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Research article

Three-Dimensional Copulas: A Generalized Convex Mixture Copulas Strategy

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Abstract: Copulas are multi-dimensional probabilistic functions that play a crucial role in modeling complex dependence structures between random variables. The theory and applications of the three-dimensional case have attracted considerable interest in recent years. This article discusses recent developments in this specific topic and innovates in some aspects. More precisely, the first part proposed to fill certain theoretical gaps concerning a modern copula called the "product Ali-Mikhail-Haq mixed" copula and to complete its knowledge. Among these gaps, the possible values of the involved parameter that make it valid are revisited. The corresponding Spearman rho is also re-calculated. Mathematical proofs and graphic work are given. The second part presents a result on generalized convex mixed copulas and shows how the product Ali-Mikhail-Haq mixed copula can be considered as a specific example. The novelty of this result is that it is general in the dimensional sense and presents simple conditions to guarantee the validity of the resulting copula. Based on it, a new variant of the three-dimensional Ali-Mikhail-Haq copula is given. In summary, for the first time, some existing three-dimensional copulas are analyzed from the point of view of generalized convex mixture copulas in multiple dimensions, thus opening new horizons for dependence modeling.

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1. Introduction

Copula is a probabilistic notion that finds its origins in the work of Sklar (see [1] and [2]). Thanks to it, flexible dependence models applicable to random vectors of varying dimensions can be constructed.

In fact, at the center of this notion, there are "copulas", defined as functions that establish a connection between the marginal distribution functions and the parent joint distribution function. The precise mathematical detail behind this claim was revealed in the famous Sklar theorem. We refer to [3], [4], [5], [6], and [7]. Copulas can be of different functional natures. We may mention the normal, log-normal, Student, Farlie-Gumbel-Morgenstern (FGM), Archimedean, piecewise, D-, and R-vine copulas (see [8] and [9]), among others. The need to construct new copulas arises from the necessity to model complex dependence structures beyond what traditional copulas can capture. This thereby ensures a more accurate and flexible representation of multivariate distributions, which is essential in various practical scenarios.

Parallel to the inexorable development of two-dimensional copulas, three-dimensional copulas have recently experienced renewed interest. They find numerous applications in various applied fields due to their ability to model the dependence structure between three random variables. For instance, in finance, they are used for portfolio optimization and risk management, where understanding the joint behavior of multiple assets is crucial. In environmental science, they help analyze the interactions between three environmental variables, such as temperature, humidity, and precipitation, to predict weather conditions or assess the impacts of climate change. Additionally, in engineering, three-dimensional copulas are used to study the connections between three components in complex systems, thereby facilitating reliability analysis and the improvement of system design. Some of the best-known of three-dimensional copulas have been recently practically involved in [10], [11], [12], [13], and [14], among others. New theoretically oriented methodologies on this topic can be found in [15], [16], [17], [18], [19], [20], [21], and [22]. A brief overview of notable results is given below.

In [19], the following three-dimensional copula is studied:

$$C(x_1, x_2, x_3) = \Pi(x_1, x_2, x_3) + \lambda x_1 x_2 (1 - x_1)(1 - x_2) x_3^{\theta}, \quad (x_1, x_2, x_3) \in [0, 1]^3,$$

where $\Pi(x_1, x_2, x_3) = x_1 x_2 x_3$ is the independence copula, $\theta \geq 1$ and $|\lambda| \leq 1/\theta$. In a certain sense, the function $D(x_1, x_2, x_3) = \lambda x_1 x_2 (1 - x_1)(1 - x_2) x_3^{\theta}$ can be thought of as a perturbation of the independence modeling. This construction presents a novel and easily manageable alternative to the three-dimensional copulas documented in the literature. In particular, it mainly differs from the standard three-dimensional FGM copula by its non-exchangeability and the presence of the power term x_3^{θ}.

In the same spirit, we can also cite the work in [21] describing the following three-dimensional copula:

$$C(x_1, x_2, x_3) = \Pi(x_1, x_2, x_3) + \lambda x_1 x_2 x_3 (x_1 - x_2)^m (1 - x_3) x_3^{\theta}, \quad (x_1, x_2, x_3) \in [0, 1]^3,$$

where m is a positive integer, $\theta \geq 0$ and $|\lambda| \leq 1/\{(\theta + 1) \max[m(m - 1), m + 1]\}$. Thus, a new perturbation mechanism of the independence copula is considered, represented by the function $E(x_1, x_2, x_3) = \lambda x_1 x_2 x_3 (x_1 - x_2)^m (1 - x_3) x_3^{\theta}$. The novelty of this perturbation function is that it is not separable with respect to x_1, x_2 and x_3 thanks to the term $(x_1 - x_2)^m$.

Other options are suggested in [22]. In particular, the following three-dimensional copula is considered:

$$C(x_1, x_2, x_3) = \Pi(x_1, x_2, x_3) - \lambda x_1 x_2 x_3 (1 - x_2)(1 - x_3), \quad (x_1, x_2, x_3) \in [0, 1]^3, \tag{1.1}$$

where $\lambda \in [-1, 1]$. It was called the product FGM mixed (PFGM) copula. Thus, the function $F(x_1, x_2, x_3) = \lambda x_1 x_2 x_3 (1 - x_2)(1 - x_3)$ aims to perturb the three-dimensional independence copula. It is separable with respect to x_1, x_2 and x_3, but not exchangeable because of the absence of the term $1 - x_1$. In addition to the PFGM copula, still in [22], the following three-dimensional copula is examined:

$$C(x_1, x_2, x_3) = \Pi(x_1, x_2, x_3)\left[2 - \frac{1}{1 - \lambda(1 - x_2)(1 - x_3)}\right], \quad (x_1, x_2, x_3) \in [0, 1]^3, \tag{1.2}$$

which can also be written as

$$C(x_1, x_2, x_3) = \Pi(x_1, x_2, x_3) - \lambda \frac{x_1 x_2 x_3 (1 - x_2)(1 - x_3)}{1 - \lambda(1 - x_2)(1 - x_3)}, \quad (x_1, x_2, x_3) \in [0, 1]^3,$$

where it is understood that λ belongs to $[-1, 1]$. It is mainly based on the Ali-Mikhail-Haq (AMH) copula and was called the product AMH mixed (PAMM) copula. Again, a new perturbation function of the three-dimensional independence copula is proposed, following a ratio-function scheme characterized by $G(x_1, x_2, x_3) = -\lambda x_1 x_2 x_3 (1-x_2)(1-x_3)/[1-\lambda(1-x_2)(1-x_3)]$. In fact, the PFGM and PAMM copulas established in [22] are based on a more general three-dimensional copula construction, which can be expressed as follows:

$$C(x_1, x_2, x_3) = 3\Pi(x_1, x_2, x_3) - x_3 C^*(x_1, x_2) - x_1 C^\dagger(x_2, x_3), \quad (x_1, x_2, x_3) \in [0, 1]^3, \tag{1.3}$$

where $C^*(x_1, x_2)$ and $C^\dagger(x_1, x_2)$ are two two-dimensional copulas (see [22, Equation (13)]). Equivalently, under a perturbation of the independence copula form, we can write it as

$$C(x_1, x_2, x_3) = \Pi(x_1, x_2, x_3) + 2x_1 x_2 x_3 - x_3 C^*(x_1, x_2) - x_1 C^\dagger(x_2, x_3), \quad (x_1, x_2, x_3) \in [0, 1]^3.$$

Based on this expression, the remarks below hold.

