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A New Mixture of Two Components of Exponentiated Family with Applications to Real Life Data Sets

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Abstract: In this paper, the mixture of two components of exponentiated family is introduced as a new family of continuous distributions. Some general properties of the proposed family are discussed such as the quantile function, moments, moment generating function and order statistics. The maximum likelihood estimation method is used to derive the estimators for the unknown parameters, reliability and hazard rate functions. The mixture of two components exponentiated inverted Kumaraswamy distribution is studied as a sub-model from the mixture exponentiated family of distributions. Some statistical properties are studied such as the quantile function, moments, moment generating function and order statistics. Also, the maximum likelihood estimators for the unknown parameters, reliability and hazard rate functions of the mixture exponentiated inverted Kumaraswamy distribution are obtained. A simulation study is conducted to assess the performance of the maximum likelihood estimators of the parameters of the mixture exponentiated inverted Kumaraswamy distribution. Finally, two real data sets are applied to ensure the simulated results.

Keywords: Mixture distribution, Exponentiated family, Inverted Kumaraswamy distribution, Identifiability property, Maximum likelihood estimation.
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1. Introduction

The mixture models were applied to model population heterogeneity, generalize distributional assumptions, clustering and classification, etc. Fields that are used successfully in mixture distributions include astronomy, biology, genetics, medicine, psychiatry, economics, engineering and marketing, among many other fields in biological, physical and social sciences.

Newcomb [30] pioneered the concept of the finite mixture distribution for modeling outliers. Pearson [32] considered a mixture of two univariate Gaussian distributions to estimate the parameters of the model using the method of moments to analyze a dataset containing ratios of the forehead to body lengths for 1,000 crabs. Several authors studied finite mixture models for different distributions. For example, Jorgensen et al. [23] studied the mixture of the inverse Gaussian distribution with its complementary reciprocal. Ahmed et al. [3] obtained approximate Bayes estimators for the parameters of the mixture of two Weibull distributions under Type-II censoring. AL-Hussaini et al. [5] concerned with the statistical properties of the finite mixture of two Gompertz lifetime models. Jaheen [22] implemented both maximum likelihood (ML) and Bayesian approaches to discuss the problem of estimating the parameters using the finite mixture of two exponential distributions based on record statistics. Shawky and Bakoban [33] used both ML and Bayesian methods to estimate the parameters of the model, reliability, and failure rate functions of two-component finite mixtures of exponentiated gamma distribution. Mahmoud and Ghazal [25] derived characteristics of a finite mixture of two components of exponentiated family of distributions based on recurrence relations for moment and conditional moment generating functions of generalized order statistics.

Ateya and Al Khald [7] studied the finite mixture of truncated generalized Cauchy distribution based on Type-II censored samples and progressively Type II censoring. In addition, Tahir et al. [34] constructed the three-component mixture of exponential distributions from the Bayesian viewpoint based on Type-II doubly censoring sampling scheme. Kharazmi et al. [24] obtained non-Bayes and Bayes estimators based on the complete sample from the two-component mixture of Topp–Leone distribution. Dey et al. [16] introduced a mixture of the Marshall–Olkin extended Weibull distribution for efficient modeling of failure, survival and COVID-19 data under non-Bayesian and Bayesian approaches based on Type-II censored data. Recently, Usman et al. [35] presented the mixture cure rate model for right censored survival data and Crisci et al. [12] provided a method based on quantiles to estimate the parameters of a finite mixture of Fréchet distributions for a large sample of dependent data.

A finite mixture model is a convex combination of two or more probability density functions. Thus, the random variable X is said to have a finite mixture distribution with k components if the probability density function (pdf) of x can be written in the following form

$$f(x) = \sum_{i=1}^{k} p_i g_i(x) . \tag{1.1}$$

Then, the cumulative distribution function (cdf) of a finite mixture distribution is

$$F(x) = \sum_{i=1}^{k} p_i G_i(x) , \tag{1.2}$$

where $p_i's$ are non-negative quantities with the sum one where

$$0 \leq p_i \leq 1 , \qquad i = 1, \cdots, k,$$

and

$$\sum_{i=1}^{k} p_i = 1.$$

The quantities $p_i's$ are called the mixing proportion or weights also the $g_i(x)$ in (1.1) and $G_i(x)$ in (1.2) are called the pdf and cdf of i^{th} components of the mixture. [See Mclachlan and Peel [27]].

Several families of distributions are constructed by adding one or more parameters to a distribution function, which let it richer and more flexible to analyze data. Some of these families of distributions are Marshall-Olkin generated family by Marshall and Olkin [26], beta-G by Eugene et al. [17], Kumaraswamy-G by Cordeiro and Castro [13], transformed transformer (T-X) family by Alzaatreh et al. [8], exponentiated T-X by Alzaghal et al. [9], Weibull-G by Bourguignon et al. [11], exponentiated half-logistic by Cordeiro et al. [14], Lomax-G by Cordeiro et al. [15], Zografos-Balakrishnan-G by Nadarajah et al. [29] and Topp-Leone G family by Hassan et al. [20].

Gupta et al. [19] defined the cdf and pdf of exponentiated distributions for a random variable X as

$$F(x) = [G(x)]^{\alpha}, \tag{1.3}$$

and

$$f(x) = \alpha g(x)[G(x)]^{\alpha-1}, \tag{1.4}$$

where

$\alpha > 0$ is a shape parameter and $G(x)$ is a cdf of the baseline distribution.

Abd AL-Fattah et al. [1] derived inverted Kumaraswamy (IKum) distribution from Kumaraswamy (Kum) distribution using the transformation $X = \frac{1}{Y} - 1$. Then, the pdf and cdf for the IKum distribution with shape parameters λ and β are, respectively, given by

$$g(x;\lambda,\beta) = \lambda\beta(1+x)^{-(\lambda+1)}[1-(1+x)^{-\lambda}]^{\beta-1}, \quad x > 0,\ \lambda,\beta > 0, \tag{1.5}$$

and

$$G(x;\lambda,\beta) = \left[1-(1+x)^{-\lambda}\right]^{\beta}, \qquad x > 0,\ \lambda,\beta > 0. \tag{1.6}$$

Identifiability is a very important concept associated with mixture models since it gives a unique representation for a class of mixtures. For more details, see, AL-Hussaini and Ahmad [4], Everitt and Hand [18], Ahmad [2] and Ateya [6] who proved the identifiability of a finite mixture of generalized exponential distributions.

In this study, the mixture of two components exponentiated (ME) family with some properties and estimation are discussed in Section 2. In Section 3, the mixture of two components exponentiated inverted Kumaraswamy (MEIK) distribution is presented as a sub-model from the ME family after verifying the identifiability property of two components from exponentiated inverted Kumaraswamy (EIK) distribution. Also, some statistical properties and the ML estimators for the unknown parameters are also obtained. In Section 4, a simulation study is conducted to assess the performance of the ML estimators of the parameters of the MEIK distribution. Finally, in Section 5, two real data sets are applied to ensure the theoretical results and to prove the flexibility and applicability of the MEIK distribution.

2. Mixture of Two Components of Exponentiated Family

A density function for the ME family with two mixing proportions ($k = 2$) in (1.1) takes the form

$$f(x) = p\alpha_1 g_1(x)[G_1(x)]^{\alpha_1-1} + \alpha_2(1-p)g_2(x)[G_2(x)]^{\alpha_2-1}, x > 0, \alpha_1,\ \alpha_2 > 0, \tag{2.1}$$

Then, the cdf of the ME is,

$$F(x) = p[G_1(x)]^{\alpha_1} + (1-p)[G_2(x)]^{\alpha_2}, \qquad x > 0, \alpha_1, \alpha_2 > 0. \tag{2.2}$$

The corresponding reliability function (rf), hazard rate function (hrf) and reversed hazard rate function (rhrf), respectively, are

$$S(x) = 1 - p[G_1(x)]^{\alpha_1} - (1-p)[G_2(x)]^{\alpha_2}, \tag{2.3}$$

$$h(x) = \frac{p\alpha_1 g_1(x)[G_1(x)]^{\alpha_1-1} + \alpha_2(1-p)g_2(x)[G_2(x)]^{\alpha_2-1}}{1 - p[G_1(x)]^{\alpha_1} - (1-p)[G_2(x)]^{\alpha_2}}, \tag{2.4}$$

and

$$rh(x) = \frac{p\alpha_1 g_1(x)[G_1(x)]^{\alpha_1-1} + \alpha_2(1-p)g_2(x)[G_2(x)]^{\alpha_2-1}}{p[G_1(x)]^{\alpha_1} + (1-p)[G_2(x)]^{\alpha_2}}. \tag{2.5}$$

2.1. Some properties of the mixture of two components exponentiated family

In this subsection, some statistical properties of the ME family are derived.

