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Statistical Properties of a Generalization Erlang Truncated Exponential Distribution with Applications and Its Bivariate Extension

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Abstract: Using power exponentiated family, this paper introduces the New Power Exponentiated Erlang-Truncated Exponential distribution as a new generalization of the Erlang-Truncated Exponential distribution. The suggested distribution has constant and increasing shapes for hazard rate function. Numerous structural characteristics are derived, including quantile function, moments, moment generating function, behavior of hazard, reversed hazard and cumulative hazard functions, entropy measures, stochastic ordering, and order statistics. The model parameters are estimated by maximum likelihood, Cramer von Mises, and Percentiles estimation methods. A numerical study is performed using simulated data to examine performance of the different estimators with varying sample size. The flexibility and potentiality of proposed model and some existing models are examined using two actual data sets and some criteria for model selection and goodness of fit test statistics. Finally, a bivariate extension of the suggested distribution called the bivariate new power exponentiated erlang-truncated exponential distribution was introduced. The recommended bivariate distribution is of type Farlie–Gumbel–Morgenstern copula. The proposed distribution has joint probability density function, the joint cumulative function, and joint survival function. In addition, Some statistical properties of the distribution are also obtained.

Keywords: Power exponentiated family, FGM coupla, moments, maximum likelihood method, joint probability density function.

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1. Introduction

In probability and statistics, a mixture distribution is a mixture of two or more probability distributions. Random variables are drawn from more than one parent population to create a new distribution.

The parent distributions can be univariate or multivariate, althougth the mixed distributions should have the same dimensionality. In addition, they should either be all discrete probability distributions or all continuous probability distributions. Mixtures of distributions have been used in numerous investigations. Some examples of generalizations of mixtures of exponential distributions were examined by Drozdenko [8]. Finite Weibull mixture distributions were employed in the reliability approximation of Bucar et al. [6]. Ben Nakhi and Kalla [2] introduced hyper Poisson distribution-generalized gamma mixes.

The Erlang truncated exponential (ETE) distribution is considered one of the mixture distribution was first described by El-Alosey [10] using a mixture of the Erlang distribution and the left truncated one parameter exponential distribution. Stochastic processes and queuing systems both make extensive use of ETE. For ω is a scale parameter and υ is a shape parameter, the cumulative distribution (Cd) and hazard function (Hf) of ETE are respectively, as follows:

$$T_{ETE}(x) = 1 - e^{-\left(1-e^{-\omega}\right)\upsilon x}, \;\; x \geq 0 \;, \; \omega, \, \upsilon > 0, \tag{1.1}$$

and

$$h_{ETE}(x) = \left(1 - e^{-\omega}\right) \upsilon \,. \tag{1.2}$$

For every $\omega \to \infty$, the ETE distribution collapses to the traditional one-parameter exponential distribution with parameter υ. The HF of the ETE distribution is constant, as shown in (1.2), which means that many complex lifetime data sets with nonconstant failure rates cannot be well modeled by it. Therefore, the primary goal of this work is to construct a new flexible model referred to as the power exponentiated Erlang-truncated exponential (PE-ETE) distribution by extending the ETE distribution using a new power exponentiated family by adding two shape parameters. The additional shape parameters play a crucial role in modeling observed data by introducing skewness, varying tail weights, and offering greater flexibility in the generalized distribution's shape. In order to enhance the ETE distribution's goodness of fit when modeling data with varying failure rates, some authors have recently proposed novel model modifications as follows: Generalized ETE distribution and Poisson exponentiated ETE distribution are introduced by Nasiru et al. [21], Okorie et al. [24] are introduced Transmuted ETE distribution, Extended ETE distribution, and Marshall-Olkin generalized ETE distribution [26]. Beta ETE distribution is defined by Shrahili et al. [28] and McDonald ETE distribution by Elbatal and Aldukeel. El-Alosey is proposed Discrete ETE distribution. Discrete extended ETE distribution [11] [25], Binomial Discrete ETE mixture and Negative Binomial Discrete ETE mixture are defined by El-Alosey and Eledum. Discrete Kumaraswamy ETE distribution is discussed by Eledum and El-Alosey, and others are among them.

Our goal in this study, introduce a noval flexible distribution called new power exponentiated-Erlang truncated exponential (NPE-ETE) distribution by using power exponentiated family of distributions. The overview of the paper include NPE-ETE density, cumulative, and hazard functions and their shapes were defined in section 2. In section 3, moments and some associated measures, Renyi and Tsallis entropy measures, order statistic of the proposed model were obtained. In addition, various estimation methods are used to examine model's parameters in section 4. A simulation study and two real data sets are performed in section 5 and 6. Furthermore, a bivariate extension of the suggested model was introduced in section 7. Finally, the conclusion remarks were proposed in section 8.

2. New Model

Modi (2021) introduced a NPE-family of probability distributions. For x > 0, σ≠1, the Cd of NPE-family is:

$$B(x) = (\sigma - 1)^{-1} \left(\sigma^{(T(x))^{\varepsilon}} - 1\right), \ \sigma > 0, \ \varepsilon > 0, \tag{2.1}$$

where σ, ε are two shape parameters. We suggest a NPE-ETE distribution utilizing the Cd defined in (1.1) and (2.1), the Cd and its corresponding probability function (Pf) of the suggested model are given, respectively, as follows:

$$B_{NPE-ETE}(x) = (\sigma - 1)^{-1} \left(\sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}} - 1\right), \ x > 0, \ \sigma, \ \varepsilon, \ \upsilon, \ \omega > 0, \ \sigma \neq 1, \tag{2.2}$$

and

$$b_{NPE-ETE}(x) = \varepsilon\, \upsilon \ (\sigma - 1)^{-1} \left(1 - e^{-\omega}\right) \ln(\sigma) \ \sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}} \left(1 - e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{\varepsilon-1} e^{-\left(1-e^{-\omega}\right)\upsilon x}. \tag{2.3}$$

Theorem 1. The Pf of the NPE-ETE distribution can be written as a mixture form of ETE distibution as follows:

$$b_{NPE-ETE}(x) = \varepsilon \ (\sigma - 1)^{-1} \ln(\sigma) \sum_{i,j,k,l=0}^{\infty} \vartheta_{i,j,k,l} \ T_{ETE}\left(x \,;\, (l+1)\,\upsilon\,,\, \omega\right), \ x > 0\,, \ \sigma \neq 1, \tag{2.4}$$

where $T_{ETE}(x, \ (l+1)\,\upsilon, \ \omega)$is the Pf of ETE distribution with parameters $(l+1)\,\upsilon$, $\omega > 0$, and

$$\vartheta_{i,j,k,l} = \frac{(\ln\sigma)^{k}\,(-1)^{i+j+l} i^{j} \ \Gamma(\varepsilon) \ \Gamma(j + \varepsilon k + 1)}{i\,!\,j\,!\,k\,!\ (l+1)\,!\Gamma(\varepsilon - i) \ \Gamma(j + \varepsilon k - l + 1)}, \ \varepsilon > i.$$

Proof

For Pf of NPE-ETE distribution as shown in (2.3), Utilizing binomial series and exponential expansion we get

$$b_{NPE-ETE}(x) = \frac{\varepsilon\,\upsilon\ (1 - e^{-\omega})\ \ln\sigma}{(\sigma - 1)} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}\, i^{j}}{j!} \binom{\varepsilon - 1}{i} \sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}} \left(1 - e^{-(1-e^{-\omega})\upsilon x}\right)^{j} e^{-\left(1-e^{-\omega}\right)\upsilon x}.$$

Applying (see Gradshteyn and Ryzhik [17])

$$\sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}} = \sum_{k=0}^{\infty} \frac{(\ln\sigma)^{k}}{\Gamma(k+1)} \left(1 - e^{-\left(1-e^{-\omega}\right)\upsilon x}\right)^{\varepsilon k}.$$

The NPE-ETE distribution's Pf can be expressed as:

$$b_{NPE-ETE}(x) = \frac{\varepsilon\,\upsilon\ (1 - e^{-\omega})\ \ln\sigma}{(\sigma - 1)} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j}\, i^{j}\ (\ln\sigma)^{k}}{j!\ \Gamma(k+1)} \binom{\varepsilon - 1}{i} \left(1 - e^{-(1-e^{-\omega})\upsilon x}\right)^{j+\varepsilon k} e^{-\left(1-e^{-\omega}\right)\upsilon x}.$$

For $0 < \left(1 - e^{-(1-e^{-\omega})\upsilon x}\right)^{j+\varepsilon k} < 1$, the result hold and it is clear from (2.4) that the NPE-ETE distribution's Pf is a weighted function of the ETE distribution with varying values.

Additionally, a quantile function can be used to simulate the NPE-ETE model; let q be a random variable with a uniform distribution on the interval [0,1]. The NPE-ETE quantile function can be found by:

$$x_q = (\upsilon\ (1 - e^{-\omega}))^{-1} \left[-\ln \left(1 - \left(\frac{\ln ((\sigma - 1)\ q + 1)}{\ln \sigma} \right)^{\frac{1}{\varepsilon}} \right) \right]. \tag{2.5}$$

The first quartile, median, and third quartile can be obtained by substituting q = 0.25, 0.5, and 0.75 , respectively into (2.5). For different parameter values, the shapes of the PE-ETE distribution's density and cumulative are presented in Fig. 1.