- The PFGM copula in Equation (1.1) is obtained by choosing $C^*(x_1, x_2) = x_1 x_2$, i.e., the two-dimensional independence copula, and $C^\dagger(x_2, x_3) = x_2 x_3 + \lambda x_2 x_3 (1-x_2)(1-x_3)$, where $\lambda \in [-1, 1]$, i.e., the two-dimensional FGM copula with parameter λ. It also corresponds to the following product copula: $C(x_1, x_2, x_3) = x_1 C^\ddagger(x_2, x_3)$, where $C^\ddagger(x_2, x_3) = x_2 x_3 - \lambda x_2 x_3 (1 - x_2)(1 - x_3)$ is the FGM copula with parameter $-\lambda$.
- The PAMM copula in Equation (1.2) is obtained by choosing again $C^*(x_1, x_2) = x_1 x_2$, and $C^\dagger(x_2, x_3) = x_2 x_3/[1 - \lambda(1 - x_2)(1 - x_3)]$, i.e., the two-dimensional AMH copula with parameter λ. In [22], it is implicitly assumed, or at least understood, that $\lambda \in [-1, 1]$; this aspect will be discussed later in the article.

Since the value variable x_1 can be put in factor, i.e., isolated from x_2 and x_3, in both the definitions of the PFGM and PAMM copulas, the associated uniform random variable, say X_1, is supposed to be independent of the other two uniform random variables, say X_2 and X_3. However, X_2 and X_3 are allowed to exhibit some stochastic connection. With this in mind, the PFGM and PAMM copulas have been applied in [22] on water quality measurements for the Chattahoochee River. The considered adequacy criteria were the Akaike information criterion, Bayesian information criterion (AIC), consistent AIC (CAIC), and Hannan-Quinn information criterion (HQIC). They gave quite convincing results, showing the importance of the PFGM and PAMM copulas in a practical data analysis scenario. The best being the PAMM copula with λ estimated as -0.8349 and with the following values of the criteria: AIC = 560.933, BIC = 574.725, CAIC = 563.421, and HQIC = 566.236. All of the above developments represent a significant advance in the construction of three-dimensional models.

The contributions to this article take their source from the original work of [22]. In particular, as implicitly mentioned in [22], the three-dimensional function in Equation (1.3) is not necessarily a valid copula. This opens up some questions, including the determination of manageable conditions that make it effectively valid. In the first part of the article, a specific example is emphasized; we show that the PAMM copula in Equation (1.2) cannot be considered valid for all possible values of λ in $[-1, 1]$. We study this aspect through mathematical developments, numerical analysis, and graphics. In particular, we show that it is valid for $\lambda \in [-1, 0]$, among others, which corresponds to the case considered in the data analysis in [22], and attest to the importance of these published results. The possible positive values for λ are also discussed. For the case $\lambda \in [-1, 0]$, we revisit the expression of the Spearman rho and show that the PAMM copula is of interest for small positive dependence. In the second part, a generalized convex mixture copula is established, along with the underlying assumptions to ensure its validity. Then we show that the construction in Equation (1.3) can be viewed as a particular example of such a mixture copula strategy, which allows us to emphasize the assumptions making it valid. Our general result goes beyond the three-dimensional case; it can be applied to any higher dimension. A simple example presenting a variant of the three-dimensional AMH copula is finally provided.

The rest of the article is composed of three complementary sections. Section 2 recalls several copula notions. Section 3 is about some new contributions to the PAMM copula. Section 4 discusses the aforementioned general convex mixture copulas strategy. A conclusion is given in Section 5.

2. Basic copula notions

To begin, let us recall the general definition of an absolutely continuous copula in multiple dimensions, thereby establishing a basis for our subsequent findings.

Definition 1. *Let $n \geq 2$ be a positive integer, representing a specific dimension. The function $C(x_1, \ldots, x_n)$, $(x_1, \ldots, x_n) \in [0, 1]^n$, is designated as an absolutely continuous n-dimensional copula if it adheres to the following properties:*

(I) $C(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$ *for any* $(x_1, \ldots, x_n) \in [0, 1]^n$ *and* $i = 1, \ldots, n$,

(II) $C(1, \ldots, 1, x, 1, \ldots, 1) = x$ *for any* $x \in [0, 1]$, *and this condition applies independently to each of the n vector components.*

(III) $\partial_{x_1,\ldots,x_n} C(x_1, \ldots, x_n) \geq 0$ *for any* $(x_1, \ldots, x_n) \in (0, 1)^n$, *where* $\partial_{x_1,\ldots,x_n} = \partial^n/(\partial x_1 \ldots \partial x_n)$ *denotes the mixed n-th order partial derivatives according to* $x_1, \ldots, x_n$ *(it is understood that* $C(x_1, \ldots, x_n)$ *is n times differentiable almost everywhere for* $(x_1, \ldots, x_n) \in (0, 1)^n$*).*

We refer to [3] for more details on the notion of an absolutely continuous n-dimensional copula. Below, to simplify the text, we omit the expression "absolutely continuous". From a mathematical point of view, the conditions **(I)** and **(II)** are often immediate; the most difficult condition to prove remains **(III)**. Naturally, based on Definition 1, we obtain a two-dimensional copula by taking $n = 2$, and a three-dimensional copula by taking $n = 3$.

In full generality, as outlined in the introduction, the notion of n-dimensional copula plays a crucial role in multivariate probability theory, providing a rigorous framework for modeling complex dependence structures between n random variables. In particular, by denoting these random variables as $U_1, \ldots, U_n$, the Sklar theorem ensures that the joint cumulative distribution function of $(U_1, \ldots, U_n)$, say $H(u_1, \ldots, u_n)$, can be decomposed as

$$H(u_1, \ldots, u_n) = C[H(u_1), \ldots, H(u_n)], \quad (u_1, \ldots, u_n) \in \mathbb{R}^n,$$

where $C(x_1, \ldots, x_n)$ denotes the corresponding n-dimensional copula and $H(u_1)$,..., $H(u_n)$ are the marginal cumulative distribution functions of $U_1, \ldots, U_n$, respectively (see [1] and [2]). Thus, the importance of such copulas lies in providing a versatile tool for analyzing and simulating joint distributions in various applied fields. Understanding their properties and applications improves our ability to effectively assess and mitigate risks in complex systems.

Complementary to the notion of n-dimensional copula, the related notion of Spearman rho is recalled below.

Definition 2. *Let $n \geq 2$ be a positive integer, and $C(x_1, \ldots, x_n)$, $(x_1, \ldots, x_n) \in [0, 1]^n$, be a n-dimensional copula as defined in Definition 1. Then we define the Spearman rho associated with $C(x_1, \ldots, x_n)$ as*

$$\rho = \frac{n+1}{2^n - (n+1)} \left[2^n \int_{[0,1]^n} C(x_1, \ldots, x_n) dx_1 \ldots, dx_n - 1 \right].$$

The Spearman rho measures how strong the association is between the underlying random variables associated with $C(x_1, \ldots, x_n)$. See [23] and [24] for more details. In the next section, a part will investigate this measure for the PAMM copula in Equation (1.2), revisiting the result in [22, Proposition 4].

