Computational Journal of Mathematical and Statistical Sciences
3(1), 177–202
DOI:10.21608/CJMSS.2024.256760.1035
https://cjmss.journals.ekb.eg/

Inference for Generalized Progressive Hybrid Type-II Censored Weibull Lifetimes under Competing Risk Data

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Abstract: The competing risks model plays a pivotal role in the analysis of various fields, including engineering, econometrics, and biology. When a product being tested is likely to fail due to multiple factors, these factors conflict with each other to precipitate product failure. This situation is referred to as the competing risks problem. This paper focuses in particular on a competing risks model that employs a generalized progressive hybrid type-II censoring scheme . It assumes that the underlying lifetime distributions of the failure causes follow Weibull distribution with different scale and common shape parameters. The paper derives maximum likelihood estimates and Bayes estimates , utilizing Markov Chain Monte Carlo techniques for computing Bayes estimates and credible intervals. Further, Bayes estimates of the parameters which obtained based on squared error and linear exponential loss functions under the assumptions of independent Gamma priors. To illustrate these concepts, the paper includes simulation studies and real-world examples.

Keywords: Competing risks, Generalized progressive hybrid type-II censoring scheme, Weibull distribution , Markov Chain Monte Carlo , Bayesian estimation.

Mathematics Subject Classification: 91B05, 62N02, 60J22.

Received: 30 December 2023; Revised: 23 January 2024; Accepted: 28 January 2024; Online: 10 February 2024.

1. Introduction:

In survival analysis, experimental units may fail due to more than one causes. The experimenter observes time of failure of the experimental units along with the corresponding causes of failure. The causes compete with each other for failure of the experimental units. This kind of studies are known as the competing risks problem in literature. The causes of failure in the competing risks data analysis can be assumed to be dependent or independent where the data consists of a failure time and the associated cause of failure. Censored data occurs when an experimenter or reliability practitioner desires to stop a life test before getting the entire sample. As a result, the available data are actually censored. The conventional censoring schemes, namely type-I and type-II have been widely used in reliability studies. The experimental time is fixed in Type-I censoring scheme, but the number of observed failures is a random variable. On the other hand, in Type-II censoring scheme, the experimental time is a random variable while the number of observed failures is fixed. Hybrid censoring is the combination of Type-I and Type-II censoring schemes. If the experiment ends at the time point $T_1^* = \min(x_{(r)}, T)$ it is referred to as hybrid type I (HT-I) proposed by Epstein [9]. The experiment is referred to as a hybrid type-II , suggested by Childs et al. [3], if it ends at the time point $T_2^* = \max(x_{(r)}, T)$. These schemes have the disadvantage of not allowing the units to be removed from the experiment at any point other than the terminal point. To deal with this problem, a more general censoring scheme called progressive type-II ((PT-II)) censoring is used. Balakrishnan and Aggarwala [1] and Wu [16] proposed another type of censoring called progressive censoring allows removal of units from the test at times other than the final termination point. Cho and Lee [5], derived point and interval estimations for the unknown parameters of exponential distribution under generalized progressive hybrid type-II (GPHT-II) censoring in presence of competing risks data when the cause of failure is known.

Childs et al. [3] proposed progressive hybrid type-II censoring scheme with the purpose of increasing the efficiency of statistical analysis as well as saving the overall testing time. The drawback of proposed progressive hybrid type-II censoring scheme is that the length of the experiment can be quite large. For this motivation, Cho et al. [4] proposed GPH type-II censoring scheme, which the experiment is guaranteed to terminate at a pre-fixed time. Kundu and Koley [12], obtained the maximum likelihood estimators for the unknown parameters based of exponential distribution basesd on GPH Type-II and also obtained confidence intervals.

Gorny and Cramer [10], introduced a new censoring scheme called GPH Type-II censoring which can be explain as follows: suppose n identical units are placed on a life testing experiment with r, T_1 and T_2 pre-fixed with $r \leq n$ and $0 < T_1 < T_2 < \infty$. The censoring scheme $R_1, R_2, ..., R_r$ (where $R_i \geq 0$, $i = 1, ..., r$) are pre-fixed integers satisfies $\sum_{i=1}^{r} R_i + r = n$. Then, at the time of the first failure $x_{(1)}$, R_1 of the remaining units are randomly removed. Similarly at the time of the second failure $x_{(2)}$, R_2 of the remaining units are removed and so on. This process continues until, immediately following the terminated time $T_2^{\bullet} = \max(T_1, \min(T_2, x_{(r)}))$, all the remaining units are removed from the experiment as a schematic illustration in Figure 1. If $x_{(r)} < T_1$, then instead of terminating the test by withdrawing the remaining R_r items after the r^{th} failure, we continue to observe failures (without any further withdrawals) up to time T_1. Therefore, $R_r = R_{r+1} = ... = R_{D_1} = 0$. If $T_1 < x_{(r)} < T_2$, terminate the test at $x_{(r)}$. If $x_{(r)} > T_2$, terminate the test at time T_2. Note that GPH type-II censored schemes modifies progressive hybrid type-II censored schemes by guaranteeing that the test will be completed by time T_2. Therefore, T_2 represents the absolute longest that the researcher is willing to allow the experiment to continue. In this scheme, the data is one of the following types