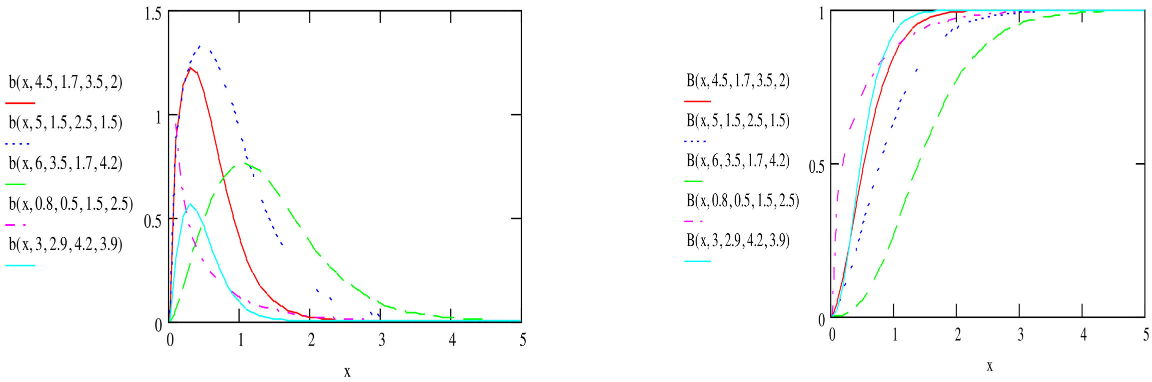

Figure 1. Pf and Cd plots for NPE-ETE model

Fig 1 indicates the plots of Pd of new model grows more leptokurtic, and the Cd is strictly growing and tends to one as the values of the two extra parameters σ, εincreases.

3. Statistical Characteristics

Numerous statistical characteristics of the NPE-ETE distribution are provided, including The moment generating function, moments, reliability measures, entropy, stochastic ordering and order statistics of suggested model are among of statistical characteristics that we derived in this section.

3.1. Moments

Moments are crucial to statistical analysis in general, and in applications in particular. Central tendency, skewness and kurtosis, dispersion, and other measures can be found using them.

Theorem 2

The moment generating function (mgf) of NPE-ETE distribution is supplied by

$$M_X(p) = 1 + \varepsilon\ (\sigma - 1)^{-1} \ln(\sigma) \sum_{h=1}^{\infty} \frac{p^h}{h\,!} \sum_{i,j,k,l,=0}^{\infty} \vartheta_{i,j,k,l}\ \frac{\Gamma(h+1)}{[(1 - e^{-\omega})\ (l+1)\ \upsilon]^{h+1}}\ , \sigma \neq 1 \tag{3.1}$$

Proof

The NPE-ETE mgf is defined by

$$M_X(p) = \int_0^{\infty} e^{px} b_{NPE-ETE}(x) \quad dx = 1 + \sum_{h=1}^{\infty} \frac{p^h}{h\,!} \mu_h^{'}$$

If X distributed NPE-ETE, using (2.4), the kth moment about zero is

$$\mu_h^{'} = \int_0^{\infty} x^h \, b_{NPE-ETE}(x) \quad dx$$

$$= \int_0^{\infty} \varepsilon\,(\sigma - 1)^{-1} \ln(\sigma)\; x^h \sum_{i,j,k,l=0}^{\infty} \vartheta_{i,j,k,l}\; T_{ETE}(x\,;\,(l+1)\,\upsilon\,,\,\omega)\; dx$$

$$= \varepsilon\,(\sigma - 1)^{-1} \ln(\sigma)\,(1 - e^{-\omega}) \sum_{i,j,k,l=0}^{\infty} \nu\,(l+1)\,\vartheta_{i,j,k,l} \int_0^{\infty} x^h\, e^{-\left(\left(1-e^{-\omega}\right)(l+1)\,\upsilon\right)x}\, dx$$

$$= \varepsilon\,(\sigma - 1)^{-1} \ln(\sigma) \sum_{i,j,k,l=0}^{\infty} \vartheta_{i,j,k,l} \frac{\Gamma(h+1)}{[(1 - e^{-\omega})\,(l+1)\,\upsilon]^{h+1}} \quad ,\ h = 1,\, 2,\, 3,\, ... \tag{3.2}$$

The expected value (mean) and variance can be obtained from (3.2) as

$$mean = \varepsilon\,(\sigma - 1)^{-1} \ln(\sigma) \sum_{i,j,k,l=0}^{\infty} \vartheta_{i,j,k,l} \frac{1}{[(1 - e^{-\omega})\,(l+1)\,\upsilon]^2} \ ,\ \sigma \neq 1.$$

$$Variance = \varepsilon\,(\sigma - 1)^{-1} \ln(\sigma) \sum_{i,j,k,l=0}^{\infty} \vartheta_{i,j,k,l} \frac{2}{[(1 - e^{-\omega})\,(l+1)\,\upsilon]^3}$$

$$- \left[\varepsilon\,(\sigma - 1)^{-1} \ln(\sigma) \sum_{i,j,k,l=0}^{\infty} \vartheta_{i,j,k,l} \frac{1}{[(1 - e^{-\omega})\,(l+1)\,\upsilon]^2}\right]^2 ,\ \sigma \neq 1\ .$$

Moreover, Utilizing the correlation between the moment about zero and the moment about the mean, the hth moment about mean

$$\mu_h = \sum_{a=0}^{h} (-1)^a \binom{h}{a} \left(\mu_1^{'}\right)^a \mu_{h-a}^{'}$$

Measures of skewness $\Im_1$ gives the direction and magnitude of the lack of symmetry whereas the kurtosis$\Im_2$gives the idea of flatness. The statistics are defined as follows:
$\Im_1 = \mu_3 / \mu_2{}^{3/2}$, and $\Im_2 = \mu_4 / \mu_2{}^2$.

3.2. Reliability measures

The survival function (Su) and Hf of the NPE-ETE are given, respectively by:

$$Su_{NPE-ETE}(x) = 1 - \left[(\sigma - 1)^{-1}\left(\sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}} - 1\right)\right] \ ,\ x > 0,\ \sigma \neq 1,$$

and

$$h_{NPE-ETE}(x)=\frac{\varepsilon\,\upsilon\,(1-e^{-\omega})\,\ln(\sigma)\,\sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}}\left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{\varepsilon-1}e^{-(1-e^{-\omega})\upsilon x}}{\sigma-\sigma^{\left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{\varepsilon}}},\ x>0,\ \sigma\neq 1.$$

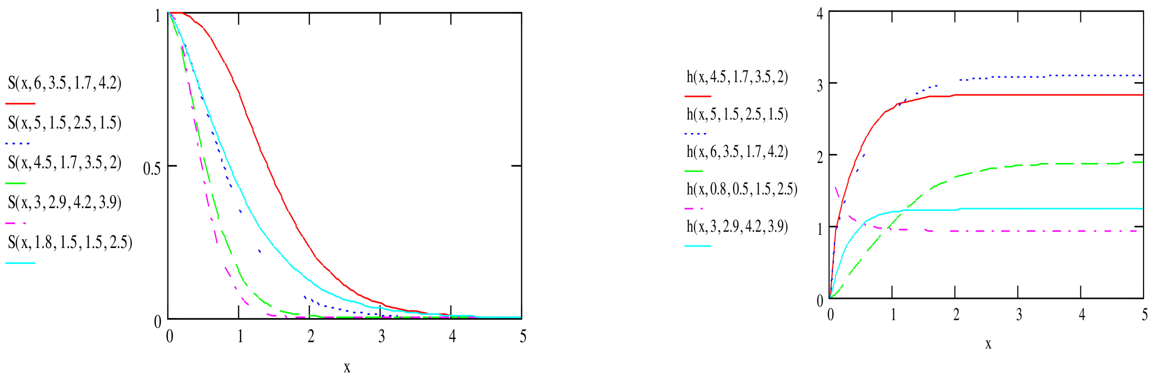

Figure 2. Su and Hf plots for PE-ETE model

Fig 2 introduces the behavior of Su function of NPE-ETE decreases when the extra parameters decreases and approaches zero fastly when x tends to ∞.The Hf plot of NPE-ETE displays two shapes, constant function for small values of extra parameters and increasing function overall with large values of extra parameters. For$\sigma\neq 1$, The cumulative hazard (Ch) function and reversed hazard (Rh) function of the recommended model are given, respectively by:

$$Ch_{NPE-ETE}(x)=-\ln\left(1-\left[(\sigma-1)^{-1}\left(\sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}}-1\right)\right]\right),$$

and

$$Rh_{NPE-ETE}(x)=\frac{\varepsilon\,\upsilon\,(\sigma-1)\,(1-e^{-\omega})\,\ln(\sigma)\,\sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}}\left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{\varepsilon-1}e^{-(1-e^{-\omega})\upsilon x}}{\sigma^{\left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{\varepsilon}}-1}.$$

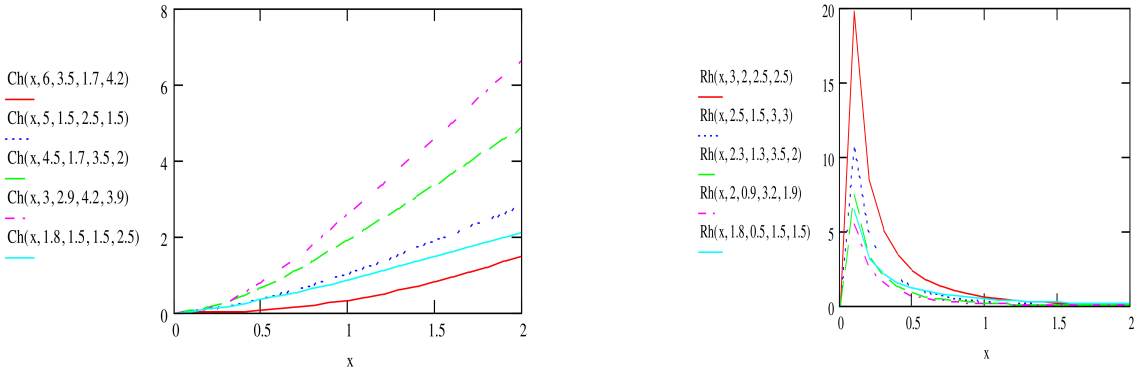

Figure 3. Ch and Rh plots for PE-ETE model

Fig 3 shows the graphs of Ch and Rh of recommended distribution. The Ch plot displays strictly increasing behavior for various parameters. The Rh plot indicates the increasing trend at initial phase. Longer tail to right shows the decreasing behavior of NPE-ETE distribution.