3. Contributions to the PAMM copula

In this section, we provide some contributions to the PAMM copula in Equation (1.2), as described in [22].

3.1. Discussion

In the proposition below, we nuance the validity of the PAMM copula with respect to the possible values of λ.

Proposition 3.1. *There exist values of λ in $[-1, 1]$ such that the three-dimensional function defined in Equation (1.2), i.e.,*

$$C(x_1, x_2, x_3) = x_1 x_2 x_3 \left[2 - \frac{1}{1 - \lambda(1 - x_2)(1 - x_3)} \right], \quad (x_1, x_2, x_3) \in [0, 1]^3,$$

is not a valid copula.

Proof. Based on Definition 1, we need to check the conditions **(I)**, **(II)** and **(III)**. Since **(I)** and **(II)** are obvious (already mentioned in [22]) and do not involve the possible values of λ, let us concentrate our efforts on **(III)**. For any $(x_1, x_2, x_3) \in (0,1)^3$, we have

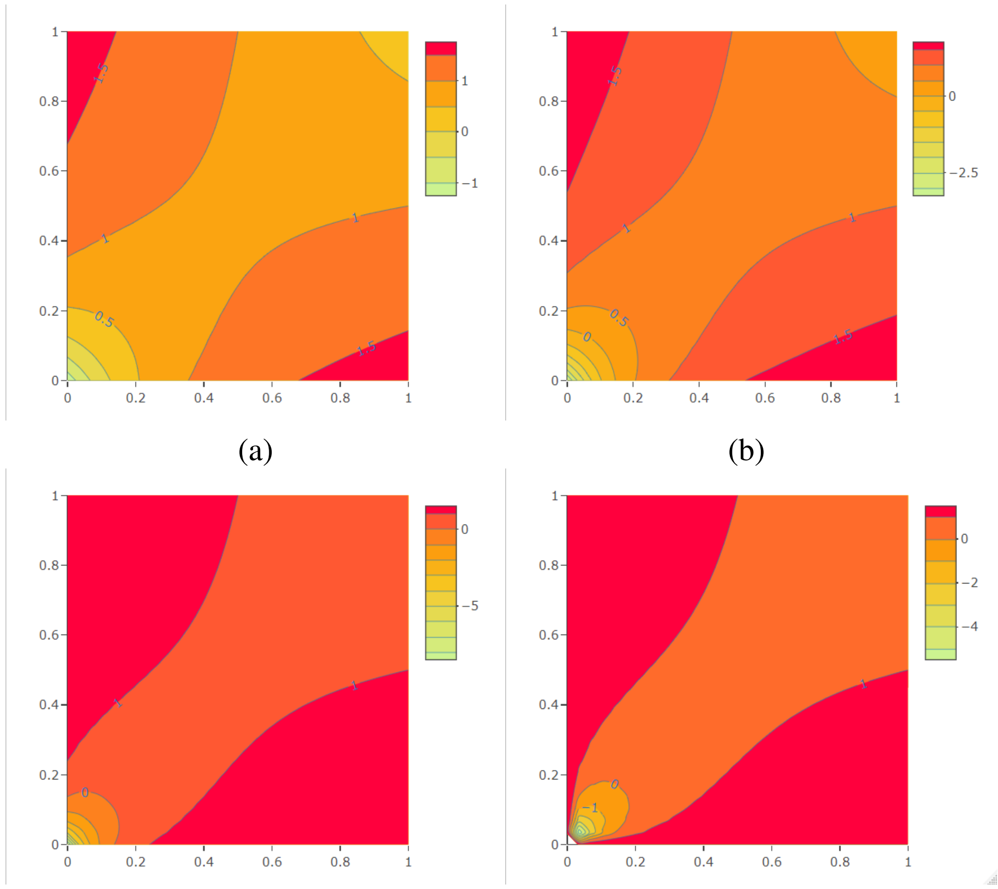
$$\partial_{x_1,x_2,x_3} C(x_1,x_2,x_3) = 2 - 2\lambda^2 \frac{x_2 x_3 (1-x_2)(1-x_3)}{[1-\lambda(1-x_2)(1-x_3)]^3} + \lambda \frac{x_2(1-x_3)}{[1-\lambda(1-x_2)(1-x_3)]^2}$$
$$+ \lambda \frac{(1-x_2)x_3}{[1-\lambda(1-x_2)(1-x_3)]^2} - \lambda \frac{x_2 x_3}{[1-\lambda(1-x_2)(1-x_3)]^2} - \frac{1}{1-\lambda(1-x_2)(1-x_3)}$$
$$= 2 - \frac{\lambda^2(1-x_2)(1-x_3) + \lambda[x_2(x_3+1) + x_3 - 2] + 1}{[1-\lambda(1-x_2)(1-x_3)]^3}.$$

This differentiation was also obtained in [22, Equation (19)]. Let us now show that it can be negative for certain $\lambda \in [-1, 1]$, and $(x_1, x_2, x_3) \in (0,1)^3$, thus relativizing its validity in this domain parameter.

- For punctual evidence, by choosing $\lambda = 0.9$, $x_1 = 0.1$ (among other values into $(0,1)$ since there is no dependence in x_1), $x_2 = 0.1$, and $x_3 = 0.1$, we numerically find that

$$\partial_{x_1,x_2,x_3} C(x_1,x_2,x_3) \approx -0.266046 < 0.$$

  Therefore, $C(x_1, x_2, x_3)$ is not a valid copula for any $\lambda \in [-1, 1]$.
- For graphical evidence, since $\partial_{x_1,x_2,x_3} C(x_1,x_2,x_3)$ does not depend on x_1, we can consider the two-dimensional function $\psi(x_2, x_3; \lambda) = \partial_{x_1,x_2,x_3} C(x_1,x_2,x_3)$ and display it as a two-dimensional curve (we thus voluntary reduce the problem to two dimensions). The aim is to analyze its positive and negative values while varying λ. Figure 1 does that from a color contour point of view for several values of λ. The software `R` combined with the package `plot3D` was used in this regard (see [25]).



(a) (b) (c) (d)

Figure 1. Plots of $\psi(x_2, x_3; \lambda)$, $(x_2, x_3) \in (0, 1)^2$ for (a) $\lambda = 0.7$, (b) $\lambda = 0.8$, (c) $\lambda = 0.9$, and (d) $\lambda = 1$

From this figure, we easily identify "light-yellow-green zones", mainly in the neighborhood of the point $(0, 0)$, corresponding to the values of x_2 and x_3 for which $\psi(x_2, x_3; \lambda) \leq 0$. This shows that $\partial_{x_1,x_2,x_3} C(x_1, x_2, x_3)$ can be negative. Hence, $C(x_1, x_2, x_3)$ is not a copula for all the values λ into $[-1, 1]$.

