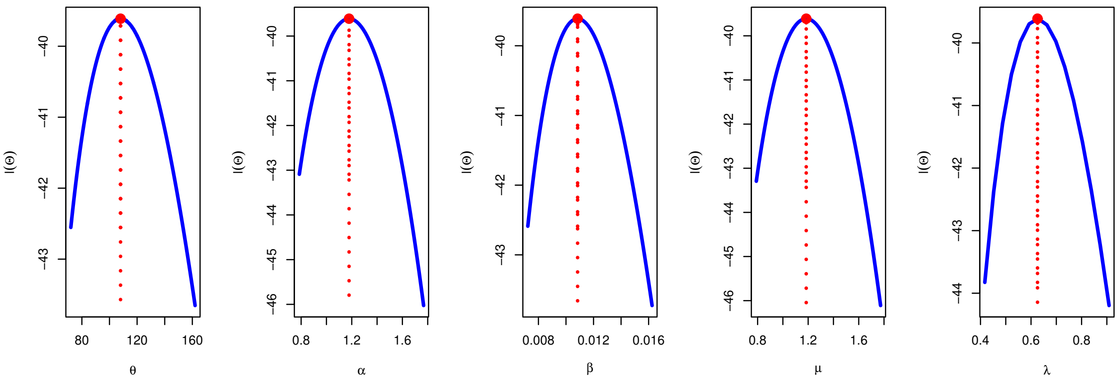


**Figure 7.** Profile for parameters of EOIW-W distribution: data set 1

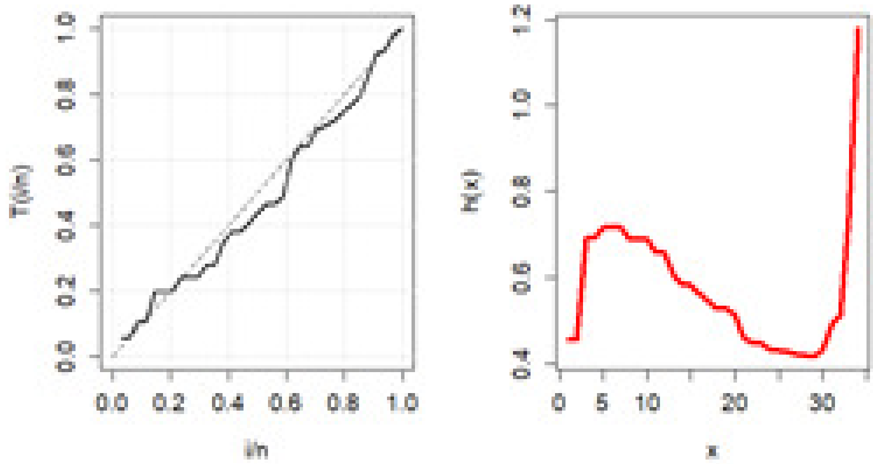


**Figure 8.** TTT plot and hazared of EOIWW distribution: data set 2

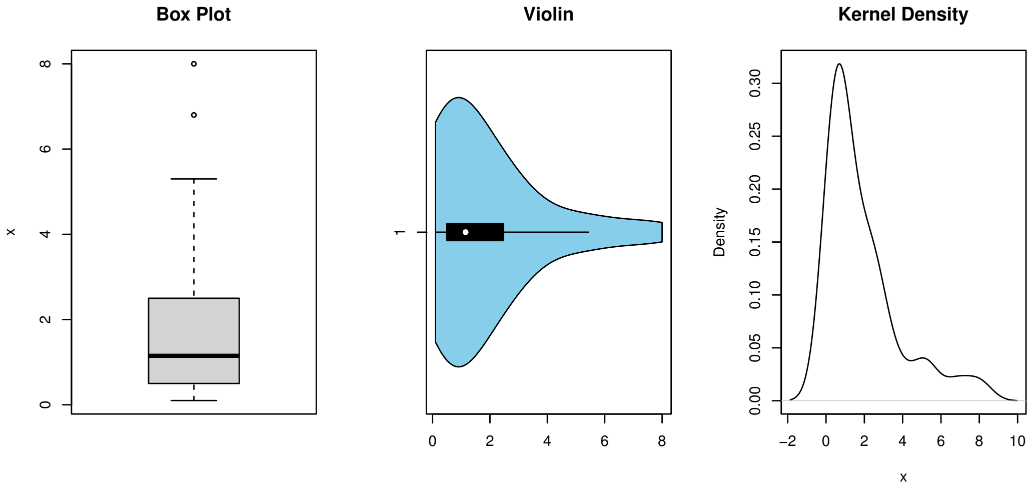


**Figure 9.** Non-parameter plots distribution for data set 2

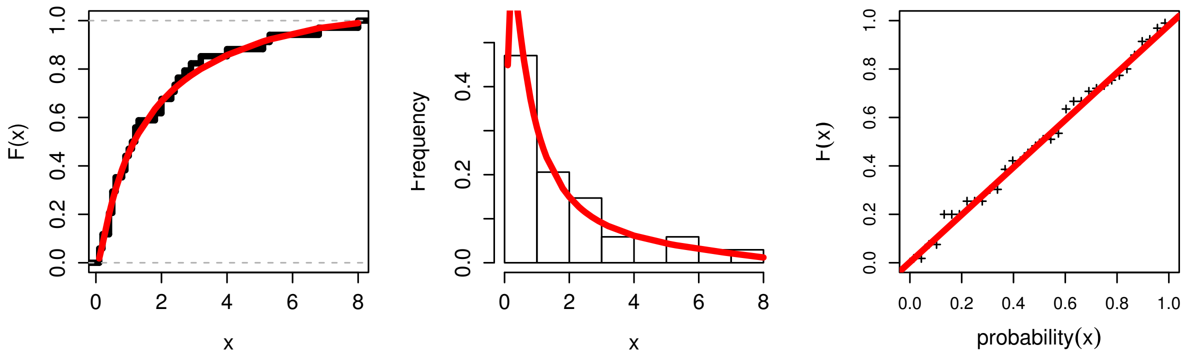


**Figure 10.** Fitting plots of EOIWW distribution: data set 2

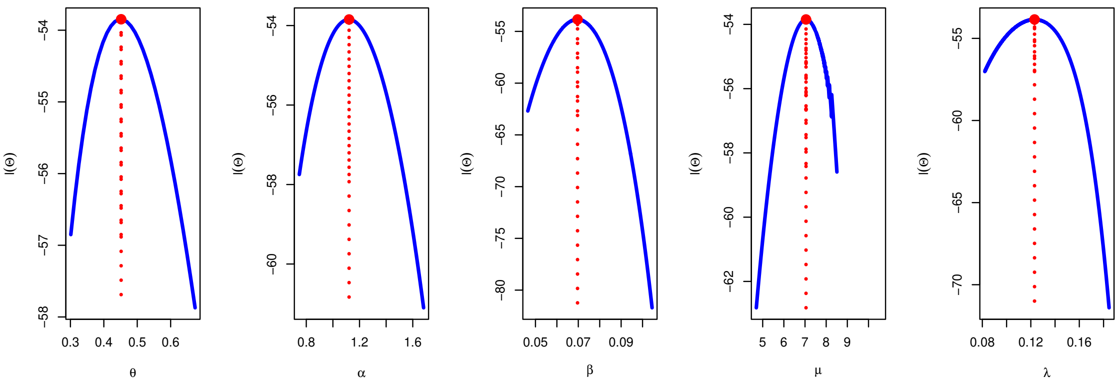


**Figure 11.** Profile for parameters of EOIWW distribution: data set 2

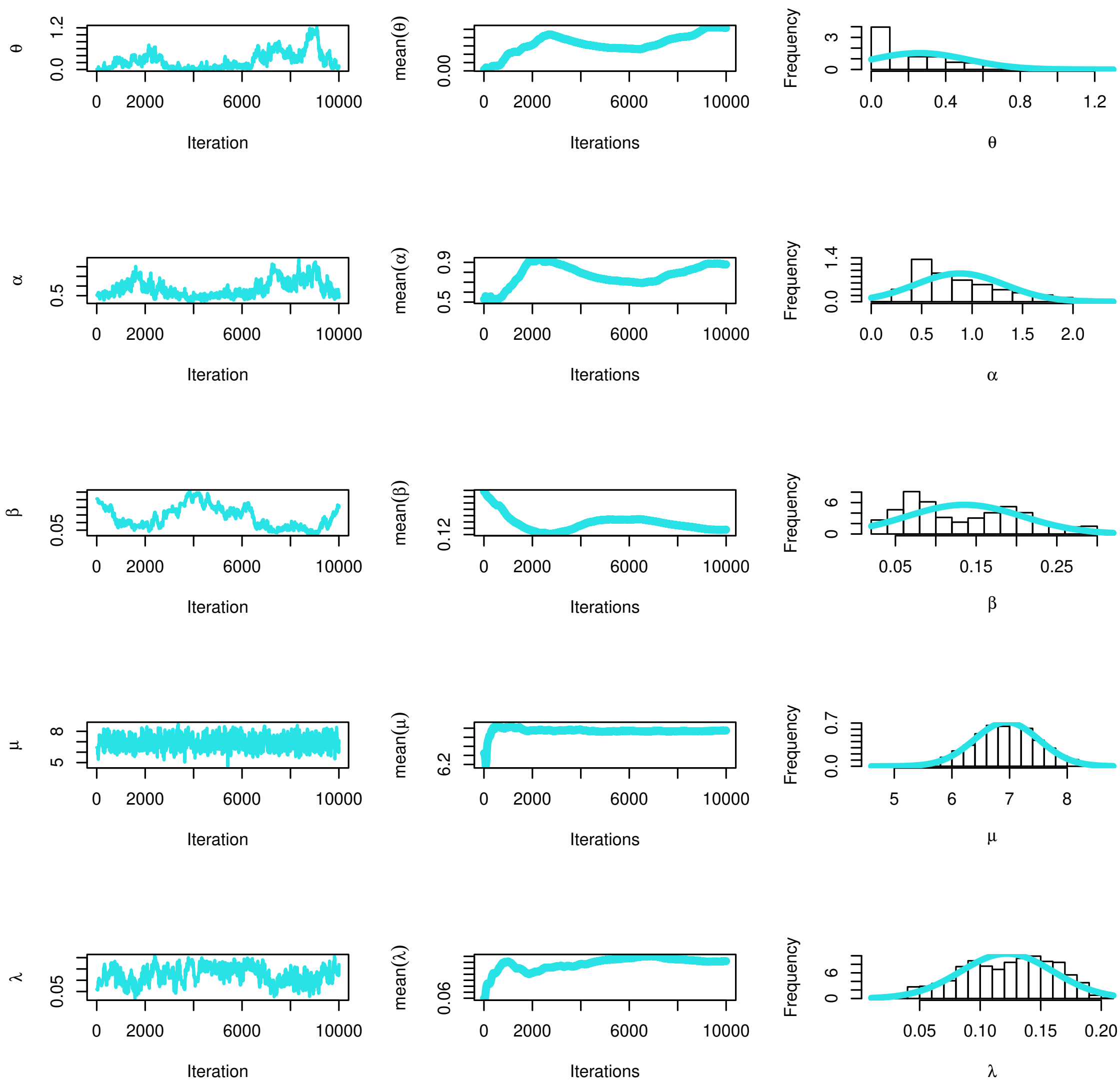


**Figure 12.** MCMC plots for parameters of EOIWW distribution: data set 2

## 7. Concluding Remarks

In this paper, we propose a novel generator to obtain better distribution flexibility called the extended odd inverse Weibull-generator. This generator is thought to be a generalization of three well-known families. Effective continuous symmetric and asymmetric models that may outperform the baseline model can be obtained from the recently created family. The statistical characteristics of the family, such as the moments, incomplete moments, entropy measure,mean deivation and density function expansion, are investigated. By employing the generated family, some popular models are offered as unique scenarios. Using the Weibull distribution as a baseline model, we study and thoroughly investigate a five-parameter special member of the extended odd inverse Weibull-G family. In order to analyse the parameters' behaviour, the parameters were examined using Bayesian and traditional methods in conjunction with a comprehensive simulation analysis. The RAB and MSEs of the estimates reduced as the sample size rose, indicating a satisfactory simulation outcome. In comparison with the MLEs, the Bayes estimates perform well. To further highlight the fitted model's adaptability and show how it performs better in practice than other well-known models according to various model selection criteria and goodness-of-fit tests, two real-world data scenarios have been presented. In closing, we suggest delving deeper into the various EOIW-G family models and associated estimating techniques, including percentile and product spacing estimation approaches, among others.

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