3.3. Entropy measures

The total quantity of data in the system can be described by uncertainty measures, we introduce some of these measures such as Renyi and Tsallis entropies.

Theorem 3

Renyi [27] is presented the Renyi entropy. For NPE-ETE distribution, $Re_{NPE-ETE}$, is defined by

$$Re_{NPE-ETE}(d) = (1-d)^{-1} \log \left[C^d \sum_{i,j,k=0}^{\infty} \Omega_{i,j,k}\, B(d,\ j+\varepsilon k+1) \right],$$

where $B(.,.)$ is the beta function, $C = \left(\varepsilon \upsilon\, (\sigma-1)^{-1} (1-e^{-\omega}) \ln(\sigma)\right)$, and $\Omega_{i,j,k} = \left[\frac{(-1)^{i+j} i^j (d \ln \sigma)^k \Gamma(d(\varepsilon-1)+1)}{\upsilon (1-e^{-\omega})\, i!\, \Gamma(j+1)\Gamma(k+1)\Gamma(d(\varepsilon-1)-i+1)} \right]$, $\nu > 0,\ \sigma \neq 1$.

Proof

By definition

$$Re_{NPE-ETE}(d) = (1-d)^{-1} \log \left[\int_0^{\infty} (b_{NPE-ETE}(x))^d \ dx \right],\ d \neq 1,\ d > 0 \tag{3.3}$$

Substituting (2.3) in (3.3), we get

$$\begin{aligned} &Re_{NPE-ETE}(d) \\ &= (1-d)^{-1} \log \left[C^d \int_0^{\infty} \sigma^{d \left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}} \left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{d(\varepsilon-1)} e^{-\upsilon d (1-e^{-\omega})\, x} dx \right] \end{aligned}$$

$$= (1-d)^{-1} \log \left[C^d \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}\, i^j}{j!} \binom{d\,(\varepsilon-1)}{i} \int_0^{\infty} \sigma^{d\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}} \left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{j+\varepsilon k} e^{-\upsilon d\,(1-e^{-\omega})\,x}\, dx \right] =$$

Hence,

$$Re_{NPE-ETE}(d) = (1-d)^{-1} \log \left[C^d \sum_{i,j,k=0}^{\infty} \Omega_{i,j,k}\, B\,(d,\ j+\varepsilon k+1) \right].$$

Tsallis entropy is defined by Tsallis [29], with respect to NPE-ETE distribution, $Ta_{NPE-ETE}$, is defined as below

$$Ta_{NPE-ETE}(d) = (1-d)^{-1} \left[1 - C^d \sum_{i,j,k=0}^{\infty} \Omega_{i,j,k}\, B\,(d,\ j+\varepsilon k+1) \right].$$

3.4. Stochastic Ordering

For evaluating the comparative behavior of positive continuous random variables, one crucial tool is their stochastic ordering. LetX_1 and X_2 be two continuous random variable distributed NPE-ETE with parameters$(\sigma,\ \varepsilon_1,\ \upsilon,\ \omega)and(\sigma,\ \varepsilon_2,\ \upsilon,\ \omega)$ respectively, X_1 is said to be stochastically smaller than X_2in the

1. Stochastic ordering $X_1 \le_{st} X_2$, if $B_{X_1}(x) \ge B_{X_2}(x)$, for all x.
2. Hazard rate order $X_1 \le_{hr} X_2$, if $h_{X_1}(x) \ge h_{X_2}(x)$, for all x.
3. Likelihood ratio order $X_1 \le_{lr} X_2$, if the ratio $b_{X_1}(x)/b_{X_2}(x)$, decreases in x.

The following implications apply if X_1 and X_2have a common finite left end-point support.

$$X_1 \le_{lr} X_2 \Rightarrow X_1 \le_{hr} X_2 \Rightarrow X_1 \le_{st} X_2$$

The ratio of the two densities is

$$\frac{b_{X_1}(x)}{b_{X_2}(x)} = \frac{\varepsilon_1}{\varepsilon_2}\, \sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon_1} - \left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon_2}} \left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{\varepsilon_1-\varepsilon_2},\ x>0$$

Taking logarithm for both sides, we get

$$\log\left(\frac{b_{X_1}(x)}{b_{X_2}(x)}\right) = \log(\varepsilon_1-\varepsilon_2) + \left[\left(1-e^{-\left(1-e^{-\omega}\right)\upsilon x}\right)^{\varepsilon_1} - \left(1-e^{-\left(1-e^{-\omega}\right)\upsilon x}\right)^{\varepsilon_2}\right] \log\sigma$$
$$+(\varepsilon_1-\varepsilon_2)\log\left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right),\ x>0$$

Differentiating both sides yields

$$\frac{d}{dx}\log\left(\frac{b_{X_1}(x)}{b_{X_2}(x)}\right) = C\,\log\sigma\left[\varepsilon_1\left(1-e^{-\left(1-e^{-\omega}\right)\upsilon x}\right)^{\varepsilon_1-1} - \varepsilon_2\left(1-e^{-\left(1-e^{-\omega}\right)\upsilon x}\right)^{\varepsilon_2-1}\right]$$
$$+C\,(\varepsilon_1-\varepsilon_2)\left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{-1},$$

where$C = \upsilon\,(1-e^{-\omega})\, e^{-\left(1-e^{-\omega}\right)\upsilon x}$.

For $\varepsilon_2 > \varepsilon_1$, $\frac{d}{dx}\log\left(\frac{b_{X_1}(x)}{b_{X_2}(x)}\right) < 0$,

This implies that $X_1 \le_{lr} X_2$,and consequently $X_1 \le_{hr} X_2$, and $X_1 \le_{st} X_2$.

3.5. Order Statistics

From the NPE-ETE distribution with Cd in (2.2) and Pf in (2.3), we determine the Pd of the joint of order statistic and the sth order statistic. Moreover, we derive the Pd of the smallest (first order statistic) and the last (largest order statistic).

For an ordered random sample of size n, $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$,

1. **The Pf of the first$X_{(1)}$and last$X_{(n)}$order statistics are given respectively as:**

$$b_{X_{(1)}}(x) = n\,[1 - B(x)]^{n-1}\, b(x)\,,$$

and

$$b_{X_{(n)}}(x) = n\,[B(x)]^{n-1}\, b(x)$$

For NPE-ETE distribution, Using (2.2) and (2.3), we get for $\sigma \neq 1$,

$$b_{X_{(1.1)}}(x) = \frac{n\,\varepsilon\,\upsilon\,\ln(\sigma)}{(\sigma-1)}\left(\sigma - \sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}}\right)^{n-1}(1-e^{-\omega})\;\sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}}\left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{\varepsilon-1} e^{-\left(1-e^{-\omega}\right)\upsilon x}\text{ ,and}$$

$$b_{X_{(n)}}(x) = \frac{n\,\varepsilon\,\upsilon\,\ln(\sigma)}{(\sigma-1)^{n+1}}\left(\sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}} - 1\right)^{n-1}(1-e^{-\omega})\;\sigma^{\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)^{\varepsilon}}\left(1-e^{-\left(1-e^{-(1-e^{-\omega})\upsilon x}\right)}\right)^{\varepsilon-1} e^{-\left(1-e^{-\omega}\right)\upsilon x}.$$

1. **The Pf of the sth order statistic is given by:**

$$b_{X_{(s)}}(x) = \frac{n\,!}{(s-1)\,!\,(n-s)\,!}\,[B(x)]^{s-1}\,[1 - B(x)]^{n-s} b(x)\,,$$

For NPE-ETE distribution,

$$b_{X_{(s)}}(x) = Q\left(\sigma^{\left(1-e^{-Kx}\right)^{\varepsilon}} - 1\right)^{s-1}\left(\sigma - \sigma^{\left(1-e^{-Kx}\right)^{\varepsilon}}\right)^{n-s}\left(1-e^{-\left(1-e^{-Kx}\right)}\right)^{\varepsilon-1}\sigma^{\left(1-e^{-Kx}\right)^{\varepsilon}} e^{-Kx},\;\; 0 \leq x < \infty\,,$$

where$K = (1-e^{-\omega})\;\upsilon$,and$Q = \frac{n\,!}{(s-1)\,!\,(n-s)\,!}\,\frac{\varepsilon K\,\ln(\sigma)}{(\sigma-1)^{n}}$.

1. **The joint Pf of two order statistics is derived as follows:**

Let $X_{(s)} = X$,and$X_{(h)} = Z\,,\;\; s < h\,,$

$$b_{X_{(s)},X_{(h)}}(x\,,\,z) = \frac{n\,!}{(s-1)\,!\,(h-s-1)\,!(n-h)\,!}\,[B(x)]^{s-1}\;[B(z) - B(x)]^{h-s-1}\,[1 - B(z)]^{n-h} b(x)\,b(z)\,,$$

Applying for NPE-ETE distribution,

$$b_{X_{(s)},X_{(h)}}(x\,,\,z) = T\left(\sigma^{\left(1-e^{-Kx}\right)^{\varepsilon}} - 1\right)^{s-1}\left(\sigma^{\left(1-e^{-Kz}\right)^{\varepsilon}} - \sigma^{\left(1-e^{-Kx}\right)^{\varepsilon}}\right)^{h-s-1}\left(\sigma - \sigma^{\left(1-e^{-Kz}\right)^{\varepsilon}}\right)^{n-h}\left(1-e^{-\left(1-e^{-Kx}\right)}\right)^{\varepsilon-1}$$
$$\left(1-e^{-\left(1-e^{-Kz}\right)}\right)^{\varepsilon-1}\sigma^{\left(1-e^{-Kx}\right)^{\varepsilon}+\left(1-e^{-Kz}\right)^{\varepsilon}} e^{-K(x+z)},\;\; 0 \leq x,\, z < \infty\,,$$

whereT $= \frac{n\,!}{(s-1)\,!\,(h-s-1)\,!(n-h)\,!}\,\frac{(\varepsilon K\,\ln(\sigma))^{2}}{(\sigma-1)^{n}}$, $\sigma \neq 1$.