This ends the proof. □

From this result, we complete the understanding of the PAMM copula as described in [22], demonstrating that it cannot be used blindly for any $\lambda \in [-1, 1]$.

3.2. Validity

In light of the previous comments on the PAMM copula, a theoretical examination of some possible ranges of values for λ is done below.

Proposition 3.2. *For either*

- $\lambda \in [-1, 0]$, *or*
- $\lambda \in [0, \lambda_0]$ *with* $\lambda_0 = 0.164$,

the three-dimensional function defined in Equation (1.2), i.e.,

$$C(x_1, x_2, x_3) = x_1 x_2 x_3 \left[2 - \frac{1}{1 - \lambda(1 - x_2)(1 - x_3)}\right], \quad (x_1, x_2, x_3) \in [0, 1]^3,$$

is a valid copula.

Proof. For any $(x_1, x_2, x_3) \in (0, 1)^3$, in the proof of Proposition 3.1, we have already shown that

$$\partial_{x_1,x_2,x_3} C(x_1, x_2, x_3) = 2 - \frac{\lambda^2(1 - x_2)(1 - x_3) + \lambda[x_2(x_3 + 1) + x_3 - 2] + 1}{[1 - \lambda(1 - x_2)(1 - x_3)]^3}.$$

Let us now study the sign of this function. To achieve this aim, we first focus on the main ratio term. For any $\lambda \in [-1, 1]$, after a differentiation work, we remark that

$$\lambda^2(1 - x_2)(1 - x_3) + \lambda[x_2(x_3 + 1) + x_3 - 2] + 1 = [1 - \lambda(1 - x_2)(1 - x_3)]^3 \partial_{x_2,x_3} C^\dagger(x_2, x_3),$$

where $C^\dagger(x_2, x_3)$ denotes the two-dimensional AMH copula with parameter λ. Therefore, since $1 - \lambda(1 - x_2)(1 - x_3) \geq 1 - |\lambda| \geq 0$ and $\partial_{x_2,x_3} C^\dagger(x_2, x_3) \geq 0$ as a basic property of any copula density function, we get

$$\lambda^2(1 - x_2)(1 - x_3) + \lambda[x_2(x_3 + 1) + x_3 - 2] + 1 \geq 0.$$

The numerator of the main ratio term is thus positive. Since $1 - \lambda(1 - x_2)(1 - x_3) \geq 0$, the denominator obviously is too. In light of this, one aim is to bound this ratio term in the sharpest way possible, then compare it to the value 2, hoping to establish that $\partial_{x_1,x_2,x_3} C(x_1, x_2, x_3) \geq 0$.

- Let us suppose that $\lambda \in [-1, 0]$. Since $(1 - x_2)(1 - x_3) \geq 0$ and $\lambda^2 \leq |\lambda| = -\lambda$, we have

$$\begin{aligned} &\lambda^2(1 - x_2)(1 - x_3) + \lambda[x_2(x_3 + 1) + x_3 - 2] + 1 \\ &\leq -\lambda(1 - x_2)(1 - x_3) + \lambda[x_2(x_3 + 1) + x_3 - 2] + 1. \end{aligned}$$

  Hence, since $[1 - \lambda(1 - x_2)(1 - x_3)]^3 \geq 1 - \lambda(1 - x_2)(1 - x_3) \geq 1$, we obtain

$$\begin{aligned} &\frac{\lambda^2(1 - x_2)(1 - x_3) + \lambda[x_2(x_3 + 1) + x_3 - 2] + 1}{[1 - \lambda(1 - x_2)(1 - x_3)]^3} \\ &\leq \frac{-\lambda(1 - x_2)(1 - x_3) + \lambda[x_2(x_3 + 1) + x_3 - 2] + 1}{1 - \lambda(1 - x_2)(1 - x_3)}. \end{aligned} \tag{3.1}$$

  Hence, we get

$$\partial_{x_1,x_2,x_3} C(x_1, x_2, x_3) \geq 2 - \frac{-\lambda(1 - x_2)(1 - x_3) + \lambda[x_2(x_3 + 1) + x_3 - 2] + 1}{1 - \lambda(1 - x_2)(1 - x_3)}.$$

  Now, since $1 - 2x_2x_3 \in [-1, 1]$, we establish that

$$\begin{aligned} &-\lambda(1 - x_2)(1 - x_3) + \lambda[x_2(x_3 + 1) + x_3 - 2] + 1 - 2[1 - \lambda(1 - x_2)(1 - x_3)] \\ &= -\lambda(1 - 2x_2x_3) - 1 \leq -\lambda - 1 \leq 0. \end{aligned}$$

This inequality implies that

$$\frac{-\lambda(1-x_2)(1-x_3)+\lambda[x_2(x_3+1)+x_3-2]+1}{1-\lambda(1-x_2)(1-x_3)} \leq 2. \tag{3.2}$$

As a result, we have

$$\partial_{x_1,x_2,x_3} C(x_1,x_2,x_3) \geq 0.$$

Thus, in the case $\lambda \in [-1,0]$, $C(x_1,x_2,x_3)$ is a valid copula.

- Let us now suppose that $\lambda \in [0,\lambda_0]$ with $\lambda_0 = 0.164$. The numerator of the ratio term can be expressed as follows:

$$\begin{aligned}
&\lambda^2(1-x_2)(1-x_3)+\lambda[x_2(x_3+1)+x_3-2]+1\\
&= \lambda^2(1-x_2)(1-x_3)+\lambda[-(1-x_2)(1-x_3)-1+2x_2x_3]+1\\
&= -\lambda(1-\lambda)(1-x_2)(1-x_3)-\lambda(1-2x_2x_3)+1.
\end{aligned}$$

Since $-\lambda(1-\lambda) \leq 0$, $(1-x_2)(1-x_3) \geq 0$, and $1-2x_2x_3 \in [-1,1]$, we obtain

$$\begin{aligned}
&-\lambda(1-\lambda)(1-x_2)(1-x_3)-\lambda(1-2x_2x_3)+1 \leq -\lambda(1-2x_2x_3)+1\\
&\leq 1+\lambda.
\end{aligned}$$

Hence, since $[1-\lambda(1-x_2)(1-x_3)]^3 \geq (1-\lambda)^3 \geq 0$, we get

$$\frac{\lambda^2(1-x_2)(1-x_3)+\lambda[x_2(x_3+1)+x_3-2]+1}{[1-\lambda(1-x_2)(1-x_3)]^3} \leq \frac{1+\lambda}{(1-\lambda)^3}.$$

For $\lambda \leq \lambda_0 = 0.164$, we have

$$\partial_{x_1,x_2,x_3} C(x_1,x_2,x_3) \geq 2 - \frac{1+\lambda}{(1-\lambda)^3} \geq 0.$$

Thus, in the case of $\lambda \in [0,\lambda_0]$, $C(x_1,x_2,x_3)$ is a valid copula.

The desired results are established. □

The case $\lambda \in [-1,0]$ is the most interesting because the obtained range of values is simple to handle, and it corresponds to the one considered in the application in [22]. Indeed, λ is estimated as $-0.8349 \in [-1,0]$ for the water quality analysis data.