$$x_{(1)} < .... < x_{(r)} < x_{(r+1)} < .... < x_{(D_1)} \quad if \quad x_{(r)} < T_1, \quad case 1$$

$$x_{(1)} < ..... < x_{(D_1)} < ..... < x_{(r)} \quad if \quad T_1 < x_{(r)} < T_2, \quad case 2$$

$$x_{(1)} < .... < x_{(D_1)} < .... < x_{(D_2)} \quad if \quad T_1 < T_2 < x_{(r)}. \quad case 3$$

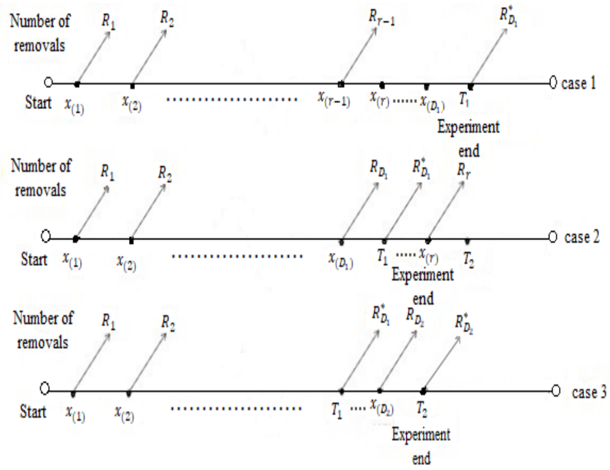

Figure 1. Schematic illustration of GPH type-II censoring scheme

The likelihood function of the GPH type-II censoring scheme can be found in Cho et al. [4] as the following form

$$L = \begin{cases} \prod_{i=1}^{D_1} \sum_{V=i}^{r} (R_V + 1) \prod_{i=1}^{D_1} f(x_{(i)})(1 - F(x_{(i)}))^{R_i} (1 - F(T_1))^{R^*_{D_1}}, & \text{case 1} \\ \prod_{i=1}^{r} \sum_{V=i}^{r} (R_V + 1) \prod_{i=1}^{r} f(x_{(i)})(1 - F(x_{(i)}))^{R_i}, & \text{case 2} \\ \prod_{i=1}^{D_2} \sum_{V=i}^{r} (R_V + 1) \prod_{i=1}^{D_2} f(x_{(i)})(1 - F(x_{(i)}))^{R_i} (1 - F(T_2))^{R^*_{D_2}}, & \text{case 3} \end{cases} \tag{1.1}$$

where $R^*_{D_1} = n - D_1 - \sum_{i=1}^{r-1} R_i$, and $R^*_{D_2} = n - D_2 - \sum_{i=1}^{D_2} R_i$.

In this paper, the research question behind it is ”How to find the estimate of competing risk under GPH type-II for Weibull distribution ?”. Therefore, we summarize our work as follows

1. the MLEs and approximate confidence intervals (ACIs) are introduced for the unknown parameters in the presence of competing risks when the cause of failure of each item is known.
2. Obtain maximum likelihood estimators (MLEs), approximate two-sided confidence intervals .
3. Bayesian estimators obtained using symmetric and asymmetric loss functions.
4. Create the highest posterior density (HPD) intervals using the Markov chain Monte Carlo (MCMC) algorithm.
5. A simulation study is conducted under various sample sizes.
6. Bayesian estimates are reached by assuming independent gamma priors and using squared error loss (SEL) function and linear exponential (LINEX) loss function. As expected, Bayesian estimates under SEL and LINEX loss functions cannot be obtained in closed forms. A real data set has been provided for illustration.

The rest article will be organized as follows: in Section 2, model description is presented. Maximum likelihood Estimation and Bayesian method are discussed in Sections 3 and 4 , respectively. In Section 5, simulation study is performed. Real data set is provided in Section 6. Finally, Section 7 contains the conclusions.