4. Estimation

In this section some methods of estimation for the unknown parameters of NPE-ETE distribution as maximum likelihood estimation (MLE), Cramer von Mises (CM), and Percentiles method (PM) will be studied. For more recently paper used estimation methods , see [1, 7, 22].

4.1. Maximum Likelihood Method

The maximum likelihood estimators (ML_{es}), $\dot{\Sigma} = (\dot{\sigma}, \dot{\varepsilon}, \dot{\upsilon}, \dot{\omega})^T$of the NPE-ETE distribution's unknown parameters were obtained in this section. If $X_1, X_2, \ldots, X_n$constitute a random sample of length n drawn from the NPE-ETE distribution with vector parameter$\Sigma = (\sigma, \varepsilon, \upsilon, \omega)^T$, $\sigma \neq 1$,the log-likelihood function can be written:

$$\begin{aligned}\ln_L(x;\Sigma) = {} & n\log(\varepsilon\upsilon(1-e^{-\omega})) + n\log(\log\sigma) - n\log(\sigma-1) - \upsilon(1-e^{-\omega})\sum_{i=1}^{n} x_i \\ & + \sum_{i=1}^{n}\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right]^{\varepsilon}\log\sigma + (\varepsilon-1)\sum_{i=1}^{n}\log\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right].\end{aligned}$$

The elements of first derivatives for log-likelihood function with respect to $\Sigma = (\sigma, \varepsilon, \upsilon, \omega)^T$are given by:

$$\frac{\partial \ln_L(x;\Sigma)}{\partial\sigma} = \frac{n}{\sigma\ln\sigma} - \frac{n}{(\sigma-1)} + \frac{1}{\sigma}\sum_{i=1}^{n}\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right]^{\varepsilon}, \tag{4.1}$$

$$\frac{\partial \ln_L(x;\Sigma)}{\partial\varepsilon} = \frac{n}{\varepsilon} + (\ln\sigma)\sum_{i=1}^{n}\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right]^{\varepsilon}\ln\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right] + \sum_{i=1}^{n}\ln\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right], \tag{4.2}$$

$$\begin{aligned}\frac{\partial \ln_L(x;\Sigma)}{\partial\upsilon} = {} & \frac{n}{\upsilon} - (1-e^{-\omega})\sum_{i=1}^{n} x_i + \varepsilon(1-e^{-\omega})(\ln\sigma)\sum_{i=1}^{n} x_i e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right]^{\varepsilon-1} \\ & + (\varepsilon-1)(1-e^{-\omega})\sum_{i=1}^{n} x_i e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right]^{-1},\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\frac{\partial \ln_L(x;\Sigma)}{\partial\omega} = {} & \frac{n e^{-\omega}}{(1-e^{-\omega})} - \upsilon e^{-\omega}\sum_{i=1}^{n} x_i + \varepsilon\upsilon(\ln\sigma)e^{-\omega}\sum_{i=1}^{n} x_i e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right]^{\varepsilon-1} \\ & + \upsilon(\varepsilon-1)e^{-\omega}\sum_{i=1}^{n} x_i e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\left[1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right]^{-1}.\end{aligned} \tag{4.4}$$

Using a Mathcad mathematical software Package to solving simultaneously the non-linear equations (4.1) through (4.4) above and setting them to zero, to determine The $ML_{es}\dot{\Sigma} = (\dot{\sigma}, \dot{\varepsilon}, \dot{\upsilon}, \dot{\omega})^T$.

The ML_{es} of NPE-ETE can be interpreted as approximately four-variates normal and a variance-covariance matrix equal to the inverse of the expected Fisher Information Matrix (FI_M) by using the standard large sample approximation. The elements of 4×4matrix $FI_M(\Sigma)$may be estimated by:

$$FI_M(\tilde{\Sigma}) = -\ln_L(x;\Sigma)_{\Sigma_i\Sigma_j}\Big|_{\Sigma=\tilde{\Sigma}}, \quad i, j = \{1, 2, 3, 4\}$$

as follows

$$FI_M(\Sigma) = -\left.\begin{pmatrix} \frac{\partial^2\ln_L(x;\Sigma)}{\partial\sigma^2} & \frac{\partial^2\ln_L(x;\Sigma)}{\partial\sigma\partial\varepsilon} & \frac{\partial^2\ln_L(x;\Sigma)}{\partial\sigma\partial\upsilon} & \frac{\partial^2\ln_L(x;\Sigma)}{\partial\sigma\partial\omega} \\ & \frac{\partial^2\ln_L(x;\Sigma)}{\partial\varepsilon^2} & \frac{\partial^2\ln_L(x;\Sigma)}{\partial\varepsilon\partial\upsilon} & \frac{\partial^2\ln_L(x;\Sigma)}{\partial\varepsilon\partial\omega} \\ & & \frac{\partial^2\ln_L(x;\Sigma)}{\partial\upsilon^2} & \frac{\partial^2\ln_L(x;\Sigma)}{\partial\upsilon\partial\omega} \\ & & & \frac{\partial^2\ln_L(x;\Sigma)}{\partial\omega^2} \end{pmatrix}\right|_{\Sigma=\dot{\Sigma}}$$

For $\rho_i = \left(1-e^{-\left(1-e^{-\omega}\right)\upsilon x_i}\right)$,Entries of the observed FI_M of NPE-ETE are:

$$\frac{\partial^2\ln_L(x;\Sigma)}{\partial\sigma^2} = \frac{-n}{\sigma^2\ln\sigma}\left[1+(\ln\sigma)^{-1}\right] - \frac{1}{\sigma^2}\sum_{i=1}^{n}\rho_i^{\varepsilon},$$

$$\frac{\partial^2 \ln_L(x;\Sigma)}{\partial\sigma\,\partial\varepsilon} = \frac{1}{\sigma}\sum_{i=1}^{n}\rho_i^{\varepsilon}\,\ln\rho_i\ ,$$

$$\frac{\partial^2 \ln_L(x;\Sigma)}{\partial\sigma\,\partial\nu} = \frac{\varepsilon}{\sigma}\,(1-e^{-\omega})\sum_{i=1}^{n}x_i\,\rho_i^{\varepsilon-1}\,(1-\rho_i)\ ,$$

$$\frac{\partial^2 \ln_L(x;\Sigma)}{\partial\sigma\,\partial\omega} = \frac{\varepsilon\,\upsilon}{\sigma}\,e^{-\omega}\sum_{i=1}^{n}x_i\,\rho_i^{\varepsilon-1}\,(1-\rho_i)\ ,$$

$$\frac{\partial^2 \ln_L(x;\Sigma)}{\partial\varepsilon^2} = \frac{-n}{\varepsilon^2} + \ln\sigma\sum_{i=1}^{n}\rho_i^{\varepsilon}\,(\ln\rho_i)^2\ ,$$

$$\frac{\partial^2 \ln_L(x;\Sigma)}{\partial\varepsilon\,\partial\nu} = (1-e^{-\omega})\left[\ln\sigma\sum_{i=1}^{n}x_i\,\rho_i^{\varepsilon-1}(1-\rho_i)\,\{1+\varepsilon\,\ln\rho_i\} + \sum_{i=1}^{n}x_i\,\rho_i^{-1}\,(1-\rho_i)\right],$$

$$\frac{\partial^2 \ln_L(x;\Sigma)}{\partial\varepsilon\,\partial\omega} = \upsilon\,e^{-\omega}\left[\ln\sigma\sum_{i=1}^{n}x_i\,\rho_i^{\varepsilon-1}\,(1-\rho_i)\ \{1+\varepsilon\,\ln\rho_i\} + \sum_{i=1}^{n}x_i\,\rho_i^{-1}\,(1-\rho_i)\right],$$

$$\begin{aligned}\frac{\partial^2 \ln_L(x;\Sigma)}{\partial\upsilon^2} = {} & \frac{-n}{\upsilon^2} + \varepsilon\,(1-e^{-\omega})^2\,\ln\sigma\,\textstyle\sum_{i=1}^{n}x_i^2\,\rho_i^{\varepsilon-1}\,(1-\rho_i)\left\{(\varepsilon-1)\,\rho_i^{-1}\,(1-\rho_i)-1\right\}\\ & -(\varepsilon-1)\,(1-e^{-\omega})^2\,\textstyle\sum_{i=1}^{n}x_i^2\,\rho_i^{-1}\,(1-\rho_i)\left\{\rho_i^{-1}\,(1-\rho_i)-1\right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ln_L(x;\Sigma)}{\partial\upsilon\,\partial\omega} = {} & \varepsilon\,e^{-\omega}\,\ln\sigma\,\textstyle\sum_{i=1}^{n}x_i\,\rho_i^{\varepsilon-1}\,(1-\rho_i)\left\{1+\upsilon\,x_i+\upsilon\,(\varepsilon-1)\,(1-e^{-\omega})\,x_i\rho_i^{-1}\right\} - e^{-\omega}\textstyle\sum_{i=1}^{n}x_i\\ & -\upsilon\,(\varepsilon-1)\,(1-e^{-\omega})\,e^{-\omega}\textstyle\sum_{i=1}^{n}x_i^2\,\rho_i^{-1}\,(1-\rho_i)\left\{1+\rho_i^{-1}\,(1-\rho_i)\right\},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \ln_L(x;\Sigma)}{\partial\omega^2} = {} & \frac{-n\,e^{-\omega}}{(1-e^{-\omega})}\left[1+\frac{e^{-\omega}}{(1-e^{-\omega})}\right] - \upsilon\,(\varepsilon-1)\,e^{-\omega}\textstyle\sum_{i=1}^{n}x_i\,\rho_i^{-1}\,(1-\rho_i)\left\{1+\upsilon\,e^{-\omega}x_i\left[\rho_i^{-1}(1-\rho_i)-1\right]\right\}\\ & +\upsilon\,e^{-\omega}\textstyle\sum_{i=1}^{n}x_i - \varepsilon\,\upsilon\,e^{-\omega}\ln\sigma\textstyle\sum_{i=1}^{n}x_i\,\rho_i^{\varepsilon-1}\,(1-\rho_i)\left\{1-\upsilon\,e^{-\omega}\,x_i\left[(\varepsilon-1)\,\rho_i^{-1}\,(1-\rho_i)\right]+1\right\}.\end{aligned}$$

Consequently, the two sided confidence intervals for $\Sigma = (\sigma,\ \varepsilon,\ \upsilon,\ \omega)^T$ of the NPE-ETE distribution are provided by:

$\dot{\sigma} \pm z_{\beta/2}\sqrt{FI_{M_{11}}^{-1}(\dot{\sigma})}$, $\dot{\varepsilon} \pm z_{\beta/2}\sqrt{FI_{M_{22}}^{-1}(\dot{\varepsilon})}$, $\dot{\upsilon} \pm z_{\beta/2}\sqrt{FI_{M_{33}}^{-1}(\dot{\upsilon})}$, and $\dot{\omega} \pm z_{\beta/2}\sqrt{FI_{M_{44}}^{-1}(\dot{\omega})}$, where the upper β^{th} percentile of the standard normal distribution is represented by z_{β}.