For the possible positive values of λ making $C(x_1,x_2,x_3)$ a valid copula, it is not claimed that the indicated value of λ_0 such that $\lambda \in [0,\lambda_0]$ is the optimal one; a greater value can certainly be determined with more sharp inequality techniques. However, this optimal value remains difficult to identify, and it is necessarily strictly inferior to 1 (and even 0.7), as discussed in Proposition 3.1.

3.3. Spearman rho

For the case $\lambda \in [-1,0]$, the expression of the Spearman rho of the PAMM copula in [22, Proposition 4] needs a revision since it involves the term $\log(\lambda-1)$ which is not well defined. The revised version is formulated below.

Proposition 3.3. *For $\lambda \in [-1,0]$, the Spearman rho of the PAMM copula defined in Equation (1.2) is expressed as*

$$\rho = 1 + 4\frac{3\lambda + 2\log(1-\lambda) - 2\lambda\log(1-\lambda) - (1+\lambda)\,\mathrm{Li}_2(\lambda)}{\lambda^2},$$

where $\mathrm{Li}_2(\lambda) = \sum_{k=1}^{\infty} \lambda^k/k^2$ is the standard dilogarithm function.

Proof. Based on Definition 2 with $n = 3$, we get

$$\rho = 8\int_{[0,1]^3} C(x_1, x_2, x_3)dx_1dx_2dx_3 - 1.$$

Thus, only the integral term needs special treatment. We have

$$\begin{aligned}&\int_{[0,1]^3} C(x_1, x_2, x_3)dx_1dx_2dx_3\\ &= \left\{\int_{[0,1]^2} x_2x_3\left[2 - \frac{1}{1-\lambda(1-x_2)(1-x_3)}\right]dx_2dx_3\right\}\left[\int_{[0,1]} x_1dx_1\right].\end{aligned}$$

A double integration and tedious developments give

$$\begin{aligned}&\int_{[0,1]^2} x_2x_3\left[2 - \frac{1}{1-\lambda(1-x_2)(1-x_3)}\right]dx_2dx_3\\ &= \frac{3\lambda + 2\log(1-\lambda) - 2\lambda\log(1-\lambda) - (1+\lambda)\,\mathrm{Li}_2(\lambda)}{\lambda^2} + \frac{1}{2}.\end{aligned}$$

Since $\int_{[0,1]} x_1dx_1 = 1/2$, by considering the constants 8 and -1 involved in ρ, we obtain the desired result. This completes the proof. □

Figure 2 displays the Spearman rho of the PAMM copula for $\lambda \in [-1,0]$ as obtained in the above proposition. Again, the software `R` was used, along with the function `polylog` of the package `copula`.

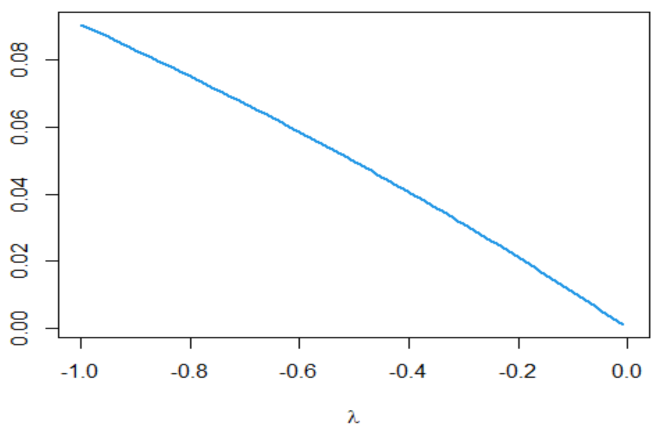


Figure 2. Plots of the Spearman rho of the PAMM copula for $\lambda \in [-1,0]$

We observe that, for $\lambda \in [-1, 0]$, the PAMM copula is able to model small positive dependence, with 0.09035489 as a maximum value. This positivity is coherent with the fact that, for $\lambda \in [-1, 0]$, we have

$$C(x_1, x_2, x_3) = \Pi(x_1, x_2, x_3) - \lambda \frac{x_1 x_2 x_3 (1 - x_2)(1 - x_3)}{1 - \lambda(1 - x_2)(1 - x_3)} \geq \Pi(x_1, x_2, x_3),$$

indicating the positive quadrant dependence of the PAMM copula.

We can also remark that the Kendall tau of the PAMM copula in [22, Proposition 3] also needs a recalculation since it involves the term $\log(\lambda - 1)$, which is not well defined for $\lambda \in [-1, 0]$.

Apart from the association measures, a more general aspect of this copula is examined in the rest of the article.

4. Generalized convex mixture copulas

In this section, we demonstrate that the three-dimensional copula strategy described in Equation (1.3), and the PAMM copula in particular, can be extended. This extension involves more intermediary copulas and higher dimensions.

4.1. Result

The main result is formulated in full generality in the theorem below.

Theorem 4.1. *We adopt the copula concept presented in Definition 1. Let m and n be positive integers, $C_1(x_1, \ldots, x_n), \ldots, C_m(x_1, \ldots, x_n)$, $(x_1, \ldots, x_n) \in [0, 1]^n$, be m n-dimensional copulas, $c_1(x_1, \ldots, x_n), \ldots, c_m(x_1, \ldots, x_n)$ be the corresponding copula densities, respectively, i.e., $c_j(x_1, \ldots, x_n) = \partial_{x_1,\ldots,x_n} C_j(x_1, \ldots, x_n)$ for $j = 1, \ldots, m$, and $(\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ (it is important to note that, for any $j = 1, \ldots, m$, ξ_j can be either negative or positive). Let us consider the following generalized convex mixture:*

$$C_{mix}(x_1, \ldots, x_n) = \sum_{j=1}^{m} \xi_j C_j(x_1, \ldots, x_n). \tag{4.1}$$

Suppose that $\sum_{j=1}^{m} \xi_j = 1$ and

$$\sum_{j=1}^{m} \xi_j \|c_j\|_{\xi_j} \geq 0,$$

where

$$\|c_j\|_{\xi_j} = \begin{cases} \inf_{(x_1,\ldots,x_n)\in(0,1)^n} c_j(x_1, \ldots, x_n) & if \ \ \xi_j > 0 \\ 0 & if \ \ \xi_j = 0 \\ \sup_{(x_1,\ldots,x_n)\in(0,1)^n} c_j(x_1, \ldots, x_n) & if \ \ \xi_j < 0 \end{cases}.$$

Then $C_{mix}(x_1, \ldots, x_n)$ is a valid n-dimensional copula.

Proof. Based on Definition 1, we need to check the conditions **(I)**, **(II)**, and **(III)**. To begin, for any $(x_1, \ldots, x_n) \in [0, 1]^n$ and $i = 1, \ldots, n$, since $C_j(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$ for any $j = 1, \ldots, m$, we have

$$C_{mix}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = \sum_{j=1}^{m} \xi_j C_j(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = \sum_{j=1}^{m} \xi_j \times 0 = 0.$$

Thus, the condition **(I)** is satisfied.