2. Model Description

Consider a lifetime experiment with n identical units where its lifetimes are described by independent and identically distributed (iid) random variables $X_1, X_2, ..X_n$. Without loss of generality, we assume that there are only two causes of failure. Let X_{ji} denotes the lifetime of the i^{th} item under the j^{th} cause of failure for $i = 1, 2, ...n$, $j = 1, 2$ and $X_i = \min(X_{1i}, X_{2i})$. The competing risks model assumes that the data consists of a failure time and an indicator denoting the cause of failure. We use the latent failure time modeling of Cox [6] and Pintilie [15] for analyzing competing risks data. In the latent failure time modeling, it is assumed that competing causes of failures are independent random variables.

Consider a population, where every units failed due to one of the two known causes; 1 and 2. A unit is selected at random from the population. Let the variable Δ_i is the indicator denoting the cause of failure of the observation, i.e, failure due to cause 1 ($\Delta_i = 1$), or failure due to cause 2 ($\Delta_i = 2$), or failure due to one of the causes 1 or 2 but it is unknown ($\Delta_i = *$). Here, $\Delta_i = j$, $j = 1, 2$ means the unit i has failed due to cause j, while $\Delta_i = *$ means that the cause of failure of unit i is unknown. Under the GPH type-II censoring scheme in presence of competing risk data, we mainly have three cases

$Case\ 1\ \ x_{(r)} < T_1,$

$Case\ 2\ \ T_1 < x_{(r)} < T_2,$

$Case\ 3\ \ T_1 < T_2 < x_{(r)}.$

Under the above three cases, we get three different sets of failure-times as

$Case\ 1\ \ (x_{(1)}, \Delta_1, R_1) < ... < (x_{(r-1)}, \Delta_{r-1}, R_{r-1}) < (x_{(r)}, \Delta_r, 0) < ... < (T_1, R^*_{D_1})$

$Case\ 2\ \ (x_{(1)}, \Delta_1, R_1) < .... < (x_{(D_1)}, \Delta_{D_1}, R_{D_1}) < ...... < (x_{(r)}, \Delta_r, R_r),$

$Case\ 3\ \ (x_{(1)}, \Delta_1, R_1) < ...... < (x_{(D_1)}, \Delta_{D_1}, R_{D_1}) < ..... < (x_{(D_2)}, \Delta_{D_2}, R^*_{D_2}).$

where

$$I(\Delta_i = j) = \begin{cases} 1 & if \quad j = 1, 2. \\ 0 & otherwise. \end{cases}$$

and

$$I(\Delta_i = *) = \begin{cases} 1 & if \quad \Delta_i = *. \\ 0 & otherwise. \end{cases}$$

The likelihood function of GPH type-II censored under competing risks when the cause of failure is known can be written as follows

$$L = \begin{cases} \gamma \prod_{i=1}^{D_1} (h_{\Delta_i}(x_{(i)})) \prod_{j=1}^{2} (\bar{F}_j(x_{(i)}))(\bar{F}(x_{(i)}))^{R_i} (\bar{F}(T_1))^{R^*_{D_1}}, & \text{case 1} \\ \gamma \prod_{i=1}^{r} (h_{\Delta_i}(x_{(i)})) \prod_{j=1}^{2} (\bar{F}_j(x_{(i)}))(\bar{F}(x_{(i)}))^{R_i}, & \text{case 2} \\ \gamma \prod_{i=1}^{D_2} (h_{\Delta_i}(x_{(i)})) \prod_{j=1}^{2} (\bar{F}_j(x_{(i)}))(\bar{F}(x_{(i)}))^{R_i} (\bar{F}(T_2))^{R^*_{D_2}}. & \text{case 3} \end{cases} \tag{2.1}$$

where $\gamma = \prod_{i=1}^{j^*} \sum_{V=i}^{r} (R_V + 1)$, $j^* = D_1$, $j^* = r$, $j^* = D_2$ for case 1,2 and 3 respectively,$R^*_{D_1} = n - D_1 - \sum_{i=1}^{r-1} R_i$, $R^*_{D_2} = n - D_2 - \sum_{i=1}^{D_2} R_i$, $\bar{F}(T) = \bar{F}_1(T)\bar{F}_2(T)$, $h_{\Delta_i}(x_{(i)})$ the hazard rate function under the cause of failure $\Delta_i = j$.