2.1.1. Quantile function

The quantile function of the ME family can be obtained from solving the following equation

$$\ln\left(p\left[G_1\left(x_q\right)\right]^{\alpha_1} + (1-p)\left[G_2\left(x_q\right)\right]^{\alpha_2}\right) - \ln(q) = 0. \tag{2.6}$$

Also, a random sample from ME family can be generated using the uniform distribution in (2.6).

2.1.2. Moments

Let $X \sim$ ME $(x, \alpha_1, \alpha_2, p)$, then the r^{th} moment of ME family is given by

$$\begin{aligned} \mu_r' = E(x^r) &= \sum_{j=1}^{2} p_j E_j(x^r) = pE_1(x^r) + (1-p)E_2(x^r) \\ &= p\int_0^\infty x^r \alpha_1 g_1(x)[G_1(x)]^{\alpha_1-1}dx + (1-p)\int_0^\infty x^r \alpha_2 g_2(x)[G_2(x)]^{\alpha_2-1}dx. \end{aligned} \tag{2.7}$$

Using the series representation as:

$$z^v = \sum_{i=0}^{\infty} \frac{(\ln z)^i}{i!} v^i. \tag{2.8}$$

Then, the cdf's $[G_1(x)]^{\alpha_1-1}$ and $[G_2(x)]^{\alpha_2-1}$ can be rewritten as follows:

$$[G_1(x)]^{\alpha_1-1} = \sum_{j_1=0}^{\infty} (\alpha_1 - 1)^{j_1} \frac{(\ln G_1(x))^{j_1}}{j_1!},$$

and

$$[G_2(x)]^{\alpha_2-1} = \sum_{j_2=0}^{\infty} (\alpha_2 - 1)^{j_2} \frac{(\ln G_2(x))^{j_2}}{j_2!}.$$

Then,

$$\mu_r' = p\alpha_1 \int_0^\infty x^r g_1(x) \sum_{j_1=0}^{\infty} \frac{(\alpha_1-1)^{j_1}}{j_1!} [\ln G_1(x)]^{j_1} dx + (1-p)\alpha_2 \int_0^\infty x^r g_2(x) \sum_{j_2=0}^{\infty} \frac{(\alpha_2-1)^{j_2}}{j_2!} [\ln G_2(x)]^{j_2} dx$$
$$\mu_r' = \sum_{j_1=0}^{\infty} \frac{(\alpha_1-1)^{j_1}}{j_1!} a_{j_1,r} + \sum_{j_2=0}^{\infty} \frac{(\alpha_2-1)^{j_2}}{j_2!} a_{j_2,r}. \tag{2.9}$$

where

$$a_{j_1,r} = p\alpha_1 \int_0^\infty x^r g_1(x) [\ln G_1(x)]^{j_1} dx,$$

and

$$a_{j_2,r} = (1-p)\alpha_2 \int_0^\infty x^r g_2(x) [\ln G_2(x)]^{j_2} dx.$$

2.1.3. Moment generating function

The moment generating function of the ME family is given by

$$\begin{aligned} M_x(t) = E(e^{tx}) &= \int_0^\infty e^{tx} f(x)\, dx \\ &= p\alpha_1 \int_0^\infty e^{tx} g_1(x) [G_1(x)]^{\alpha_1-1} dx + (1-p)\alpha_2 \int_0^\infty e^{tx} g_2(x) [G_2(x)]^{\alpha_2-1} dx. \end{aligned} \tag{2.10}$$

Expanding (e^{tx}) using Taylor series as given below

$$e^{(tx)} = \sum_{i=0}^{\infty} \frac{(tx)^i}{i!} = \sum_{i=0}^{\infty} \frac{t^i}{i!} x^i, \tag{2.11}$$

then,

$$\begin{aligned} M_x(t) &= p\alpha_1 \int_0^\infty \sum_{i=0}^{\infty} \frac{t^i}{i!} x^i g_1(x) [G_1(x)]^{\alpha_1-1} dx + (1-p)\alpha_2 \int_0^\infty \sum_{i=0}^{\infty} \frac{t^i}{i!} x^i g_2(x) [G_2(x)]^{\alpha_2-1} dx \\ M_x(t) &= \sum_{i=0}^{\infty} \frac{t^i}{i!} \mu_i'. \end{aligned} \tag{2.12}$$

2.1.4. Order statistics

The pdf of the i^{th} order statistic for a random sample $x_1, x_2, x_3, \cdots, x_n$ from the ME family is

$$\begin{aligned} f_{i,n}(x) &= \frac{n!}{(n-i)!\,(i-1)!} f(x) [F(x)]^{i-1} [1-F(x)]^{n-i} \\ &= \frac{n!}{(n-i)!\,(i-1)!} \sum_{r=0}^{n-i} (-1)^r \binom{n-i}{r} f(x) [F(x)]^{i+r-1}, \end{aligned} \tag{2.13}$$

where,

$$[1 - F(x)]^{n-i} = \sum_{r=0}^{n-i} (-1)^r \binom{n-i}{r} [F(x)]^r.$$

Then, the pdf of the i^{th} order statistics for the ME family can be obtained by substituting (2.1) and (2.2) into (2.13), thus

$$f_{i,n}(x) = \frac{n!}{(n-i)!\,(i-1)!} \sum_{r=0}^{n-i} (-1)^r \binom{n-i}{r} \left\{ p\alpha_1 g_1(x) [G_1(x)]^{\alpha_1 - 1} + \alpha_2 (1-p) g_2(x) [G_2(x)]^{\alpha_2 - 1} \right\} \\ \times \left[p[G_1(x)]^{\alpha_1} + (1-p) [G_2(x)]^{\alpha_2} \right]^{i+r-1}. \tag{2.14}$$

The smallest order statistics and the largest order statistics are, respectively, given by

$$f_{1,n}(x) = n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} \left\{ p\alpha_1 g_1(x) [G_1(x)]^{\alpha_1 - 1} + \alpha_2 (1-p) g_2(x) [G_2(x)]^{\alpha_2 - 1} \right\} \\ \times \left[p[G_1(x)]^{\alpha_1} + (1-p) [G_2(x)]^{\alpha_2} \right]^{r}, \tag{2.15}$$

and

$$f_{n,n}(x) = n \left\{ p\alpha_1 g_1(x) [G_1(x)]^{\alpha_1 - 1} + \alpha_2 (1-p) g_2(x) [G_2(x)]^{\alpha_2 - 1} \right\} \\ \times \left[p[G_1(x)]^{\alpha_1} + (1-p) [G_2(x)]^{\alpha_2} \right]^{n-1}. \tag{2.16}$$

2.2. Estimation of the mixture of two components exponentiated family

In this subsection, the ML approach can be applied to estimate the unknown parameters of the two-components ME family.

Let $x_1, x_2, \; x_3, \cdots, x_n$ be a random sample from the ME family defined by (2.1). Then, the ML estimators of $\underline{\hat{\theta}} = (\hat{p}, \hat{\alpha}_1, \; \hat{\alpha}_2)$ are obtained as the solution of the nonlinear likelihood equations:

$$\frac{\partial lnL(\underline{\theta}; \; \underline{x})}{\partial \underline{\theta}} = 0, \tag{2.17}$$

where

$$L(\underline{\theta}; \; \underline{x}) = \prod_{j=1}^{n} f\left(x_j; \underline{\theta}\right),$$

is the likelihood function formed under the assumption of independent and identically distributed random sample $x_1, x_2, \; x_3, \cdots, x_n$.