For any $x \in [0, 1]$, since $C_j(1, \ldots, 1, x, 1, \ldots, 1) = x$ and $\sum_{j=1}^{m} \xi_j = 1$, we have

$$\begin{aligned} C_{mix}(1, \ldots, 1, x, 1, \ldots, 1) &= \sum_{j=1}^{m} \xi_j C_j(1, \ldots, 1, x, 1, \ldots, 1) \\ &= \sum_{j=1}^{m} \xi_j \times x = x \times 1 = x. \end{aligned}$$

Hence, the condition **(II)** holds.

Let us now examine the condition **(III)**. For any $(x_1, \ldots, x_n) \in (0, 1)^n$, since $\sum_{j=1}^{m} \xi_j ||c_j||_{\xi_j} \geq 0$, thanks to the definition of $||c_j||_{\xi_j}$, we have

$$\begin{aligned} \partial_{x_1,\ldots,x_n} C_{mix}(x_1, \ldots, x_n) &= \sum_{j=1}^{m} \xi_j \partial_{x_1,\ldots,x_n} C_j(x_1, \ldots, x_n) \\ &= \sum_{j=1}^{m} \xi_j c_j(x_1, \ldots, x_n) \geq \sum_{j=1}^{m} \xi_j ||c_j||_{\xi_j} \geq 0. \end{aligned}$$

The condition **(III)** is satisfied. Therefore, $C_{mix}(x_1, \ldots, x_n)$ is validated as a n-dimensional copula. □

In fact, the "standard" convex mixture copulas considering that $\xi_j \geq 0$ for any $j = 1, \ldots, m$ is a well-known topic (see [3] and [26]). Hence, the proposition above fixes the ideas and assumptions on the validity of the generalized case that includes possible negative values for ξ_j. The existing theory is thus slightly modified, mainly with the introduction of the "parameter-norm-like" of a copula density, i.e., $||c_j||_{\xi_j}$.

4.2. Application to the three-dimensional case

In the proposition below, we show that the potential copula in Equation (1.3) is a particular case of the generalized convex mixture copula in Equation (4.1). In addition, clear assumptions about the intermediary copulas are given.

Proposition 4.2. *Let $C^*(x_1, x_2)$ and $C^\dagger(x_1, x_2)$, $(x_1, x_2) \in [0, 1]^2$, be two two-dimensional copulas, and $c^*(x_1, x_2)$ and $c^\dagger(x_1, x_2)$ be the corresponding copula densities, respectively. Let us suppose that*

$$||c^*||_\infty + ||c^\dagger||_\infty \leq 3,$$

where $||c^*||_\infty = \sup_{(x_1,x_2)\in(0,1)^2} c^*(x_1, x_2)$ and $||c^\dagger||_\infty = \sup_{(x_1,x_2)\in(0,1)^2} c^\dagger(x_1, x_2)$. Then the three-dimensional function

$$C(x_1, x_2, x_3) = 3\Pi(x_1, x_2, x_3) - x_3 C^*(x_1, x_2) - x_1 C^\dagger(x_2, x_3), \quad (x_1, x_2, x_3) \in [0, 1]^3,$$

is a valid copula.

Proof. Let us make the link between this result and Theorem 4.1. Let us consider the generalized convex mixture function as described in Equation (4.1) with the following configuration: $n = 3$, $m = 3$, $\xi_1 = 3$, $C_1(x_1, x_2, x_3) = \Pi(x_1, x_2, x_3)$, $\xi_2 = -1$, $C_2(x_1, x_2, x_3) = x_3 C^*(x_1, x_2)$, $\xi_3 = -1$ and $C_3(x_1, x_2, x_3) = x_1 C^\dagger(x_2, x_3)$, noticing that $C_2(x_1, x_2, x_3)$ and $C_3(x_1, x_2, x_3)$ are well-defined three-dimensional copulas based on a standard product scheme. Then the function in Equation (4.1) becomes the desired candidate expression, i.e.,

$$C_{mix}(x_1, x_2, x_3) = 3\Pi(x_1, x_2, x_3) - x_2 C^*(x_1, x_2) - x_1 C^\dagger(x_2, x_3).$$

Let us check that the conditions in Theorem 4.1 are satisfied. It is obvious that

$$\sum_{j=1}^{3} \xi_j = 3 - 1 - 1 = 1.$$

In addition, we have $\partial_{x_1,x_2,x_3} C_1(x_1, x_2, x_3) = 1$, $\partial_{x_1,x_2,x_3} C_2(x_1, x_2, x_3) = \partial_{x_1,x_2} C^*(x_1, x_2) = c^*(x_1, x_2)$, and $\partial_{x_1,x_2,x_3} C_3(x_1, x_2, x_3) = \partial_{x_2,x_3} C^\dagger(x_2, x_3) = c^\dagger(x_2, x_3)$. As a result, since $\xi_1 = 3 > 0$ (so different to 0), we have $||\partial_{x_1,x_2,x_3} C_1||_{\xi_1} = 1$, since $\xi_2 = -1 < 0$, we have $||\partial_{x_1,x_2,x_3} C_2||_{\xi_2} = ||c^*||_\infty$ and, since $\xi_3 = -1 < 0$, we have $||\partial_{x_1,x_2,x_3} C_3||_{\xi_3} = ||c^\dagger||_\infty$. Therefore, if $||c^*||_\infty + ||c^\dagger||_\infty \leq 3$, we obtain

$$\begin{aligned}\sum_{j=1}^{3} \xi_j ||\partial_{x_1,x_2,x_3} C_j||_{\xi_j} &= 3 \times 1 + (-1) \times ||c^*||_\infty + (-1) \times ||c^\dagger||_\infty \\ &= 3 - ||c^*||_\infty - ||c^\dagger||_\infty \geq 0.\end{aligned}$$

It follows from Theorem 4.1 that $C_{mix}(x_1, x_2, x_3)$, so $C(x_1, x_2, x_3)$, is a valid copula. □

This proposition thus offers the theoretical basis of Equation (1.3) as described in [22]. In particular, the PFGM copula in Equation (1.1) is obtained by choosing $C^*(x_1, x_2) = x_1 x_2$, i.e., $||c^*||_\infty = 1$, and $C^\dagger(x_2, x_3) = x_2 x_3 + \lambda x_2 x_3 (1 - x_2)(1 - x_3)$, where $\lambda \in [-1, 1]$, i.e.,

$$||c^\dagger||_\infty = \sup_{(x_2,x_3)\in(0,1)^2} [1 + \lambda(1 - 2x_2)(1 - 2x_3)] = 1 + |\lambda|.$$

Hence, it is clear that

$$||c^*||_\infty + ||c^\dagger||_\infty \leq 1 + 1 + |\lambda| \leq 3.$$

The assumptions in Proposition 4.2 are thus satisfied, ensuring that the PFGM copula is valid.

The PAMM copula in Equation (1.2) is obtained by choosing $C^*(x_1, x_2) = x_1 x_2$, i.e., $||c^*||_\infty = 1$, and $C^\dagger(x_2, x_3) = x_2 x_3 / [1 - \lambda(1 - x_2)(1 - x_3)]$, but the condition $||c^*||_\infty + ||c^\dagger||_\infty \leq 3$ is not satisfied for all $\lambda \in [-1, 1]$, as discussed in Proposition 3.1.