In this paper, we make inference under the assumption that the latent failure times follow two parameter Weibull distribution with different scale parameters $\lambda_1, \lambda_2 > 0$, $\lambda_1 \neq \lambda_2$, the same shape parameter β and the cause of failure is known. The probability density function (pdf) , the cumulative distribution function (cdf) , survival function (SF) and hazard function (HF) of X_{ji} from j^{th} cause of failure are, respectively, given by

$$f_j(x; \lambda_j, \beta) = \lambda_j \beta x^{\beta-1} e^{-\lambda_j x^\beta}; \qquad x, \lambda_j, \beta > 0 \tag{2.2}$$

$$F_j(x; \lambda_j, \beta) = 1 - e^{-\lambda_j x^\beta}, \tag{2.3}$$

$$\bar{F}_j(x; \lambda_j, \beta) = e^{-\lambda_j x^\beta}, \tag{2.4}$$

and

$$h_j(x; \lambda_j, \beta) = \lambda_j \beta x^{\beta-1}. \tag{2.5}$$

3. Maximum Likelihood Estimation

In this section, the MLEs and the corresponding ACIs of the unknown parameters β, λ_j, $j = 1, 2$ are presented under GPH type-II in presence of competing risks. Also, the relative risk rates due to cause 1 and cause 2 are derived.

3.1. Point Estimation

Based on equations (2),(5) and (6), the natural logarithm of the likelihood function of GPH type-II in presence of competing risks can be written as

$$\ln L = \ln\gamma + j^* \ln\beta + j_1 \ln\lambda_1 + j_2 \ln\lambda_2 + (\beta - 1)\sum_{i=1}^{j^*} \ln x_{(i)} - (\lambda_1 + \lambda_2)w. \tag{3.1}$$

where

$$\gamma = \begin{cases} \prod_{i=1}^{D_1} \sum_{V=i}^{r} (R_V + 1), & case1 \\ \prod_{i=1}^{r} \sum_{V=i}^{r} (R_V + 1), & case2, \\ \prod_{i=1}^{D_2} \sum_{V=i}^{r} (R_V + 1), & case3 \end{cases}$$

$$j^* = \begin{cases} D_1 = \sum_{i=1}^{D_1} I(\Delta_i = j), j = 1, 2 & case1 \\ r = \sum_{i=1}^{r} I(\Delta_i = j), j = 1, 2 & case2, \\ D_2 = \sum_{i=1}^{D_2} I(\Delta_i = j), j = 1, 2 & case3 \end{cases}$$

$$j_1 = \begin{cases} d_1 = \sum_{i=1}^{D_1} I(\Delta_i = 1), & case1 \\ r_1 = \sum_{i=1}^{r} I(\Delta_i = 1), & case2, \\ e_1 = \sum_{i=1}^{D_2} I(\Delta_i = 1), & case3 \end{cases}$$

$$j_2 = \begin{cases} d_2 = \sum_{i=1}^{D_1} I(\Delta_i = 2), & case1 \\ r_2 = \sum_{i=1}^{r} I(\Delta_i = 2), & case2. \\ e_2 = \sum_{i=1}^{D_2} I(\Delta_i = 2), & case3 \end{cases}$$

and

$$w = \begin{cases} \sum_{i=1}^{D_1} x_{(i)}^{\beta}(R_i+1) + R_{D_1}^{*} T_1^{\beta}, & case1 \\ \sum_{i=1}^{r} x_{(i)}^{\beta}(R_i+1), & case2. \\ \sum_{i=1}^{D_2} x_{(i)}^{\beta}(R_i+1) + R_{D_2}^{*} T_2^{\beta}. & case3 \end{cases}$$

where $R_{D_1}^{*} = n - D_1 - \sum_{i=1}^{r-1} R_i$, and $R_{D_2}^{*} = n - D_2 - \sum_{i=1}^{D_2} R_i$.

Differentiating equation (8) with respect to β, λ_1 and λ_2, respectively, we obtained

$$\frac{\partial \ln L}{\partial \beta} = \frac{j^*}{\beta} + \sum_{i=1}^{j^*} \ln x_{(i)} - (\lambda_1 + \lambda_2) w',$$
$$\frac{\partial \ln L}{\partial \lambda_1} = \frac{j_1}{\lambda_1} - w,$$
$$\frac{\partial \ln L}{\partial \lambda_2} = \frac{j_2}{\lambda_2} - w.$$

where

$$w' = \begin{cases} \sum_{i=1}^{D_1} x_{(i)}^{\beta}(R_i+1)\ln x_{(i)} + R_{D_1}^{*} T_1^{\beta} \ln T_1,, & case1 \\ \sum_{i=1}^{r} x_{(i)}^{\beta}(R_i+1)\ln x_{(i)} & case2. \\ \sum_{i=1}^{D_2} x_{(i)}^{\beta}(R_i+1)\ln x_{(i)} + R_{D_2}^{*} T_2^{\beta} \ln T_2, & case3 \end{cases}$$