4.2. Cramer-von Mises Method

For an ordered random sample of size n, $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$, drawn from NPE-ETE distribution (2.3), the CM estimators (CM_{es}) $\ddot{\Sigma} = (\ddot{\sigma},\ \ddot{\varepsilon},\ \ddot{\upsilon},\ \ddot{\omega})^T$ of $\Sigma = (\sigma,\ \varepsilon,\ \upsilon,\ \omega)^T$ can be obtained by minimizing

$$CM\,(x\,;\Sigma) = \frac{1}{12\,n} + \sum_{i=1}^{n}\left[(\sigma-1)^{-1}\left(\sigma^{\left(1-e^{-(1-e^{-\omega})\,\upsilon\,x_{(i)}}\right)^{\varepsilon}} - 1\right) - \frac{2i-1}{2\,n}\right]^2$$

With respect to $\Sigma = (\sigma,\ \varepsilon,\ \upsilon,\ \omega)^T$, are given by

$$\frac{\partial CM(x;\Sigma)}{\partial\sigma} = 2\sum_{i=1}^{n}(\sigma-1)^{-1}\left[\varphi^{\varepsilon}\,\sigma^{\varphi^{\varepsilon}-1}-(\sigma-1)^{-1}\left(\sigma^{\varphi^{\varepsilon}}-1\right)\right]\left[(\sigma-1)^{-1}\left(\sigma^{\varphi^{\varepsilon}}-1\right)-\frac{2i-1}{2n}\right], \tag{4.5}$$

$$\frac{\partial CM(x;\Sigma)}{\partial\varepsilon} = 2\sum_{i=1}^{n}(\sigma-1)^{-1}\left[\varphi^{\varepsilon}\,\sigma^{\varphi^{\varepsilon}}\ln\sigma\,\ln\varphi\right]\left[(\sigma-1)^{-1}\left(\sigma^{\varphi^{\varepsilon}}-1\right)-\frac{2i-1}{2n}\right], \tag{4.6}$$

$\frac{\partial CM(x;\Sigma)}{\partial\upsilon} = 2\sum_{i=1}^{n}(\sigma-1)^{-1}\left[\varepsilon\,x_{(i)}\,(1-e^{-\omega})\,(1-\varphi)\,\varphi^{\varepsilon-1}\,\sigma^{\varphi^{\varepsilon}}\ln\sigma\right]\left[(\sigma-1)^{-1}\left(\sigma^{\varphi}-1\right)-\frac{2i-1}{2n}\right]$, and

$$\frac{\partial CM(x;\Sigma)}{\partial\omega} = 2\sum_{i=1}^{n}(\sigma-1)^{-1}\left[\varepsilon\,\upsilon\,x_{(i)}\,e^{-\omega}\,(1-\varphi)\,\varphi^{\varepsilon-1}\,\sigma^{\varphi^{\varepsilon}}\ln\sigma\right]\left[(\sigma-1)^{-1}\left(\sigma^{\varphi}-1\right)-\frac{2i-1}{2n}\right], \tag{4.7}$$

Taking $\varphi = \left(1-e^{-\left(1-e^{-w}\right)\upsilon\,x_{(i)}}\right)$. By equating non-linear equations (4.5)-(4.7) to zero, equations have no explicit solution, so numerical iterative technique is used to obtain the CM$_{es}$ $\ddot{\Sigma} = (\ddot{\sigma}, \ddot{\varepsilon}, \ddot{\upsilon}, \ddot{\omega})^{T}$.

4.3. Percentiles Method

In this subsection the percentiles method (PM) for estimating unknown parameters will be considered. Let $X_1 \le X_2 \le ... \le X_n$ be a random sample from NPE-ETE distribution function with parameters vector $\Sigma = (\sigma, \varepsilon, \upsilon, \omega)^T$ and $X_{(i)}$ denotes the ith order statistic, $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$. If p_i denotes some estimates of $B(x_{(i)};\Sigma)$, Then PM estimators (PM$_{es}$) are obtained by minimizing the following equation with respect to $\Sigma = (\sigma, \varepsilon, \upsilon, \omega)^T$.

$$PM(x;\Sigma) = \sum_{i=1}^{n}\left[\ln(p_i) - \ln\left(B(x_{(i)};\Sigma)\right)\right]^2$$

In our study, the formula $p_i = (i/(n+1))$ will be used. PM estimators (PM$_{es}$) $\dddot{\Sigma} = \left(\dddot{\sigma}, \dddot{\varepsilon}, \dddot{\upsilon}, \dddot{\omega}\right)^T$ of NPE-ETE distribution can be obtained by minimizing:

$$PM(x;\Sigma) = \sum_{i=1}^{n}\left[\ln(p_i) - \ln\left((\sigma-1)^{-1}\left(\sigma^{\left(1-e^{-\left(1-e^{-\omega}\right)\upsilon\,x_{(i)}}\right)^{\varepsilon}}-1\right)\right)\right]^2$$

With respect to $\Sigma = (\sigma, \varepsilon, \upsilon, \omega)^T$, as follows:

$$\frac{\partial PM(x;\Sigma)}{\partial\sigma} = -2\sum_{i=1}^{n}\left[\ln(p_i) - \ln\left((\sigma-1)^{-1}\left(\sigma^{\varphi^{\varepsilon}}-1\right)\right)\right]\left[\frac{\varphi^{\varepsilon}\,\sigma^{\varphi^{\varepsilon}-1}}{\sigma^{\varphi^{\varepsilon}}-1}-(\sigma-1)^{-1}\right], \tag{4.8}$$

$$\frac{\partial PM(x;\Sigma)}{\partial\varepsilon} = -2\sum_{i=1}^{n}\left[\ln(p_i) - \ln\left((\sigma-1)^{-1}\left(\sigma^{\varphi^{\varepsilon}}-1\right)\right)\right]\left[\frac{\varphi^{\varepsilon}\,\sigma^{\varphi^{\varepsilon}}\ln\sigma\,\ln\varphi}{\sigma^{\varphi^{\varepsilon}}-1}\right], \tag{4.9}$$

$\frac{\partial PM(x;\Sigma)}{\partial \upsilon} = -2\sum_{i=1}^{n}\left[\ln\,(p_i) - \ln\left(\,(\sigma-1)^{-1}\left(\sigma^{\varphi^{\varepsilon}}-1\right)\,\right)\,\right]\left[\frac{\varepsilon\,x_{(i)}\,\varphi^{\varepsilon-1}\,\sigma^{\varphi^{\varepsilon}}\left(1-e^{-\omega}\right)(1-\varphi)\,\ln\sigma}{\sigma^{\varphi^{\varepsilon}}-1}\right]$, and

$$\frac{\partial PM\,(x\,;\Sigma)}{\partial\omega} = -2\sum_{i=1}^{n}\left[\ln\,(p_i) - \ln\left(\,(\sigma-1)^{-1}\left(\sigma^{\varphi^{\varepsilon}}-1\right)\,\right)\,\right]\left[\frac{\varepsilon\,\upsilon\,x_{(i)}\,\varphi^{\varepsilon-1}\,\sigma^{\varphi^{\varepsilon}}\,e^{-\omega}\,(1-\varphi)\,\ln\sigma}{\sigma^{\varphi^{\varepsilon}}-1}\right], \tag{4.10}$$

Where $\varphi = \left(1 - e^{-\left(1-e^{-w}\right)\upsilon\,x_{(i)}}\right)$. Percentiles estimators of NPE-ETE distribution can be obtained by setting equations (4.8)-(4.10) equal to zero and solved them numerically.

5. Numerical Study

In this section a numerical study is performed to compare the different estimators proposed in the previous sections. The maximum likelihood (ML), carmer-von mises (CM), and percentiles (PM) estimators of four unknown parameters of NPE-ETE distribution are derived. In terms of mean square errors (MSEs), absolute biased (AB), standarad error (SE), and length of confidence interval, we evaluate the performance of the different estimators for different sample sizes and for different parametric values.