The likelihood function corresponding to the mixture density in (2.1), is given by

$$L(\underline{\theta}; \; \underline{x}) = \prod_{j=1}^{n} \left[p\alpha_1 g_1\left(x_j\right) \left[G_1\left(x_j\right)\right]^{\alpha_1 - 1} + \alpha_2 (1-p) g_2\left(x_j\right) \left[G_2\left(x_j\right)\right]^{\alpha_2 - 1} \right]. \tag{2.18}$$

The corresponding log-likelihood function $\ell \equiv lnL(\underline{\theta}|\ \underline{x})$ is

$$\ell = \sum_{j=1}^{n} ln\left[p\alpha_1 g_1\left(x_j\right)\left[G_1\left(x_j\right)\right]^{\alpha_1-1} + \alpha_2\left(1-p\right)g_2\left(x_j\right)\left[G_2\left(x_j\right)\right]^{\alpha_2-1}\right]. \tag{2.19}$$

By differentiating ℓ in (2.19) with respect to $p,\ \alpha_1,\ \alpha_2$, then

$$\frac{\partial \ell}{\partial p} = \sum_{j=1}^{n} \omega\left(x_j,\ \underline{\hat{\theta}}\right) = 0, \tag{2.20}$$

where, $\omega\left(x_j,\ \underline{\hat{\theta}}\right) = \frac{\hat{\alpha}_1 g_1\left(x_j\right)\left[G_1\left(x_j\right)\right]^{\hat{\alpha}_1-1} - \hat{\alpha}_2 g_2\left(x_j\right)\left[G_2\left(x_j\right)\right]^{\hat{\alpha}_2-1}}{\hat{p}\hat{\alpha}_1 g_1\left(x_j\right)\left[G_1\left(x_j\right)\right]^{\hat{\alpha}_1-1} + \hat{\alpha}_2(1-\hat{p})g_2\left(x_j\right)\left[G_2\left(x_j\right)\right]^{\hat{\alpha}_2-1}}$,

$$\frac{\partial \ell}{\partial \alpha_1} = \sum_{j=1}^{n} \omega_1\left(x_j,\ \underline{\hat{\theta}}\right) = 0, \tag{2.21}$$

where, $\omega_1\left(x_j,\ \underline{\hat{\theta}}\right) = \frac{\hat{p}\hat{\alpha}_1 g_1\left(x_j\right)\left[G_1\left(x_j\right)\right]^{\hat{\alpha}_1-1} ln\left(G_1\left(x_j\right)\right) + \hat{p} g_1(x_j)\left[G_1\left(x_j\right)\right]^{\hat{\alpha}_1-1}}{\hat{p}\hat{\alpha}_1 g_1\left(x_j\right)\left[G_1\left(x_j\right)\right]^{\hat{\alpha}_1-1} + \hat{\alpha}_2(1-\hat{p})g_2\left(x_j\right)\left[G_2\left(x_j\right)\right]^{\hat{\alpha}_2-1}}$,
and

$$\frac{\partial \ell}{\partial \alpha_2} = \sum_{j=1}^{n} \omega_2\left(x_j,\ \underline{\hat{\theta}}\right) = 0, \tag{2.22}$$

where, $\omega_2\left(x_j,\ \underline{\hat{\theta}}\right) = \frac{\hat{\alpha}_2(1-\hat{p})g_2\left(x_j\right)\left[G_2\left(x_j\right)\right]^{\hat{\alpha}_2-1} ln\left(G_2\left(x_j\right)\right) + (1-\hat{p})g_2\left(x_j\right)\left[G_2\left(x_j\right)\right]^{\hat{\alpha}_2-1}}{\hat{p}\hat{\alpha}_1 g_1\left(x_j\right)\left[G_1\left(x_j\right)\right]^{\hat{\alpha}_1-1} + \hat{\alpha}_2(1-\hat{p})g_2\left(x_j\right)\left[G_2\left(x_j\right)\right]^{\hat{\alpha}_2-1}}$.

Thus, the ML estimators of $\hat{p}, \hat{\alpha}_1,\ \hat{\alpha}_2$ can be obtained by solving (2.20)-(2.22) after replacing the basic functions with different distributions.

3. Mixture of Two Components Exponentiated Inverted Kumaraswamy Distribution

In this section, the MEIK distribution is presented as a sub-model from the ME family. Some statistical properties of MEIK are studied.

3.1. Description of the distribution

The MEIK distribution can be obtained by substituting $g\left(x\right)$ and $G\left(x\right)$, respectively, from (1.5) and (1.6) into (2.1) and (2.2) after indexing (β, λ) by $i, i = 1, 2$.

Thus, the pdf and cdf of MEIK distribution are

$$f_M\left(x\right) = p\alpha_1\lambda_1\beta_1(1+x)^{-(\lambda_1+1)}\left[1-(1+x)^{-\lambda_1}\right]^{\beta_1\alpha_1-1} + \alpha_2\lambda_2\beta_2\left(1-p\right)\left(1+x\right)^{-(\lambda_2+1)}\left[1-(1+x)^{-\lambda_2}\right]^{\beta_2\alpha_2-1}, \tag{3.1}$$

and

$$F_M\left(x\right) = p\left[1-(1+x)^{-\lambda_1}\right]^{\alpha_1\beta_1} + \left(1-p\right)\left[1-(1+x)^{-\lambda_2}\right]^{\alpha_2\beta_2}. \tag{3.2}$$

The corresponding rf, hrf and rhrf, respectively, are

$$S_M(x) = 1 - p\left[1-(1+x)^{-\lambda_1}\right]^{\alpha_1\beta_1} - (1-p)\left[1-(1+x)^{-\lambda_2}\right]^{\alpha_2\beta_2}, \tag{3.3}$$

$$h_M(x) = \frac{p\alpha_1\beta_1\lambda_1(1+x)^{-(\lambda_1+1)}\left[1-(1+x)^{-\lambda_1}\right]^{\alpha_1\beta_1-1} + \alpha_2\beta_2\lambda_2(1-p)(1+x)^{-(\lambda_2+1)}\left[1-(1+x)^{-\lambda_2}\right]^{\alpha_2\beta_2-1}}{1 - p\left[1-(1+x)^{-\lambda_1}\right]^{\alpha_1\beta_1} - (1-p)\left[1-(1+x)^{-\lambda_2}\right]^{\alpha_2\beta_2}}, \tag{3.4}$$

and

$$rh_M(x) = \frac{p\alpha_1\lambda_1\beta_1(1+x)^{-(\lambda_1+1)}\left[1-(1+x)^{-\lambda_1}\right]^{\beta_1\alpha_1-1} + \alpha_2\lambda_2\beta_2(1-p)(1+x)^{-(\lambda_2+1)}\left[1-(1+x)^{-\lambda_2}\right]^{\beta_2\alpha_2-1}}{1 - p\left[1-(1+x)^{-\lambda_1}\right]^{\alpha_1\beta_1} - (1-p)\left[1-(1+x)^{-\lambda_2}\right]^{\alpha_2\beta_2}}. \tag{3.5}$$

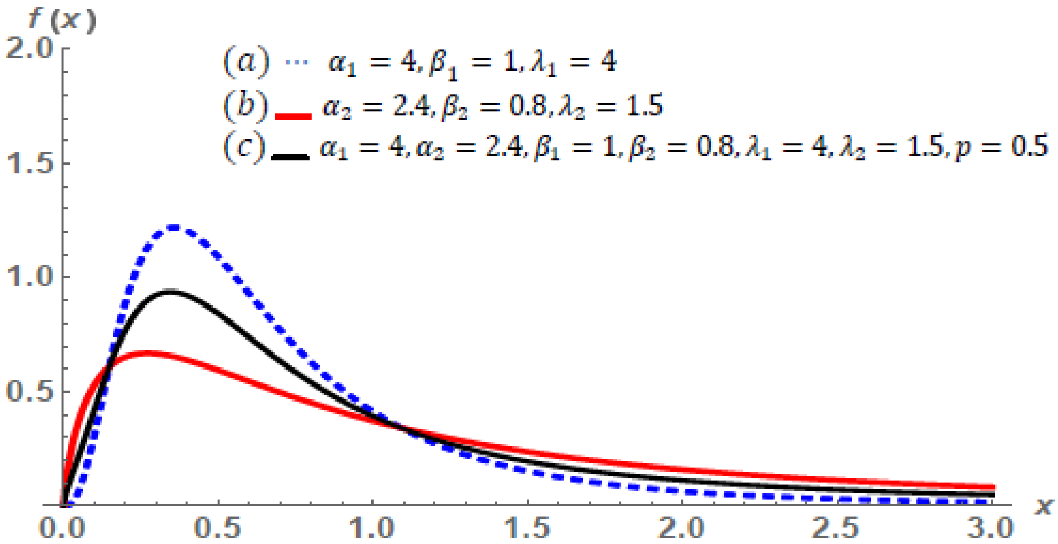

Figure 1. Plots of the two components exponentiated inverted Kumaraswamy distribution and their mixture at different values of the parameters

Figure 1 exhibit the pdf of the first component $(\alpha_1, \beta_1, \lambda_1)$ in (a), the second component $(\alpha_2, \beta_2, \lambda_2)$ in (b) and the MEIK with parameters $(\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2, p)$ in (c).