Of course, several other copula configurations are possible, beyond the standard ones. Two examples are given below.

Example 1: We can consider the Celebioglu-Cuadras copula established in [27]. It is defined by

$$C(x_1, x_2) = x_1 x_2 \exp[\mu(1 - x_1)(1 - x_2)], \quad (x_1, x_2) \in [0, 1]^2,$$

with $\mu \in [-1, 1]$. Then the corresponding copula density is indicated a

$$c(x_1, x_2) = \exp[\mu(1 - x_1)(1 - x_2)] \times$$

$$\left[\mu^2 x_1 x_2(1-x_1)(1-x_2) + \mu x_1(3x_2-1) - \mu x_2 + 1\right], \quad (x_1, x_2) \in (0,1)^2.$$

Thanks to the simplicity of this expression, we can bound it and modulate the bound by tuning μ. For instance, a direct bound (far from the optimal one) gives $\|c\|_\infty \le \mu^2/4 + |\mu| + 1$, and we can modulate μ to have the desired condition.

Example 2: We can consider the new ratio copula established in [28]. It is specified as

$$C(x_1, x_2) = x_1 x_2 \frac{(1+\nu)(1+\nu x_1 x_2)}{(1+\nu x_1)(1+\nu x_2)}, \quad (x_1, x_2) \in [0,1]^2,$$

with $\nu \ge -1/4$. Then the corresponding copula density is obtained as

$$c(x_1, x_2) = (1+\nu)\frac{1+\nu x_1 x_2(2+\nu x_1)(2+\nu x_2)}{(1+\nu x_1)^2(1+\nu x_2)^2}, \quad (x_1, x_2) \in (0,1)^2.$$

We can easily bound it and modulate the result via ν. For instance, for $\nu \ge 0$, a direct bound (far from the optimal one) gives $\|c\|_\infty \le (1+\nu)[1+\nu(2+\nu)^2]$, and we can select ν to have the desired condition.

The originality of the generalized convex mixture copula strategy in Theorem 4.1 is concretized by a new three-dimensional copula approach, as shown in the proposition below.

Proposition 4.3. *Let $\alpha \ge 0$, $\beta \ge 0$, $\gamma \ge 0$ and $C^*(x_1, x_2)$, $C^\dagger(x_1, x_2)$ and $C^\triangle(x_1, x_2)$, $(x_1, x_2) \in [0,1]^2$, be three two-dimensional copulas, and $c^*(x_1, x_2)$, $c^\dagger(x_1, x_2)$ and $c^\triangle(x_1, x_2)$ be the corresponding copula densities, respectively. Let us suppose that*

$$\alpha\|c^*\|_\infty + \beta\|c^\dagger\|_\infty + \gamma\|c^\triangle\|_\infty \le \kappa,$$

where $\kappa = 1 + \alpha + \beta + \gamma$. Then the three-dimensional function

$$\begin{aligned} C(x_1, x_2, x_3) = \kappa\Pi(x_1, x_2, x_3) - \alpha x_1 C^*(x_2, x_3) - \beta x_2 C^\dagger(x_1, x_3) - \gamma x_3 C^\triangle(x_2, x_1), \\ (x_1, x_2, x_3) \in [0,1]^3, \end{aligned} \tag{4.2}$$

is a valid copula.

Proof. It is an application of Theorem 4.1. To prove this claim, we consider the generalized convex mixture function as described in Equation (4.1) with $n = 3$, $m = 4$, $\xi_1 = \kappa$, $C_1(x_1, x_2, x_3) = \Pi(x_1, x_2, x_3)$, $\xi_2 = -\alpha$, $C_2(x_1, x_2, x_3) = x_1 C^*(x_2, x_3)$, $\xi_3 = -\beta$, $C_3(x_1, x_2, x_3) = x_2 C^\dagger(x_1, x_3)$, $\xi_4 = -\gamma$ and $C_4(x_1, x_2, x_3) = x_3 C^\triangle(x_2, x_1)$. Thus defined, it is clear that $C_2(x_1, x_2, x_3)$, $C_3(x_1, x_2, x_3)$ and $C_4(x_1, x_2, x_3)$ are well-defined three-dimensional copulas based on the product scheme. Then the function in Equation (4.1) becomes the desired expression, i.e.,

$$C_{mix}(x_1, x_2, x_3) = \kappa\Pi(x_1, x_2, x_3) - \alpha x_1 C^*(x_2, x_3) - \beta x_2 C^\dagger(x_1, x_3) - \gamma x_3 C^\triangle(x_2, x_1).$$

By the definition of κ, let us notice that

$$\sum_{j=1}^{4} \xi_j = \kappa - \alpha - \beta - \gamma = 1.$$

Furthermore, we have $\partial_{x_1,x_2,x_3}C_1(x_1,x_2,x_3) = 1$, $\partial_{x_1,x_2,x_3}C_2(x_1,x_2,x_3) = \partial_{x_2,x_3}C^*(x_2,x_3) = c^*(x_2,x_3)$, $\partial_{x_1,x_2,x_3}C_3(x_1,x_2,x_3) = \partial_{x_1,x_3}C^\dagger(x_1,x_3) = c^\dagger(x_1,x_3)$, and $\partial_{x_1,x_2,x_3}C_4(x_1,x_2,x_3) = \partial_{x_2,x_1}C^\vartriangle(x_2,x_1) = c^\vartriangle(x_2,x_1)$. As a result, since $\xi_1 = \kappa > 0$ (so different to 0), we have $||\partial_{x_1,x_2,x_3}C_1||_{\xi_1} = 1$, since $\xi_2 = -\alpha \le 0$, we have $||\partial_{x_1,x_2,x_3}C_2||_{\xi_2} = ||c^*||_\infty$ or $||\partial_{x_1,x_2,x_3}C_2||_{\xi_2} = 0$ if $\alpha = 0$, since $\xi_3 = -\beta \le 0$, we have $||\partial_{x_1,x_2,x_3}C_3||_{\xi_3} = ||c^\dagger||_\infty$ or $||\partial_{x_1,x_2,x_3}C_3||_{\xi_3} = ||c^\dagger||_\infty = 0$ if $\beta = 0$, and since $\xi_4 = -\gamma \le 0$, we have $||\partial_{x_1,x_2,x_3}C_4||_{\xi_4} = ||c^\vartriangle||_\infty$ or $||\partial_{x_1,x_2,x_3}C_4||_{\xi_4} = ||c^\vartriangle||_\infty = 0$ if $\gamma = 0$. Therefore, owing to the assumption $\alpha||c^*||_\infty + \beta||c^\dagger||_\infty + \gamma||c^\vartriangle||_\infty \le \kappa$, we have

$$\begin{aligned}\sum_{j=1}^{4}\xi_j||\partial_{x_1,x_2,x_3}C_j||_{\xi_j} &= \kappa\times 1 + (-\alpha)\times||c^*||_\infty + (-\beta)\times||c^\dagger||_\infty + (-\gamma)\times||c^\vartriangle||_\infty\\ &= \kappa - \alpha||c^*||_\infty - \beta||c^\dagger||_\infty - \gamma||c^\vartriangle||_\infty \ge 0.\end{aligned}$$

It follows from Theorem 4.1 that $C_{mix}(x_1,x_2,x_3)$, so $C(x_1,x_2,x_3)$, is a valid copula. □

We can write the copula in Equation (4.2) a perturbation of the three-dimensional independence copula, in the following form:

$$\begin{aligned}C(x_1,x_2,x_3) &= \Pi(x_1,x_2,x_3) + (\kappa-1)x_1x_2x_3 - \alpha x_1C^*(x_2,x_3) - \beta x_2C^\dagger(x_1,x_3)\\ &\quad - \gamma x_3C^\vartriangle(x_2,x_1), \quad (x_1,x_2,x_3)\in[0,1]^3.\end{aligned}$$

The next result illustrates one aspect of the findings of Proposition 4.3 by proposing a variant of the three-dimensional AMH copula (or PAMM copula) through the use of three intermediary two-dimensional AMH copulas.