For ML: $MSEs = E\left(\dot{\Sigma}-\Sigma\right)^2$, $AB = \left|\dot{\Sigma}-\Sigma\right|$, $SE = \sqrt{var(\dot{\Sigma})}$, LCI = upper CI($\dot{\Sigma}$) − lower CI $\left(\dot{\Sigma}\right)$

For CM: $MSEs = E\left(\ddot{\Sigma}-\Sigma\right)^2$, $AB = \left|\ddot{\Sigma}-\Sigma\right|$, $SE = \sqrt{var(\ddot{\Sigma})}$, LCI = upper CI($\ddot{\Sigma}$) − lower CI $\left(\ddot{\Sigma}\right)$

For PM: $MSEs = E\left(\dddot{\Sigma}-\Sigma\right)^2$, $AB = \left|\dddot{\Sigma}-\Sigma\right|$, $SE = \sqrt{var(\dddot{\Sigma})}$, LCI = upper CI($\dddot{\Sigma}$) − lower CI $\left(\dddot{\Sigma}\right)$

The algorithm is described through the following steps:

1. Generate random sample $X_1, X_2, \ldots, X_n$of sizes $(n = 20(10)50,\ 70,\ 100)$from NPE-ETE distribution.
2. Different values of parameters combinations will be selected as, $\sigma = (0.1\,,\,0.3\,,1.5)$, $\varepsilon = (0.2,\ 0.3)$, $\upsilon = (0.2,\ 0.3)$, and $\omega = (0.1,\ 0.3,\ 0.5)$.
3. The ML$_{es}$ $\dot{\Sigma} = (\dot{\sigma},\ \dot{\varepsilon},\ \dot{\upsilon},\ \dot{\omega})^T$will be obtained by solving non-linear equations (4.1)-(4.4). The CM$_{es}$ $\ddot{\Sigma} = (\ddot{\sigma},\ \ddot{\varepsilon},\ \ddot{\upsilon},\ \ddot{\omega})^T$ are obtained from equations (4.5), (4.6), (4.2), and (4.7) respectively. The PM$_{es}$ $\dddot{\Sigma} = \left(\dddot{\sigma},\ \dddot{\varepsilon},\ \dddot{\upsilon},\ \dddot{\omega}\right)^T$ can be obtained by solving non-linear equations (4.8)-(4.10).
4. Repeat Steps (1 upto 3) in 1,000 (N) iterations.
5. For each n and for selected sets of parameters, compute MSEs, AB, SE, and LCI of the sequence of 1,000 parameter estimates.

Tables 1, 2, 3, and 4 summarise the simulation results of the proposed distribution. In general it is observed that the MSEs, AB and SE of all estimators decrease as sample size increases. In addition, in almost cases, the Cramer-von Mises method does well in estimating parameters of the NPE-ETE distribution.

6. Data Analysis

We analyze the potentiality of the suggested distribution (NPE-ETE) by means of two real life data sets. Using some empirical goodness of fit measures namely $-\log_L$, Akaike Information criterion (AIC), corrected Akaike Information Criteria (CAIC), Bayesian information criterion (BIC), Anderson-Darling (A^*) and Cramer-von Mises (W^*) are displayed to compare the fitted distributions. The distribution that has lower values of $-\log_L$, AIC, CAIC, BIC, A^*, and W^* is regarded as the best distribution.

Data Set 1: (customers waiting times (min) before service data)

The first data set gives the waiting times in minutes of 100 bank customers in a queue before service. The data set was first published in Ghitany et al. [16], Merovci and Elbatal [19] and Bhati et al. [4] have also fitted the data for diferent models. The first data tabulates in Table 5.

The formulated distribution is compared with some competing sub-distributions, which are Erlang Truncated Exponential (ETE) distribution, Exponentiated Burr XII (Exp Burr XII) distribution, and Exponentiated Frechet (Exp Frechet) distribution. The Pfs of these distributions are given by:

1. ETE distribution
$$b(x) = \upsilon\,(1 - e^{-\omega})\,e^{-\left(1-e^{-\omega}\right)\upsilon\,x}\,,\; x > 0,\; \upsilon,\; \omega > 0\,.$$
2. Exp Burr XII distribution
$$b(x) = \alpha\,k\,c\,x^{c-1}\,(1 + x^{c})^{-k-1}\left[1 - (1 + x^{c})^{-k}\right]^{\alpha-1}\,,\; x > 0,\; \alpha,\; k,\; c > 0\,.$$
3. Exp Frechet
$$b(x) = \frac{\alpha\,\upsilon}{s}\left(\frac{x}{s}\right)^{-\upsilon-1}\,e^{-\alpha\left(\frac{x}{s}\right)^{-\upsilon}}\,,\; x > 0,\; \alpha,\; \upsilon,\; s > 0\,.$$

Table 6 displays the goodness of fit measures of NPE-ETE and competing models. It is clear that, the NPE-ETE distribution has smaller values of all measures compared with other distributions. So, the NPE-ETE provides a better fit to data set than other competing models.

Data Set 2: (Survival times of guinea pigs)

The second data set consists of the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, the data were reported by Bjerkedal [5] and displays in Table 7.

Comparison was made between the model's goodness of fit and that Poisson ETE (PETE) distribution (Nasiru et al. [23]), Inverse Weibull Poisson (IWP) distribution (Bera [3]) and Exponentiated Kumaraswamy Dagum (EKD) distribution (Huang and Oluyede [18]) as shown in Table 8.

It is evident that, in comparison to the other fitted models, the NPE-ETE distribution yields a more accurate fit to the second data based on the selected analytical measures as shown in Table 8.

7. Two-Variable Expansion

Bivariate distributions can be easily described using a copula function. Assume that the pair (X_1, X_2) represents bivariate random variables distributed a NPE-ETE with Copula C. The Farlie Gumbel Morgenstern (FGM) copula is a widely used parametric families of copulas, it is Cd function defines as follows:

$$C\,(u\,,\,v) = \;u\,v\;[\,1 + \,\tau\;(1 - u)\;(1 - v)\,]\;,\;\;|\,\tau\,|\;<\,1,$$

Table 1. Estimated parameters: Set of Parameters 1 ($\sigma = 0.1$, $\varepsilon = 0.3$, $\upsilon = 0.2$, $\omega = 0.3$)

n	Measures	PM				CM				MLE			
		$\ddot{\omega}$	$\ddot{\upsilon}$	$\ddot{\varepsilon}$	$\ddot{\alpha}$	$\dot{\omega}$	$\dot{\upsilon}$	$\dot{\varepsilon}$	$\dot{\alpha}$	ω	υ	ε	α
20	MSE	5.929	0.033	0.077	0.003	2.736	0.032	0.079	0.0006	3.468	2.506	2.591	5.211
	AB	2.2	0.181	0.277	0.045	1.473	0.181	0.281	0.024	1.697	1.572	1.597	3.064
	SE	1.043	0.03	0.051	0.038	0.752	0.027	0.048	0.023	0.411	0.337	0.576	1.328
	LCI	3.089	0.118	0.2	0.148	2.947	0.186	0.189	0.091	3.093	0.782	0.659	0.793
30	MSE	2.855	0.031	0.073	0.001	1.156	0.032	0.075	0.0005	1.642	2.101	2.214	3.629
	AB	1.504	0.177	0.269	0.027	0.96	0.177	0.273	0.021	1.278	1.445	1.482	2.451
	SE	0.769	0.017	0.03	0.023	0.484	0.013	0.025	0.013	0.267	0.259	0.315	0.955
	LCI	2.015	0.067	0.118	0.089	1.896	0.052	0.096	0.051	1.56	0.664	0.444	0.526
40	MSE	2.281	0.027	0.062	0.0007	0.721	0.029	0.067	0.0004	0.932	1.987	0.814	3.152
	AB	1.393	0.164	0.247	0.023	0.765	0.17	0.259	0.017	1.098	1.245	1.256	1.741
	SE	0.583	0.012	0.021	0.016	0.368	0.009	0.017	0.0092	0.091	0.188	0.202	0.651
	LCI	1.285	0.046	0.081	0.063	1.441	0.039	0.068	0.036	0.845	0.554	0.231	0.311
50	MSE	1.237	0.026	0.057	0.0002	0.394	0.027	0.061	0.0003	0.714	0.957	0.638	3.012
	AB	1.022	0.159	0.237	0.01	0.569	0.164	0.246	0.012	0.702	0.837	0.956	0.923
	SE	0.438	0.0074	0.013	0.01	0.264	0.006	0.012	0.0062	0.073	0.118	0.132	0.301
	LCI	0.716	0.029	0.053	0.041	1.037	0.025	0.045	0.024	0.622	0.321	0.091	0.102
70	MSE	0.878	0.018	0.04	0.00008	0.317	0.019	0.045	0.00019	0.609	0.655	0.523	2.781
	AB	0.88	0.132	0.193	0.0057	0.527	0.137	0.194	0.0054	0.526	0.641	0.784	0.823
	SE	0.324	0.0049	0.0085	0.0075	0.199	0.003	0.007	0.0042	0.014	0.084	0.903	0.145
	LCI	0.269	0.019	0.033	0.03	0.78	0.016	0.028	0.017	0.41	0.111	0.074	0.084
100	MSE	0.453	0.012	0.027	0.00001	0.174	0.012	0.032	0.00015	0.389	0.355	0.321	0.935
	AB	0.638	0.078	0.098	0.0014	0.395	0.078	0.082	0.0084	0.341	0.421	0.654	0.684
	SE	0.213	0.0029	0.0054	0.0043	0.133	0.002	0.004	0.0024	0.009	0.057	0.075	0.088
	LCI	0.083	0.012	0.022	0.017	0.521	0.01	0.017	0.009	0.233	0.081	0.045	0.054

where $u = B_{X_1}(x_1)$, and $v = B_{X_2}(x_2)$ are the marginal functions, $\tau$is a dependence parameter. We shall use the BNPE-ETE distribution to represent the two-variable NPE-ETE distribution.