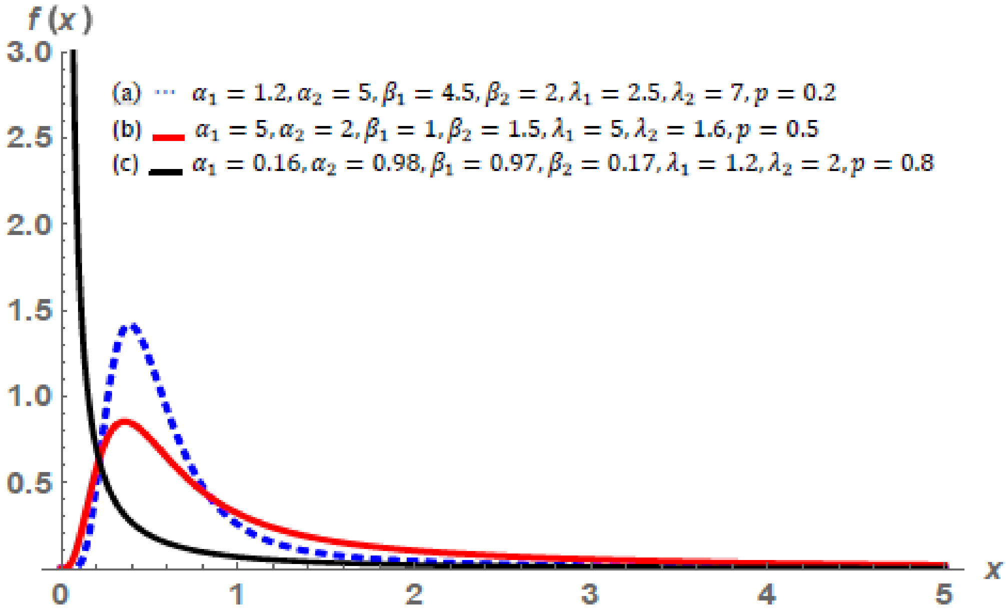

Figure 2. Plots of the pdf of mixture exponentiated inverted Kumaraswamy distribution at different values of the parameters

Figure 2 displays different shapes of the pdf for the MEIK distribution. The densities in (a) and (b) have a unimodal curve and right skewed and the density in (c) is decreasing.

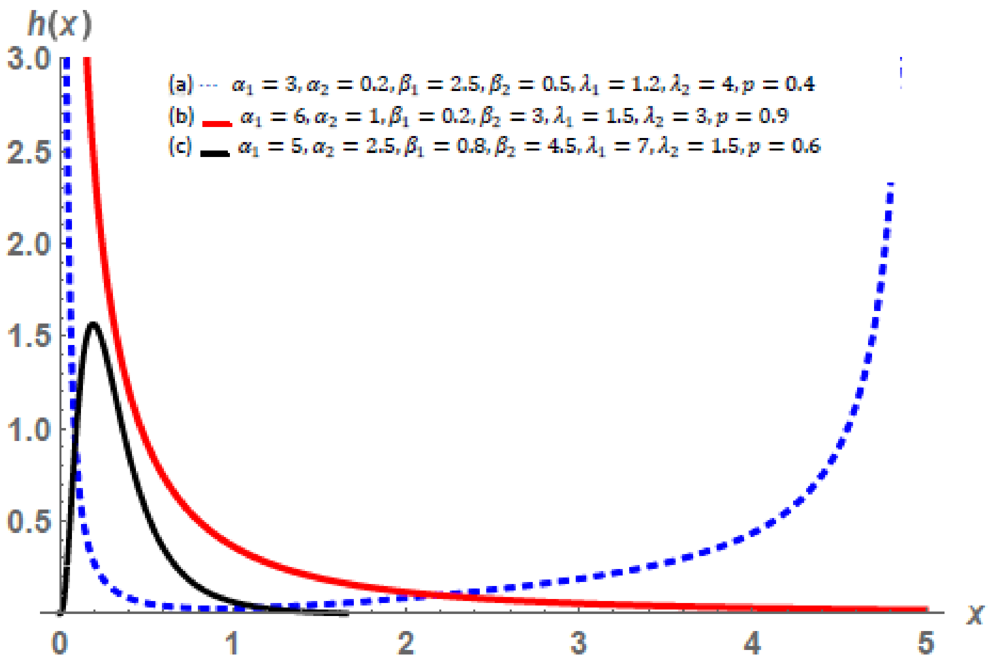

Figure 3. Plots of the hrf of mixture exponentiated inverted Kumaraswamy distribution at different values of the parameters

Figure 3 presents different shapes of the hrf for the MEIK distribution which permit high degree for flexibility of the MEIK distribution The hrf in (a) has a bathtub shape, in (b) the hrf is decreasing and in (c) it is unimodal and right skewed.

3.2. Some properties of the proposed distribution

3.2.1. Quantile function

The quantile function of the MEIK distribution can be obtained by substituting $G_1\left(x_q\right)$ and $G_2\left(x_q\right)$ from (1.6) into (2.6) and solving the following equation

$$ln\left[p\left(1-\left(1+x_q\right)^{-\lambda_1}\right)^{\alpha_1\beta_1}+(1-p)\left(1-\left(1+x_q\right)^{-\lambda_2}\right)^{\alpha_2\beta_2}\right]-lnq\ =0. \tag{3.6}$$

Also, a random sample from MEIK distribution can be generated using uniform distribution in (3.6).

3.2.2. Moments

Let $X\sim$ MEIK $\left(x, p_j, \alpha_j, \beta_j, \lambda_j\right)$ distribution, then the r^{th} moment of a MEIK distribution is given by

$$\begin{aligned}\mu'_{r(M)} &=E\left(x^r\right)=\sum_{j=1}^{2}p_jE_j\left(x^r\right)=\sum_{j=1}^{2}p_j\int_0^\infty x^rf(x)dx\\ &=\sum_{j=1}^{2}p_j\alpha_j\lambda_j\beta_j\int_0^\infty x^r(1+x)^{-(\lambda_j+1)}\left[1-(1+x)^{-\lambda_j}\right]^{\alpha_j\beta_j-1}dx.\end{aligned} \tag{3.7}$$

Then,

$$\mu'_{r(M)}=\sum_{k=0}^{\infty}\sum_{j=1}^{2}(-1)^k\binom{\alpha_j\beta_j-1}{k}p_j\alpha_j\beta_j\lambda_j\boldsymbol{B}(r+1,\lambda_j(1+k)-r). \tag{3.8}$$

where $\boldsymbol{B}(.,.)$ is the beta function and $\lambda_j(1+k)>r$.

Let, $r=1$ in (3.8), the mean of the MEIK distribution is

$$\mu_M=\mu'_{1(M)}=E(x)=\sum_{k=0}^{\infty}\sum_{j=1}^{2}(-1)^k\binom{\alpha_j\beta_j-1}{k}p_j\alpha_j\beta_j\lambda_j\boldsymbol{B}(2,\lambda_j(1+k)-1). \tag{3.9}$$

where $\boldsymbol{B}(.,.)$ is the beta function and $\lambda_j(1+k)>1$.

3.2.3. Moment generating function

The moment generating function of the MEIK distribution can be derived by substituting $\mu'_{r(M)}$ of the MEIK distribution in (3.8) into (2.12), then

$$\begin{aligned}M_{x(M)}(t)&=E(e^{tx})\sum_{i=0}^{\infty}\frac{t^i}{i!}\mu'_i\\ M_{x(M)}(t)&=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{2}(-1)^k\binom{\alpha_j\beta_j-1}{k}\frac{t^i}{i!}p_j\alpha_j\beta_j\lambda_j\boldsymbol{B}(r+1,\lambda_j(1+k)-r).\end{aligned} \tag{3.10}$$

where $\boldsymbol{B}(.,.)$ is the beta function and $\lambda_j(1+k)>r$.

3.2.4. Order statistics

The i^{th} order statistics for the MEIK distribution can be obtained by substituting (3.1) and (3.2) into (2.13); thus

$$f_{i,n(M)}(x) = \frac{n!}{(n-i)!\,(i-1)!}\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}\Big\{p\alpha_1\lambda_1\beta_1(1+x)^{-(\lambda_1+1)}\left[1-(1+x)^{-\lambda_1}\right]^{\alpha_1\beta_1-1}+ \alpha_2\lambda_2\beta_2(1-p)(1+x)^{-(\lambda_2+1)}\left[1-(1+x)^{-\lambda_2}\right]^{\alpha_2\beta_2-1}\Big\}\times \sum_{r=0}^{n-i}\left\{p\left[1-(1+x)^{-\lambda_1}\right]^{\alpha_1\beta_1}+(1-p)\left[1-(1+x)^{-\lambda_2}\right]^{\alpha_2\beta_2}\right\}^{i+r-1}. \tag{3.11}$$

Then, the corresponding smallest and largest order statistics can be obtained when $i = 1$ and $i = n$.

3.3. *Maximum likelihood estimation*

This subsection focuses on deriving the ML estimators for the parameter vector $\underline{\theta} = (p_i, \alpha_i, \lambda_i, \beta_i)$ of the MEIK density based on a random sample of size n, where $i = 1,\ 2$ and $p_2 = 1 - p_1$.