Proposition 4.4. *Let $\alpha \ge 0$, $\beta \ge 0$, $\gamma \ge 0$ such that $\alpha+\beta+\gamma \le 1$, and $\lambda^* \in [-1,0]$, $\lambda^\dagger \in [-1,0]$, and $\lambda^\vartriangle \in [-1,0]$. Then the three-dimensional function*

$$\begin{aligned}C(x_1,x_2,x_3) &= \Pi(x_1,x_2,x_3) - \alpha\lambda^*\frac{x_1x_2x_3(1-x_2)(1-x_3)}{1-\lambda^*(1-x_2)(1-x_3)}\\ &\quad - \beta\lambda^\dagger\frac{x_1x_2x_3(1-x_1)(1-x_3)}{1-\lambda^\dagger(1-x_1)(1-x_3)} - \gamma\lambda^\vartriangle\frac{x_1x_2x_3(1-x_2)(1-x_1)}{1-\lambda^\vartriangle(1-x_2)(1-x_1)}, \quad (x_1,x_2,x_3)\in[0,1]^3,\end{aligned} \tag{4.3}$$

is a valid copula.

Proof. The proof can be viewed as a consequence of Proposition 4.3, using the two-dimensional AMH copula. Indeed, let $C^*(x_1,x_2)$, $C^\dagger(x_1,x_2)$ and $C^\vartriangle(x_1,x_2)$ be the AMH copula with parameters λ^*, $\lambda^\dagger$ and $\lambda^\vartriangle$, respectively, i.e.,

$$C^*(x_1,x_2) = \frac{x_1x_2}{1-\lambda^*(1-x_1)(1-x_2)},$$
$$C^\dagger(x_1,x_2) = \frac{x_1x_2}{1-\lambda^\dagger(1-x_1)(1-x_2)},$$

and

$$C^\vartriangle(x_1,x_2) = \frac{x_1x_2}{1-\lambda^\vartriangle(1-x_1)(1-x_2)}, \quad (x_1,x_2)\in[0,1]^2.$$

By setting $\kappa = 1+\alpha+\beta+\gamma$, we have

$$C(x_1,x_2,x_3) = \kappa\Pi(x_1,x_2,x_3) - \alpha x_1C^*(x_2,x_3) - \beta x_2C^\dagger(x_1,x_3) - \gamma x_3C^\vartriangle(x_2,x_1)$$

$$= \Pi(x_1, x_2, x_3) - \alpha\lambda^* \frac{x_1x_2x_3(1-x_2)(1-x_3)}{1-\lambda^*(1-x_2)(1-x_3)}$$
$$-\beta\lambda^\dagger \frac{x_1x_2x_3(1-x_1)(1-x_3)}{1-\lambda^\dagger(1-x_1)(1-x_3)} - \gamma\lambda^\triangle \frac{x_1x_2x_3(1-x_2)(1-x_1)}{1-\lambda^\triangle(1-x_2)(1-x_1)}.$$

Thus, it is enough to check if the assumptions of Proposition 4.3 are satisfied. The copula density associated with $C^*(x_2, x_3)$ is given by

$$c^*(x_2, x_3) = \frac{(\lambda^*)^2(1-x_2)(1-x_3) + \lambda^*[x_2(x_3+1) + x_3 - 2] + 1}{[1-\lambda^*(1-x_2)(1-x_3)]^3}.$$

Since $\lambda^* \in [-1, 0]$, by proceeding as in Equations (3.1) and (3.2), we establish that $||c^*||_\infty \leq 2$. Similarly, since $\lambda^\dagger \in [-1, 0]$ and $\lambda^\triangle \in [-1, 0]$, we obtain $||c^\dagger||_\infty \leq 2$, and $||c^\triangle||_\infty \leq 2$. By using the assumption $\alpha + \beta + \gamma \leq 1$, we get

$$\alpha||c^*||_\infty + \beta||c^\dagger||_\infty + \gamma||c^\triangle||_\infty \leq 2(\alpha + \beta + \gamma) \leq 1 + \alpha + \beta + \gamma = \kappa.$$

We can apply Proposition 4.3, ensuring that $C(x_1, x_2, x_3)$ is a valid copula. □

To the best of our knowledge, the variant of the three-dimensional AMH copula indicated in Equation (4.3) is new in the literature. It can also be viewed as an original copula based on the perturbation of the three-dimensional independence copula. Since it depends on 6 parameters, its practical use may necessitate considering some fixed values to avoid the overparameterization phenomenon, which mainly appears in a parametric estimation scenario. In particular, one can think to consider $\alpha = \zeta/2$, $\beta = \zeta/2$ and $\gamma = 1 - \zeta$ with $\zeta \in [0, 1]$, and $\lambda^* = \lambda^\dagger = \lambda^\triangle = \iota$, with $\iota \in [-1, 0]$, which yields a manageable two-parameter three-dimensional copula. However, its application in a real data analysis scenario remains a challenge to explore in another study.

5. Conclusion

In conclusion, the ideas presented in this article arise from the work of [22] on various constructions of three-dimensional copulas. Specifically, it was noted that the general three-dimensional function in [22, Equation (13)] may not be a copula stricto sensu. This raises questions about the assumptions ensuring its validity. It is especially true for the PAMM copula, which remains one of the most innovative examples. Through mathematical derivations, numerical analyses, and graphical representations, one of our contributions revealed that the PAMM copula with the parameter λ is not valid for any $\lambda \in [-1, 1]$. However, a more in-depth investigation has shown that it is for $\lambda \in [-1, \lambda_0]$, with $\lambda_0 < 0.7$. Additionally, we revisited the expression of the Spearman rho, demonstrating the usefulness of the PAMM copula in scenarios of small positive dependence. In another part, we introduced a generalized convex mixture copula strategy, along with the assumptions required for its validity. The function in [22, Equation (13)] emerged as a specific case of such a strategy. Notably, our general result may depend on many intermediary copulas and goes beyond the three-dimensional scenario; it is applicable to higher-dimensional copulas. Finally, we applied it to construct a variant of the three-dimensional AMH copula, providing insight into the practical implications of our theoretical framework. With this work, we hope to further advance the topic of copula theory and contribute to the development of three-dimensional dependence models in particular.

Conflict of interest

The authors state that they have no financial or other conflicts of interest to disclose with connection to this research.

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