BNPE-ETE Model

As stated by FGM, the bivariate NPE-ETE distribution's joint Cd function is provided by:

$$
\begin{aligned}
B_{X_1X_2}(x_1\,,\,x_2) &= C\left(B_{X_1}(x_1)\,,\,B_{X_2}(x_2)\right) \\
&= K^{-1}\left(\sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}} - 1\right)\left(\sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}} - 1\right) \\
&\quad \times \left[1 + \tau K^{-1}\left(\sigma_1 - \sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}}\right)\left(\sigma_2 - \sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}}\right)\right] \\
&\qquad ,\; x_1\,, x_2 > 0,\, \sigma_1,\, \sigma_2,\, \varepsilon_1,\, \varepsilon_2,\, \upsilon_1,\, \upsilon_2,\, \omega_1,\, \omega_2 > 0\,,\, \sigma_1,\, \sigma_2 \neq 1.
\end{aligned}
$$

The corresponding joint Pf function of BNPE-ETE distribution can be obtained from the following

Table 2. Estimated parameters: Set of Parameters 2 ($\sigma = 0.3$, $\varepsilon = 0.2$, $\upsilon = 0.2$, $\omega = 0.1$)

n	Measures	PM				CM				MLE			
		$\ddot{\omega}$	$\ddot{\upsilon}$	$\ddot{\varepsilon}$	$\ddot{\alpha}$	$\dot{\omega}$	$\dot{\upsilon}$	$\dot{\varepsilon}$	$\dot{\alpha}$	ω	υ	ε	α
20	MSE	2.591	2.506	3.468	4.211	0.014	0.032	0.0076	2.822	0.0075	0.03	0.0072	3.878
	AB	1.597	1.572	1.697	4.066	0.117	0.177	0.088	1.354	0.087	0.174	0.085	1.667
	SE	0.576	0.531	0.267	1.328	0.023	0.052	0.021	0.801	0.034	0.054	0.019	0.808
	LCI	0.791	0.738	2.006	4.124	0.092	0.205	0.084	3.139	0.134	0.211	0.074	3.109
30	MSE	2.214	2.101	2.772	3.629	0.013	0.03	0.0072	1.334	0.0068	0.027	0.0067	2.778
	AB	1.482	1.445	1.451	3.289	0.114	0.172	0.085	1.019	0.082	0.165	0.082	1.458
	SE	0.315	0.337	0.114	1.089	0.013	0.027	0.011	0.544	0.021	0.034	0.012	0.808
	LCI	0.519	0.461	1.355	3.663	0.052	0.108	0.042	2.132	0.084	0.132	0.048	2.156
40	MSE	1.988	1.987	1.642	2.451	0.011	0.025	0.0063	0.73	0.0061	0.021	0.0054	2.127
	AB	1.36	1.341	1.344	3.145	0.104	0.157	0.079	0.754	0.07	0.143	0.073	1.324
	SE	0.202	0.259	0.091	0.955	0.009	0.018	0.0081	0.402	0.016	0.023	0.0092	0.612
	LCI	0.219	0.258	0.978	1.657	0.038	0.072	0.031	1.576	0.063	0.09	0.036	1.401
50	MSE	1.101	1.455	1.31	1.791	0.01	0.021	0.0055	0.44	0.0051	0.017	0.0048	1.08
	AB	1.256	1.254	1.278	2.171	0.101	0.142	0.074	0.588	0.067	0.127	0.069	0.933
	SE	0.132	0.188	0.073	0.804	0.006	0.013	0.0078	0.307	0.011	0.016	0.0056	0.458
	LCI	0.102	0.175	0.742	0.843	0.025	0.05	0.021	1.202	0.041	0.062	0.022	0.796
70	MSE	0.814	0.861	0.932	0.891	0.007	0.011	0.0032	0.328	0.0049	0.012	0.0038	0.63
	AB	0.956	0.837	1.089	1.597	0.084	0.089	0.052	0.525	0.067	0.095	0.059	0.723
	SE	0.844	0.118	0.014	0.651	0.004	0.008	0.0054	0.231	0.0075	0.0098	0.0036	0.328
	LCI	0.081	0.084	0.658	0.571	0.018	0.032	0.013	0.901	0.03	0.038	0.014	0.287
100	MSE	0.638	0.568	0.655	0.623	0.005	0.011	0.0028	0.158	0.0034	0.012	0.0023	0.32
	AB	0.742	0.602	0.702	0.911	0.035	0.048	0.036	0.365	0.0015	0.014	0.033	0.522
	SE	0.421	0.654	0.009	0.301	0.002	0.005	0.002	0.159	0.0044	0.0056	0.0023	0.221
	LCI	0.052	0.062	0.311	0.236	0.011	0.02	0.008	0.624	0.013	0.022	0.009	0.086

relation:

$$b_{X_1X_2}(x_1\,,\,x_2) = \frac{\partial^2}{\partial x_1\,\partial x_2} B_{X_1X_2}(x_1\,,\,x_2)$$

$$= \Psi e^{-\left[\left(1-e^{-\omega_1}\right)\upsilon_1 x_1 + \left(1-e^{-\omega_2}\right)\upsilon_2 x_2\right]} \left(1 - e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1 - 1} \left(1 - e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2 - 1} \sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}} \sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}}$$
$$\left[1 + \tau K^{-1}\left(\sigma_1 - 2\sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}} + 1\right)\left(\sigma_2 - 2\sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}} + 1\right)\right],$$

where$K = (\sigma_1 - 1)\,(\sigma_2 - 1)$, $\Psi = \frac{\varepsilon_1\varepsilon_2\upsilon_1\upsilon_2 \ln\sigma_1 \ln\sigma_2\left(1-e^{-\omega_1}\right)\left(1-e^{-\omega_2}\right)}{K}$.

Table 3. Estimated parameters: Set of Parameters 3 ($\sigma = 1.5$, $\varepsilon = 0.3$, $\upsilon = 0.2$, $\omega = 0.3$)

n	Measures	PM				CM				MLE			
		$\ddot{\omega}$	$\ddot{\upsilon}$	$\ddot{\varepsilon}$	$\ddot{\alpha}$	$\dot{\omega}$	$\dot{\upsilon}$	$\dot{\varepsilon}$	$\dot{\alpha}$	ω	υ	ε	α
20	MSE	1.185	0.075	0.035	0.046	2.354	0.07	0.035	0.056	3.948	3.564	2.112	0.692
	AB	1.932	0.274	0.187	0.214	1.252	0.264	0.187	0.236	2.394	2.856	2.217	1.109
	SE	0.761	0.041	0.037	0.038	0.886	0.038	0.037	0.024	1.651	0.287	0.305	0.573
	LCI	1.996	0.159	0.145	0.148	1.473	0.148	0.144	0.095	1.393	1.124	1.197	2.245
30	MSE	0.792	0.072	0.033	0.043	1.01	0.068	0.033	0.054	3.101	2.411	1.991	0.613
	AB	1.322	0.267	0.181	0.208	0.808	0.26	0.182	0.232	2.284	2.71	2.023	0.817
	SE	0.6	0.024	0.022	0.023	0.598	0.019	0.018	0.014	0.988	0.181	0.138	0.159
	LCI	0.748	0.093	0.088	0.091	0.744	0.076	0.069	0.054	0.453	0.157	0.943	1.622
40	MSE	0.593	0.069	0.03	0.042	0.476	0.066	0.032	0.053	2.571	1.396	0.978	0.576
	AB	1.256	0.263	0.174	0.204	0.527	0.256	0.178	0.23	1.688	1.197	1.844	0.642
	SE	0.43	0.017	0.015	0.017	0.445	0.015	0.013	0.009	0.736	0.097	0.08	0.083
	LCI	0.687	0.066	0.058	0.066	0.742	0.058	0.05	0.038	0.236	0.091	0.741	0.966
50	MSE	0.581	0.061	0.026	0.036	0.293	0.06	0.028	0.049	1.681	0.976	0.833	0.422
	AB	0.844	0.247	0.161	0.19	0.412	0.245	0.167	0.222	0.94	0.977	0.963	0.431
	SE	0.307	0.011	0.011	0.011	0.351	0.009	0.008	0.006	0.581	0.073	0.0611	0.064
	LCI	0.352	0.041	0.041	0.043	0.376	0.037	0.034	0.026	0.156	0.066	0.557	0.721
70	MSE	0.233	0.057	0.023	0.035	0.231	0.057	0.025	0.048	0.932	7.311	0.654	0.166
	AB	0.638	0.237	0.15	0.186	0.386	0.239	0.158	0.218	0.711	0.652	0.786	0.281
	SE	0.211	0.006	0.006	0.007	0.286	0.006	0.005	0.004	0.322	0.052	0.035	0.045
	LCI	0.262	0.027	0.025	0.031	0.344	0.025	0.02	0.018	0.016	0.025	0.157	0.622
100	MSE	0.073	0.043	0.016	0.029	0.08	0.042	0.016	0.04	0.714	5.111	0.456	0.087
	AB	0.373	0.202	0.119	0.167	0.209	0.201	0.121	0.121	0.533	0.436	0.531	0.132
	SE	0.022	0.004	0.003	0.004	0.189	0.004	0.003	0.002	0.101	0.034	0.018	0.023
	LCI	0.108	0.018	0.015	0.018	0.123	0.016	0.013	0.01	0.003	0.015	0.054	0.122

The joint Su function of two variables distributed NPE-ETE is described as:

$$
\begin{aligned}
Su_{X_1X_2}(x_1\,,\,x_2) = 1 &- (\sigma_1-1)^{-1}\left(\sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}}-1\right) - (\sigma_2-1)^{-1}\left(\sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}}-1\right)\\
&+\frac{1}{K^2}\left(\sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}}-1\right)\left(\sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}}-1\right)\\
&\times\left[K+\tau\left(\sigma_1-\sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}}\right)\left(\sigma_2-\sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}}\right)\right]\\
&\quad ,\ x_1\,,x_2>0,\ \sigma_1,\ \sigma_2,\ \varepsilon_1,\ \varepsilon_2,\ \upsilon_1,\ \upsilon_2,\ \omega_1,\ \omega_2>0\,,\ \sigma_1,\ \sigma_2\neq 1.
\end{aligned}
$$

Marginal Distributions

The marginal Cd functions of $(X_1,\ X_2)$ distributed NPE-ETE are:

Table 4. Estimated parameters: Set of Parameters 4 ($\sigma = 0.1$, $\varepsilon = 0.2$, $\upsilon = 0.3$, $\omega = 0.5$)

n	Measures	PM				CM				MLE			
		$\ddot{\omega}$	$\ddot{\upsilon}$	$\ddot{\varepsilon}$	$\ddot{\alpha}$	$\dot{\omega}$	$\dot{\upsilon}$	$\dot{\varepsilon}$	$\dot{\alpha}$	ω	υ	ε	α
20	MSE	1.185	0.075	0.035	0.046	2.354	0.07	0.035	0.056	3.948	3.564	2.112	0.692
	AB	1.932	0.274	0.187	0.214	1.252	0.264	0.187	0.236	2.394	2.856	2.217	1.109
	SE	0.761	0.041	0.037	0.038	0.886	0.038	0.037	0.024	1.651	0.287	0.305	0.573
	LCI	1.996	0.159	0.145	0.148	1.473	0.148	0.144	0.095	1.393	1.124	1.197	2.245
30	MSE	0.792	0.072	0.033	0.043	1.01	0.068	0.033	0.054	3.101	2.411	1.991	0.613
	AB	1.322	0.267	0.181	0.208	0.808	0.26	0.182	0.232	2.284	2.71	2.023	0.817
	SE	0.6	0.024	0.022	0.023	0.598	0.019	0.018	0.014	0.988	0.181	0.138	0.159
	LCI	0.748	0.093	0.088	0.091	0.744	0.076	0.069	0.054	0.453	0.157	0.943	1.622
40	MSE	0.593	0.069	0.03	0.042	0.476	0.066	0.032	0.053	2.571	1.396	0.978	0.576
	AB	1.256	0.263	0.174	0.204	0.527	0.256	0.178	0.23	1.688	1.197	1.844	0.642
	SE	0.43	0.017	0.015	0.017	0.445	0.015	0.013	0.009	0.736	0.097	0.08	0.083
	LCI	0.687	0.066	0.058	0.066	0.742	0.058	0.05	0.038	0.236	0.091	0.741	0.966
50	MSE	0.581	0.061	0.026	0.036	0.293	0.06	0.028	0.049	1.681	0.976	0.833	0.422
	AB	0.844	0.247	0.161	0.19	0.412	0.245	0.167	0.222	0.94	0.977	0.963	0.431
	SE	0.307	0.011	0.011	0.011	0.351	0.009	0.008	0.006	0.581	0.073	0.0611	0.064
	LCI	0.352	0.041	0.041	0.043	0.376	0.037	0.034	0.026	0.156	0.066	0.557	0.721
70	MSE	0.233	0.057	0.023	0.035	0.231	0.057	0.025	0.048	0.932	7.311	0.654	0.166
	AB	0.638	0.237	0.15	0.186	0.386	0.239	0.158	0.218	0.711	0.652	0.786	0.281
	SE	0.211	0.006	0.006	0.007	0.286	0.006	0.005	0.004	0.322	0.052	0.035	0.045
	LCI	0.262	0.027	0.025	0.031	0.344	0.025	0.02	0.018	0.016	0.025	0.157	0.622
100	MSE	0.073	0.043	0.016	0.029	0.08	0.042	0.016	0.04	0.714	5.111	0.456	0.087
	AB	0.373	0.202	0.119	0.167	0.209	0.201	0.121	0.121	0.533	0.436	0.531	0.132
	SE	0.022	0.004	0.003	0.004	0.189	0.004	0.003	0.002	0.101	0.034	0.018	0.023
	LCI	0.108	0.018	0.015	0.018	0.123	0.016	0.013	0.01	0.003	0.015	0.054	0.122

$$B_{X_1}(x_1) = (\sigma_1 - 1)^{-1}\left(\sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}} - 1\right),\; x_1 > 0,\, \sigma_1,\, \varepsilon_1,\, \upsilon_1,\, \omega_1 > 0\,, B_{X_2}(x_2) = (\sigma_2 - 1)^{-1}\left(\sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}}\right.$$

Similarly, the marginal Pd functions for BNPE-ETE are:

$$b_{X_1}(x_1) = \frac{\varepsilon_1\,\upsilon_1\,(1-e^{-\omega_1})\,\ln(\sigma_1)}{(\sigma_1 - 1)}\,\sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}}\left(1 - e^{-\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)}\right)^{\varepsilon_1 - 1} e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1},\; x_1 > 0,$$

$$b_{X_2}(x_2) = \frac{\varepsilon_2\,\upsilon_2\,(1-e^{-\omega_2})\,\ln(\sigma_2)}{(\sigma_2 - 1)}\,\sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}}\left(1 - e^{-\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)}\right)^{\varepsilon_2 - 1} e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2},\; x_2 > 0$$

Conditional distributions

Table 5. Data set of 100 bank customers waiting times (min) before service

0.8	2.9	4.3	5	6.7	8.2	9.7	11.9	14.1	19.9
0.8	3.1	4.3	5.3	6.9	8.6	9.8	12.4	15.4	20.6
1.3	3.2	4.4	5.5	7.1	8.6	10.7	12.5	15.4	21.3
1.5	3.3	4.4	5.7	7.1	8.6	10.9	12.9	17.3	21.4
1.8	3.5	4.6	5.7	7.1	8.8	11	13	17.3	21.9
1.9	3.6	4.7	6.1	7.1	8.8	11	13.1	18.1	23
1.9	4	4.7	6.2	7.4	8.9	11.1	13.3	18.2	27
2.1	4.1	4.8	6.2	7.6	8.9	11.2	13.6	18.4	31.6
2.6	4.2	4.9	6.2	7.7	9.5	11.2	13.7	18.9	33.1
2.7	4.2	4.9	6.3	8	9.6	11.5	13.9	19	38.5

Table 6. Goodness of fit measures for customer waiting times data

Model	MLE		-logL	AIC	CAIC	BIC	A*	W*
EXP Frechet	$\alpha =$	4.20008	334.381	674.762	675.0123	682.5775	2.5041	0.3822
	$\upsilon =$	1.1629						
	s	1.4619						
ETE	$\upsilon =$	0.2428	329.0209	662.1655	662.2892	667.2521	4.229	27.897
	$\omega =$	0.5394						
EXP Burr XII	$\alpha =$	39.8824	327.5301	661.0601	661.3101	668.8757	1.5611	0.2297
	k=	2.8291						
	c=	0.5959						
NPE-ETE	$\alpha =$	0.5963	**317.0234**	**642.0391**	**642.4601**	**634.0391**	**0.1278**	**0.0179**
	$\varepsilon =$	2.2852						
	$\upsilon =$	0.1551						
	$\omega =$	3.0214						

Table 7. Data set of 72 survival times of guinea pigs

0.1	0.92	1.07	1.16	1.36	1.63	1.97	2.4	3.27
0.33	0.93	1.08	1.2	1.39	1.68	2.02	2.45	3.42
0.44	0.96	1.08	1.21	1.44	1.71	2.13	2.51	3.47
0.56	1	1.08	1.22	1.46	1.72	2.15	2.53	3.61
0.59	1	1.09	1.22	1.53	1.76	2.16	2.54	4.02
0.72	1.02	1.12	1.24	1.59	1.83	2.22	2.54	4.32
0.74	1.05	1.13	1.3	1.6	1.95	2.3	2.78	4.58
0.77	1.07	1.15	1.34	1.63	1.96	2.31	2.93	5.55

Table 8. Goodness of fit measures for survival times of guinea pigs

Model	MLE		-logL	AIC	CAIC	BIC	A*	W*
	$\alpha =$	100.69						
PETE	$\upsilon =$	150.83	113.27	232.5525	233.1495	239.3825	0.5908	0.0954
	s	0.0000373						
	$\theta =$	8.321						
IWP	$\varepsilon =$	0.165	104.85	215.7	216	222.5	2.098	0.31
	$\upsilon =$	1.645						
	$\theta =$	2.931						
EKD	$\varepsilon =$	1.864	95.5	197	197.9	208.4	0.415	0.0628
	$\upsilon =$	5.997						
	$\alpha =$	0.8631						
NPE-ETE	$\varepsilon =$	3.6722	**94.237**	**196.4742**	**197.071**	**205.5824**	**0.3165**	**0.0557**
	$\upsilon =$	1.2634						
	$\omega =$	2.1028						

The two conditional Cd functions of BNPE-ETE are respectively, given as:

$$B_{X_1/X_2}(X_1 \mid X_2) = \frac{B_{X_1}(x_1)}{K}\left[K + \tau\left(\sigma_1 - \sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}}\right)\left(\sigma_2 - 2\sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}} + 1\right)\right],$$

$$B_{X_2/X_1}(X_2 \mid X_1) = \frac{B_{X_2}(x_2)}{K}\left[K + \tau\left(\sigma_2 - \sigma_2^{\left(1-e^{-\left(1-e^{-\omega_2}\right)\upsilon_2 x_2}\right)^{\varepsilon_2}}\right)\left(\sigma_1 - 2\sigma_1^{\left(1-e^{-\left(1-e^{-\omega_1}\right)\upsilon_1 x_1}\right)^{\varepsilon_1}} + 1\right)\right].$$

8. Concluding Remarks

This paper introduced a new formula of erlang truncated exponential model using new power exponentiated family, called the NPE-ETE distribution. The bivariate new power exponentiated erlang truncated exponential distribution was suggested. The properties of the proposed distribution are discussed, including quantile, moments, moment generating functions, entropy measures, stochastic ordering, and order statistics. Estimation problem of unknown parameters from NPE-ETE based on different methods are provided. Utilizing maximum likelihood, cramer-von mises, and percentiles methods of estimation.The AB, MSEs, and SE will be calculated, to compare performance of different estimators, via simulation study. Two real data applications demonstrate that NPE-ETE distribution frequently provides a better fit than other competing distributions.

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