The likelihood function corresponding to the MEIK density in (3.1) is given by

$$L_M(\underline{\theta}; \underline{x}) = \prod_{j=1}^{n}\left[\begin{array}{c} p\alpha_1\lambda_1\beta_1\left(1+x_j\right)^{-(\lambda_1+1)}\left[1-\left(1+x_j\right)^{-\lambda_1}\right]^{\alpha_1\beta_1-1} \\ +\alpha_2\lambda_2\beta_2(1-p)\left(1+x_j\right)^{-(\lambda_2+1)}\left[1-\left(1+x_j\right)^{-\lambda_2}\right]^{\alpha_2\beta_2-1}\end{array}\right]. \tag{3.12}$$

The natural logarithm of the likelihood function is given by

$$\ell_M \equiv lnL\left(\underline{\theta};\ \underline{x}\right) = \sum_{j=1}^{n} ln\left[\begin{array}{c} p\alpha_1\lambda_1\beta_1\left(1+x_j\right)^{-(\lambda_1+1)}\left[1-\left(1+x_j\right)^{-\lambda_1}\right]^{\alpha_1\beta_1-1} \\ +\alpha_2\lambda_2\beta_2(1-p)\left(1+x_j\right)^{-(\lambda_2+1)}\left[1-\left(1+x_j\right)^{-\lambda_2}\right]^{\alpha_2\beta_2-1}\end{array}\right]. \tag{3.13}$$

By differentiating ℓ_M with respect to the unknown parameters $\underline{\theta} = (p_i, \alpha_i, \lambda_i, \beta_i)$ of the MEIK distribution, the first derivatives are obtained as follows:

$$\frac{\partial \ell_M}{\partial p} = \sum_{j=1}^{n}\frac{f_1\left(x_j, \underline{\hat{\theta}}\right) - f_2\left(x_j, \underline{\hat{\theta}}\right)}{f_M\left(x_j, \underline{\hat{\theta}}\right)} = 0,$$

$$\frac{\partial \ell_M}{\partial \alpha_i} = \sum_{j=1}^{n}\frac{p_i\zeta_i\left(x_j, \underline{\hat{\theta}}\right)}{f_M\left(x_j, \underline{\hat{\theta}}\right)} = 0,$$

$$\frac{\partial \ell_M}{\partial \beta_i} = \sum_{j=1}^{n}\frac{p_i\eta_i\left(x_j, \underline{\hat{\theta}}\right)}{f_M\left(x_j, \underline{\hat{\theta}}\right)} = 0, \tag{3.14}$$

and

$$\frac{\partial \ell_M}{\partial \lambda_i} = \sum_{j=1}^{n}\frac{p_i\phi_i\left(x_j, \underline{\hat{\theta}}\right)}{f_M\left(x_j, \underline{\hat{\theta}}\right)} = 0,$$

where $f_1\left(x_j,\underline{\theta}\right)$, $f_2\left(x_j,\underline{\theta}\right)$, $\zeta_i\left(x_j,\underline{\theta}\right)$, $\eta_i\left(x_j,\underline{\theta}\right)$ and $\phi_i\left(x_j,\underline{\theta}\right)$ are

$$f_1\left(x_j,\underline{\theta}\right)=\alpha_1\lambda_1\beta_1\left(1+x_j\right)^{-(\lambda_1+1)}\left[1-\left(1+x_j\right)^{-\lambda_1}\right]^{\alpha_1\beta_1-1}, \tag{3.15}$$

$$f_2\left(x_j,\underline{\theta}\right)=\alpha_2\lambda_2\beta_2\left(1+x_j\right)^{-(\lambda_2+1)}\left[1-\left(1+x_j\right)^{-\lambda_2}\right]^{\alpha_2\beta_2-1}, \tag{3.16}$$

$$\zeta_i\left(x_j,\underline{\theta}\right)=\lambda_i\beta_i\left(1+x_j\right)^{-(\lambda_i+1)}\left[1-\left(1+x_j\right)^{-\lambda_i}\right]^{\alpha_i\beta_i-1}\left[\beta_i\alpha_i ln\left(1-\left(1+x_j\right)^{-\lambda_i}\right)+1\right], \tag{3.17}$$

$$\eta_i\left(x_j,\underline{\theta}\right)=\alpha_i\lambda_i\left(1+x_j\right)^{-(\lambda_i+1)}\left[1-\left(1+x_j\right)^{-\lambda_i}\right]^{\alpha_i\beta_i-1}\left[\beta_i\alpha_i ln\left(1-\left(1+x_j\right)^{-\lambda_i}\right)+1\right], \tag{3.18}$$

and

$$\begin{aligned}\phi_i\left(x_j,\underline{\theta}\right)=&\left\{\alpha_i\lambda_i\beta_i\left(\alpha_i\beta_i-1\right)\left(1+x_j\right)^{-2\lambda_i-1}\left[1-\left(1+x_j\right)^{-\lambda_i}\right]^{\alpha_i\beta_i-2}\ln\left(1+x_j\right)\right\}\\&+\left\{\alpha_i\beta_i\left(1+x_j\right)^{-\lambda_i}\left[1-\left(1+x_j\right)^{-\lambda_i}\right]^{\alpha_i\beta_i-1}\left[1-\lambda_i ln\left(1+x_j\right)\right]\right\}.\end{aligned} \tag{3.19}$$

Solving the nonlinear likelihood equations in (3.14) numerically one can obtain the ML estimates of the unknown parameters.

The ML estimators of the rf and hrf can be obtained by replacing the parameters (p, α_1, α_2, β_1, β_2, $\lambda_1 and$ λ_2) in (3.3) and (3.4) by their corresponding ML estimators. Hence, the ML estimators of $S_M(x)$ and $h_M(x)$ are, respectively, given by

$$\hat{S}_M(x)=1-\hat{p}\left[1-(1+x)^{-\hat{\lambda}_1}\right]^{\hat{\alpha}_1\hat{\beta}_1}-(1-\hat{p})\left[1-(1+x)^{-\hat{\lambda}_2}\right]^{\hat{\alpha}_2\hat{\beta}_2}, \tag{3.20}$$

and

$$\hat{h}_M(x)=\frac{\hat{p}\hat{\alpha}_1\hat{\beta}_1\hat{\lambda}_1(1+x)^{-(\hat{\lambda}_1+1)}\left[1-(1+x)^{-\hat{\lambda}_1}\right]^{\hat{\alpha}_1\hat{\beta}_1-1}+\hat{\alpha}_2\hat{\beta}_2\hat{\lambda}_2(1-\hat{p})(1+x)^{-(\hat{\lambda}_2+1)}\left[1-(1+x)^{-\hat{\lambda}_2}\right]^{\hat{\alpha}_2\hat{\beta}_2-1}}{1-\hat{p}\left[1-(1+x)^{-\hat{\lambda}_1}\right]^{\hat{\alpha}_1\hat{\beta}_1}-(1-\hat{p})\left[1-(1+x)^{-\hat{\lambda}_2}\right]^{\hat{\alpha}_2\hat{\beta}_2}}. \tag{3.21}$$

4. Simulation Study

In this section, a simulation study is performed to investigate the efficiency of the ML estimates. A random variable X from MEIK $(p_i,\alpha_i,\lambda_i,\beta_i)$ distribution is generated using Mathematica 11, for different samples of size (n=30, 50, 100, 150 and 200) using number of replications (NR)=1000. The averages, estimated risks (ERs), Biases of the ML estimates of the parameters, rf and hrf are computed for each model parameters and for each sample size as follows:

1. Average $=\frac{\sum_{i=1}^{NR}(estimate)}{NR}$,
2. ERs $=\frac{\sum_{i=1}^{NR}(estimated\ value-true\ value)^2}{NR}$,
3. $(\text{Bias})^2=(estimated\ value-true\ value)^2$.

Table 1, 2, 3 and 4 display the ML averages, ERs, Biases of the ML estimates and 95% confidence intervals (CIs) of the unknown parameters $p_i, \alpha_i, \lambda_i, \beta_i$,rf and hrf for different samples of size (n=30, 50, 100, 150 and 200) where the population parameter values are (α_1=2, β_1=2.5, λ_1=1.4, p=0.6, α_2=0.8, β_2=1.6, λ_2=1.9) and (α_1=1.5, β_1=3, λ_1=2, p=0.8, α_2=1, β_2=3, λ_2=1.5).

From Tables 1, 2, 3, and 4, one can observe that when the sample size n increases, the ERs and biases of the ML estimates for the parameters $p_i, \alpha_i, \lambda_i, \beta_i$, rf and hrf decrease in almost cases. Moreover, the lengths of the CI become narrower as the sample size increases.

Table 1. ML averages, estimated risks, biases and 95% confidence intervals of the parameters, rf and hrf from MEIK distribution for different sample sizes n and ($\boldsymbol{\alpha}_1$=2, $\boldsymbol{\beta}_1$=2.5, $\boldsymbol{\lambda}_1$=1.4, p=0.6, $\boldsymbol{\alpha}_2$=0.8, $\boldsymbol{\beta}_2$=1.6, $\boldsymbol{\lambda}_2$=1.9 and NR=1000)

n	Parameters	Averages	ER	Bias	UI	LI	Length
30	α_1	2.0459	0.1982	0.0021	2.9138	1.1781	1.7357
	β_1	2.5574	0.3096	0.0033	3.6422	1.4726	2.1696
	λ_1	1.4196	0.0783	0.0004	1.9667	0.8725	1.0941
	p	0.5992	0.0103	0.0000	0.7982	0.4001	0.3981
	α_2	0.8298	0.0187	0.0009	1.0917	0.5680	0.5237
	β_2	1.6597	0.0749	0.0036	2.1833	1.1360	1.0473
	λ_2	1.9084	0.2991	0.0001	2.9802	0.8366	2.1436
	R	0.5329	0.0094	0.0002	0.7205	0.3454	0.3751
	H	0.5280	0.2273	0.0029	1.4565	0.0000	1.4565
50	α_1	2.0398	0.1666	0.0016	2.8359	1.2436	1.5923
	β_1	2.5497	0.2603	0.0025	3.5449	1.5545	1.9904
	λ_1	1.4114	0.0612	0.0001	1.8956	0.9272	0.9684
	p	0.5928	0.0079	0.0001	0.7673	0.4183	0.3490
	α_2	0.8226	0.0131	0.0005	1.0429	0.6022	0.4407
	β_2	1.6451	0.0526	0.0020	2.0858	1.2044	0.8814
	λ_2	1.8815	0.2464	0.0003	2.8538	0.9092	1.9446
	R	0.5419	0.0070	0.0000	0.7057	0.3782	0.3275
	H	0.5041	0.1488	0.0009	1.2580	0.0000	1.2580

5. Applications

In this subsection, two real data sets are applied to illustrate the flexibility and applicability of the MEIK distribution in real life. To check the validity of the fitted model, Kolmogorov- Smirnov goodness of fit test is performed for the data sets where the p value indicates that the model fits the data well. Therefore, a comparison is provided between the proposed distribution and other fitted distributions such as mixture of two components inverted Kumaraswamy (MIK) presented by Noor et al. [31], generalized inverted Kumaraswamy (GIK) by Iqbal et al. [21] and Topp-Leone-inverted Kumaraswamy (TL-IK) by Behairy et al. [10]. The ML estimates of the unknown parameters, rf and hrf, the values of log likelihood (LL), Akaike information criterion (AIC), Bayesian information criterion (BIC) and corrected Akaike information criterion (CAIC) for MEIK, MIK, GIK and TL-IK

Table 2. Continued Table 1

n	**Parameters**	Averages	ER	Bias	UI	LI	Length
100	α_1	2.0388	0.1056	0.0015	2.6710	1.4064	1.2646
	β_1	2.5484	0.1649	0.0023	3.3388	1.7580	1.5808
	λ_1	1.4108	0.0322	0.0001	1.7618	1.0597	0.7021
	p	0.5917	0.0058	0.0001	0.7398	0.4436	0.2962
	α_2	0.8118	0.0072	0.0001	0.9762	0.6475	0.3287
	β_2	1.6237	0.0287	0.0006	1.9523	1.2950	0.6573
	λ_2	1.8723	0.1484	0.0008	2.6255	1.1191	1.5064
	R	0.5468	0.0035	0.0000	0.6631	0.4303	0.2328
	H	0.4874	0.0098	0.0002	0.6793	0.2956	0.3837
150	α_1	2.0294	0.0847	0.0009	2.5970	1.4617	1.1353
	β_1	2.5368	0.1324	0.0014	3.2463	1.8272	1.4191
	λ_1	1.4065	0.0198	0.0000	1.6817	1.1313	0.5504
	p	0.5949	0.0054	0.0000	0.7386	0.4514	0.2872
	α_2	0.8086	0.0052	0.0001	0.9494	0.6678	0.2816
	β_2	1.6172	0.0209	0.0003	1.8988	1.3357	0.5631
	λ_2	1.8806	0.1151	0.0004	2.5443	1.2168	1.3275
	R	0.5481	0.0021	0.0000	0.6387	0.4573	0.1814
	H	0.4831	0.0053	0.0001	0.6252	0.3409	0.2843
200	α_1	2.0451	0.0674	0.0020	2.5461	1.5441	1.0020
	β_1	2.5564	0.1053	0.0030	3.1826	1.9301	1.2525
	λ_1	1.4179	0.0162	0.0003	1.6651	1.1706	0.4945
	p	0.5945	0.0040	0.0000	0.7186	0.4703	0.2483
	α_2	0.8071	0.0043	0.0001	0.9343	0.6797	0.2546
	β_2	1.6141	0.0171	0.0002	1.8687	1.3594	0.5093
	λ_2	1.8713	0.1094	0.0008	2.5171	1.2253	1.2918
	R	0.5467	0.0019	0.0000	0.6340	0.4593	0.1747
	H	0.4790	0.0039	0.0000	0.6017	0.3563	0.2454

distributions are given in Table 4.

$AIC = 2m - 2\mathcal{L}$, $BIC = mln(n) - 2\mathcal{L}andCAIC = AIC + 2\left(\frac{m(m+1)}{n-m-1}\right)$, where $\mathcal{L}$ is the natural logarithm of the value of the likelihood function evaluated at the ML estimates,n is the number of the observations and m is the number of the estimated parameters. The best distribution corresponds to the lowest values of AIC, BIC and CAIC.

Data set I: The first real data set was provided by Iqbal et. al [21] which are the prices of wooden toys for 30 children in April 1991 at Suffolk craft shop. The data are:

4.2, 1.12, 1.39, 2, 3.99, 2.15, 1.74, 5.81, 1.7, 0.5, 0.99, 11.5, 5.12, 0.9, 1.99, 6.24, 2.6, 3, 12.2, 7.36, 4.75, 11.59, 8.69, 9.8, 1.85, 1.99, 1.35,10, 0.65, 1.45.

Table 5 presents the ML estimates of the parameters and standard errors (SEs).

Figures 4, 5, 6, 7, and 8 display the PP-plot, QQ-plot, empirical scaled TTT-transform plot, Boxplot and empirical histogram of the MEIK distribution for the first real data set.

The plot of the empirical scaled TTT-transform indicates that the first real data has a bathtub hazard function, boxplot and the histogram of the data. One can notice that this data is right-skewed. P-P plot, Q-Q plot and the fitted MEIK distribution plots indicate that MEIK distribution provides a better fit to this data.

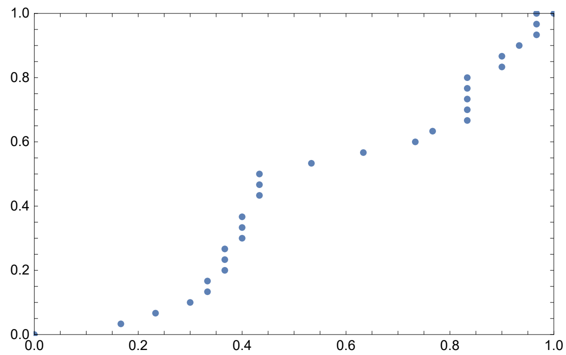

Figure 4. PP-plot of the MEIK distribution for the real data set

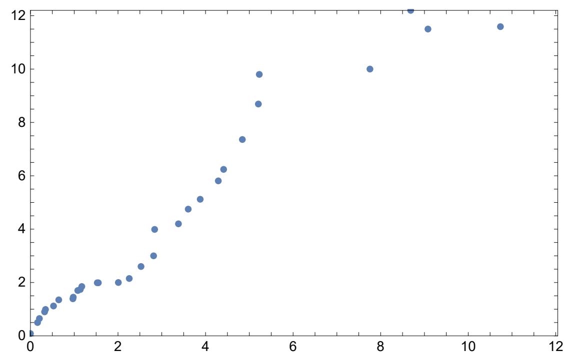

Figure 5. QQ-plot of the MEIK distribution for the real data set

Data set II: The second real data set represents the time between failures for a repairable item. It was presented by Murthy et al. [28]. The data are: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86 and 1.17.

Figures 9, 10, 11, 12, and 13 show the PP-plot, QQ-plot, empirical scaled TTT-transform plot, Boxplot and empirical histogram of the MEIK distribution for the second real data set.

The plot of the empirical scaled TTT-transform implies that this data has an increasing hazard function, the boxplot implies that this data is right skewed. The P-P plot, Q-Q plot and the fitted MEIK distribution plots indicate that MEIK distribution gives better fit for this data.

Table 5 presents the ML estimates of the parameters and standard errors (SEs). Table 6 shows that MEIK distribution is the best among the compared distributions because it has the smallest value of LL, AIC, BIC and CAIC for the two real data sets.

6. Concluding Remarks

In this paper, the ME family is introduced as a new family of continuous distributions. Some important properties are studied and the ML estimation of this proposed mixture family is obtained. The

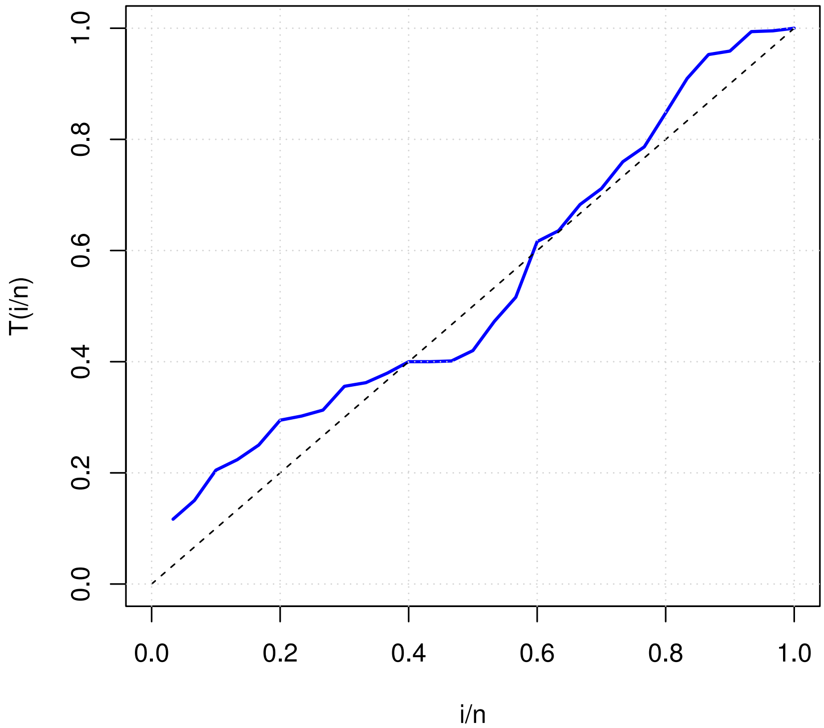

Figure 6. The empirical scaled TTT- transform plot for the real data set

MEIK distribution is proposed as a sub-model from the mixture of two components exponentiated family. Some characteristics and the ML estimators for the unknown parameters of the MEIK distribution are derived. A simulation study is conducted to assess the performance of the ML estimators of the parameters of the MEIK distribution. Finally, two real data sets are applied. The results indicate that the values of some information criteria are the lowest for the MEIK as compared to MIK, GIK and TL-IK distributions, therefore, this proposed distribution is superior to other distributions.

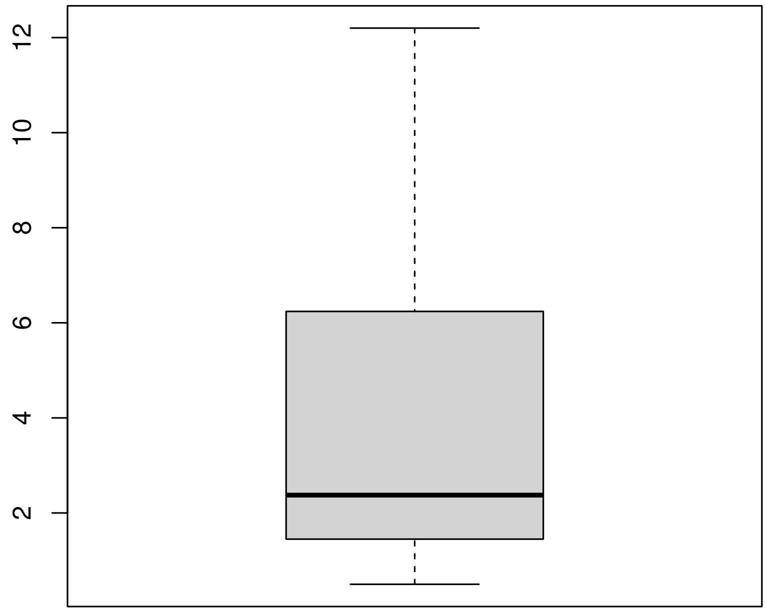

Figure 7. Boxplot for the real data set

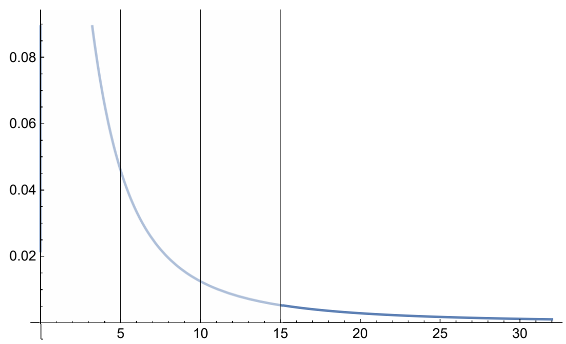

Figure 8. Empirical histogram plot for the real data set

Table 3. ML averages, estimated risks, biases and 95% confidence intervals of the parameters, rf and hrf from MEIK distribution for different sample sizes n and (α_1=1.5, β_1=3, λ_1=2, p=0.8, α_2=1, β_2=3, λ_2=1.5 and NR=1000)

n	Parameters	Averages	ER	Bias	UI	LI	Length
30	α_1	1.51390	0.05380	0.00020	1.96770	1.06030	0.90740
	β_1	3.02790	0.21510	0.00080	3.93540	2.12060	1.81480
	λ_1	2.00130	0.11490	0.00000	2.66580	1.33680	1.32900
	p	0.78280	0.00600	0.00030	0.93110	0.63460	0.29650
	α_2	1.10260	0.05240	0.01050	1.50380	0.70150	0.80230
	β_2	3.30790	0.47180	0.09480	4.51140	2.10450	2.40690
	λ_2	1.67890	0.18940	0.03200	2.45660	0.90130	1.55530
	R	0.55030	0.01370	0.00020	0.77830	0.32220	0.45610
	H	0.82520	0.40580	0.00940	2.05930	0.00000	2.05930
50	α_1	1.49550	0.03490	0.00000	1.86170	1.12940	0.73230
	β_1	2.99110	0.13970	0.00000	3.72340	2.25880	1.46460
	λ_1	1.97420	0.07820	0.00070	2.51980	1.42860	1.09120
	p	0.78690	0.00500	0.00020	0.92310	0.65060	0.27250
	α_2	k1.08120	0.03670	0.00660	1.42120	0.74130	0.67990
	β_2	3.24370	0.33010	0.05940	4.26360	2.22390	2.03970
	λ_2	1.65010	0.14660	0.02250	2.34040	0.95980	1.38060
	R	0.56050	0.00830	0.00000	0.73880	0.38220	0.35660
	H	0.75990	0.06500	0.00100	1.25590	0.26380	0.99210
100	α_1	1.48420	0.01640	0.00030	1.73320	1.23510	0.49810
	β_1	2.96830	0.06560	0.00100	3.46640	2.47010	0.99630
	λ_1	1.96940	0.03450	0.00090	2.32880	1.61010	0.71870
	p	0.78780	0.00480	0.00010	0.92160	0.65390	0.26770
	α_2	1.06960	0.02100	0.00480	1.31900	0.82010	0.49890
	β_2	3.20860	0.18930	0.04350	3.95700	2.46030	1.49670
	λ_2	1.63550	0.08310	0.01840	2.13440	1.13670	0.99770
	R	0.56640	0.00330	0.00000	0.67900	0.45370	0.22530
	H	0.73270	0.01830	0.00000	0.99760	0.46790	0.52970

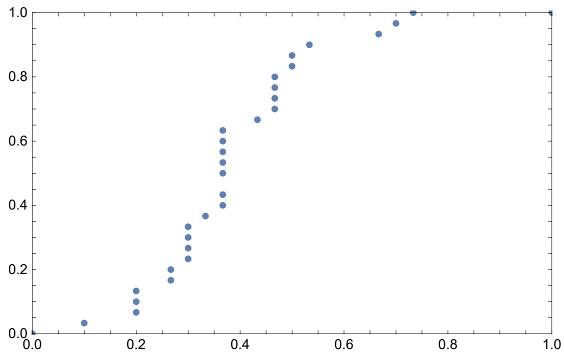

Figure 9. PP-plot of the MEIK distribution for the real data set

Table 4. Continued Table 3

n	Parameters	Averages	ER	Bias	UI	LI	Length
150	α_1	1.4838	0.0146	0.0003	1.7182	1.2493	0.4689
	β_1	2.9676	0.0583	0.0011	3.4365	2.4986	0.9379
	λ_1	1.9743	0.0302	0.0007	2.3112	1.6374	0.6738
	p	0.7875	0.0044	0.0002	0.9147	0.6602	0.2545
	α_2	1.0714	0.0218	0.0051	1.3249	0.8178	0.5071
	β_2	3.2142	0.1965	0.0459	3.9748	2.4536	1.5212
	λ_2	1.6356	0.0886	0.0184	2.1551	1.1161	1.0390
	R	0.5646	0.0028	0.0000	0.6692	0.4601	0.2091
	H	0.7365	0.0155	0.0000	0.9803	0.4927	0.4876
200	α_1	1.4836	0.0117	0.0003	1.6929	1.2742	0.4187
	β_1	2.9671	0.0467	0.0011	3.3859	2.5484	0.8375
	λ_1	1.9723	0.0238	0.0008	2.2698	1.6748	0.5950
	p	0.7863	0.0047	0.0002	0.9178	0.6549	0.2629
	α_2	1.0638	0.0145	0.0041	1.2643	0.8632	0.4011
	β_2	3.1914	0.1308	0.0366	3.7930	2.5897	1.2033
	λ_2	1.6249	0.0642	0.0156	2.0560	1.1926	0.8644
	R	0.5674	0.0020	0.0000	0.6551	0.4796	0.1755
	H	0.7279	0.0092	0.0000	0.9162	0.5396	0.3766

Table 5. ML estimates of the parameters, rf and hrf of MEIK distribution for the two real data sets

Application	Parameters	Estimates	SE	Application	Parameters	Estimates	SE
Application I	α_1	2.3375	0.1138	Application II	α_1	1.4596	0.292
	β_1	2.9218	0.1779		β_1	1.8245	0.4563
	λ_1	1.7158	0.0997		λ_1	1.0816	0.1014
	p	0.9142	0.0987		p	0.4968	0.0106
	α_2	0.9089	0.0118		α_2	0.9436	0.0206
	β_2	1.8179	0.0475		β_2	1.8873	0.0825
	λ_2	1.3541	0.298		λ_2	1.9445	0.0019
	R	0.4062	0.0199		R	0.5867	0.0015
	H	0.4594	0.0002		H	0.5945	0.0145

Table 6. ML estimates and information criteria for the two real data sets

Application	Parameters	MEIK	MIK	GIK	TL-IK
Application I	α_1	2.3375	1.3512	2.4496	0.4526
	β_1	2.9218	5.302	7.9995	2.3388
	λ_1	1.7157	--	5.3774	1.2116
	p	0.9143	0.3378	--	--
	α_2	0.9089	0.3595	--	--
	β_2	1.8179	0.9986	--	--
	λ_2	1.3541	--	--	--
	R	0.4063	0.8597	0.9974	0.9632
	H	0.4594	0.1778	0.0875	0.0488
	Information criteria				
	LL	145.466	165.405	1117.17	246.613
	AIC	159.466	175.405	1123.17	252.613
	BIC	169.274	182.411	1127.37	256.817
	CAIC	164.557	177.905	1124.09	253.536
Application	Parameters	MEIK	MIK	GIK	TL-IK
Application II	α_1	1.4596	1.1196	0.6829	0.4
	β_1	1.8245	2.7936	0.9181	2.0408
	λ_1	1.0816	--	2.1222	1.0599
	p	0.4968	0.2342	--	--
	α_2	0.9436	1.3713	--	--
	β_2	1.8873	0.9907	--	--
	λ_2	1.9445	--	--	--
	R	0.5867	0.5434	0.6877	0.9306
	H	0.5945	0.5824	0.3499	0.0854
	Information criteria				
	LL	94.424	100.371	149.133	174.31
	AIC	108.424	110.371	155.133	180.31
	BIC	118.232	117.377	159.337	184.513
	CAIC	113.515	112.871	156.056	181.233

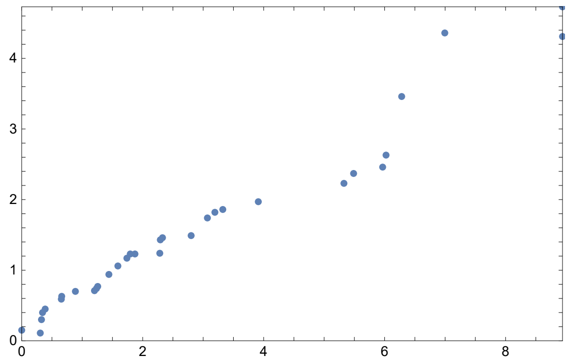

Figure 10. QQ-plot of the MEIK distribution for the real data set

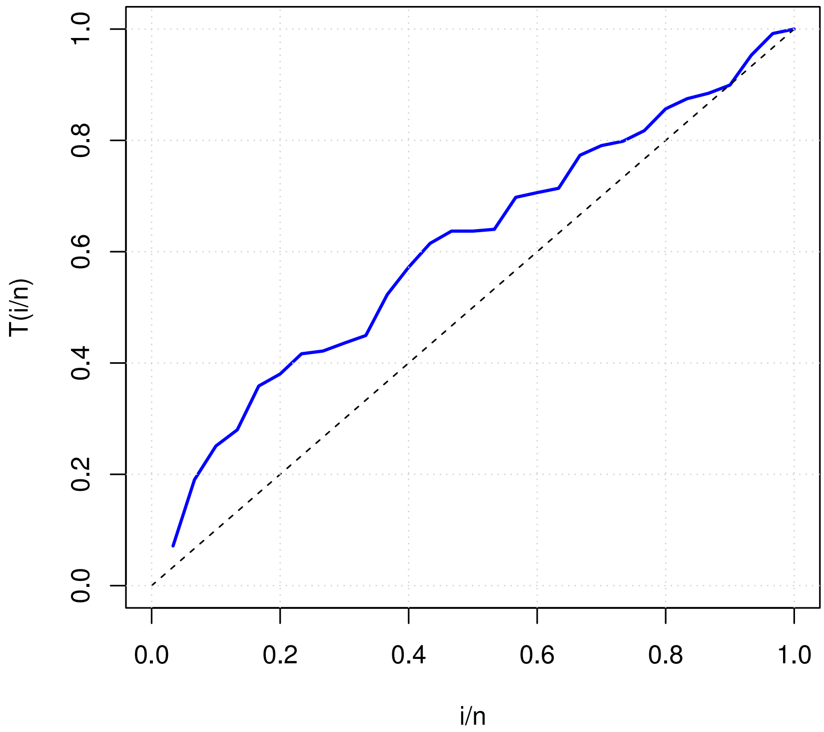

Figure 11. The empirical scaled TTT- transform plot for the real data set

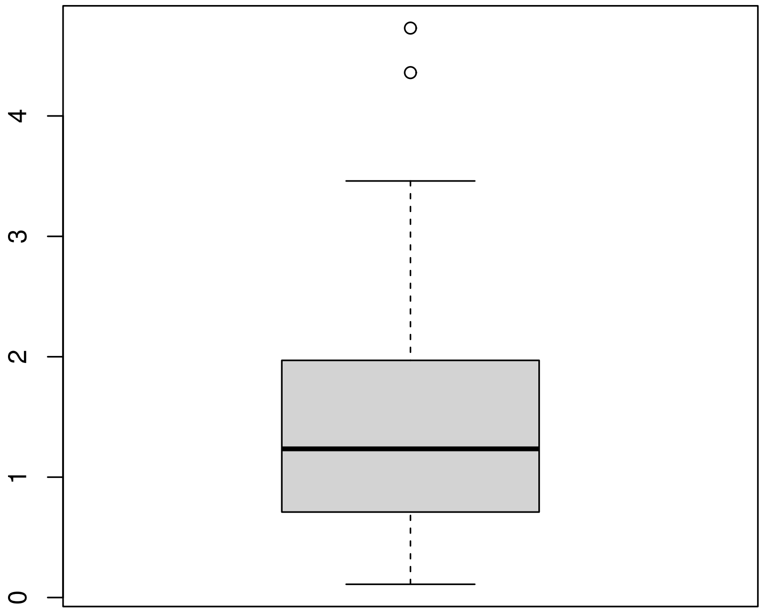

Figure 12. Boxplot for the real data set

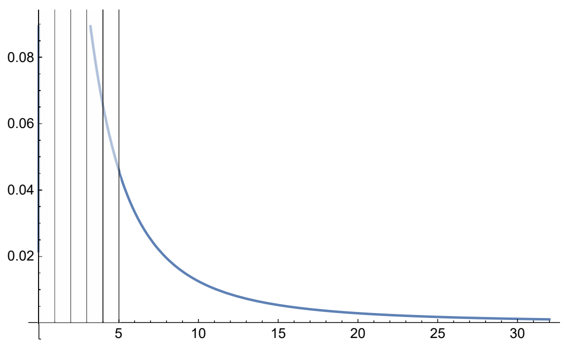

Figure 13. Empirical histogram plot for the real data set

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