The MLEs of β, λ_j where $j = 1, 2$ cannot be expressed in closed form. So we need to employ some required numerical approach for computing the MLEs of β, λ_j. We present the relative risk rates, RR_1 and RR_2 due to causes 1 and 2, respectively, in closed forms. The relative risk related to cause 1 is calculated as follows:

$$RR_1 = p(X_{1i} < X_{2i}) = \int_0^\infty f_1(x)\bar{F}_2(x)dx,$$
$$= \int_0^\infty f_1(x)e^{-\lambda_2 x^\beta}dx,$$
$$= \lambda_1\beta \int_0^\infty x^{\beta-1}e^{-(\lambda_1+\lambda_2)x^\beta}dx.$$

Relative risk related to cause 2 is computed as

$$RR_2 = 1 - RR_1,$$

$$= 1 - \lambda_1\beta \int_0^{\infty} x^{\beta-1} e^{-(\lambda_1+\lambda_2)x^{\beta}} dx.$$

3.2. Interval Estimation

Because the MLEs of the unknown parameters β, λ_1 and λ_2 are not obtained in closed form, it is not possible to derive the exact distribution of the MLE. In this section, we derive the confidence intervals of the parameters based on the asymptotic variance covariance matrix J_0^{-1} for the maximum likelihood of the parameters to build the ACIs. The asymptotic variance-covariance matrix is given by

$$J_0^{-1} \cong \begin{bmatrix} -\dfrac{\partial^2 \ln L}{\partial \lambda_1^2} & -\dfrac{\partial^2 \ln L}{\partial \lambda_1 \partial \lambda_2} & -\dfrac{\partial^2 \ln L}{\partial \lambda_1 \partial \beta} \\ -\dfrac{\partial^2 \ln L}{\partial \lambda_2 \partial \lambda_1} & -\dfrac{\partial^2 \ln L}{\partial \lambda_2^2} & -\dfrac{\partial^2 \ln L}{\partial \lambda_2 \partial \beta} \\ -\dfrac{\partial^2 \ln L}{\partial \beta \partial \lambda_1} & -\dfrac{\partial^2 \ln L}{\partial \beta \partial \lambda_2} & -\dfrac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix}^{-1}_{\beta=\hat{\beta},\lambda_1=\hat{\lambda}_1,\lambda_2=\hat{\lambda}_2}$$

$$\cong \begin{bmatrix} var(\hat{\lambda}_1) & cov(\hat{\lambda}_1, \hat{\lambda}_2) & .... & cov(\hat{\lambda}_1, \hat{\beta}) \\ cov(\hat{\lambda}_2, \hat{\lambda}_1) & var(\hat{\lambda}_2) & .... & cov(\hat{\lambda}_2, \hat{\beta}) \\ \vdots & \vdots & :::: & \vdots \\ cov(\hat{\lambda}_s, \hat{\lambda}_1) & cov(\hat{\beta}, \hat{\lambda}_2) & .... & var(\hat{\beta}) \end{bmatrix}_{\beta=\hat{\beta},\lambda_1=\hat{\lambda}_1,\lambda_2=\hat{\lambda}_2}$$

where

$$\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \lambda_2} = \frac{\partial^2 \ln L}{\partial \lambda_2 \partial \lambda_1} = 0,$$
$$\frac{\partial^2 \ln L}{\partial \lambda_1^2} = -\frac{j_1}{\lambda_1^2},$$
$$\frac{\partial^2 \ln L}{\partial \lambda_2^2} = -\frac{j_2}{\lambda_2^2},$$
$$\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \beta} = \frac{\partial^2 \ln L}{\partial \lambda_2 \partial \beta} = -w',$$
$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{j^*}{\beta^2} - (\lambda_1 + \lambda_2)w'',$$

where

$$w'' = \begin{cases} \sum_{i=1}^{D_1} x_{(i)}^{\beta}(R_i + 1)(\ln x_{(i)})^2 + R_{D_1}^* T_1^{\beta}(\ln T_1)^2, & case1 \\ \sum_{i=1}^{r} x_{(i)}^{\beta}(R_i + 1)(\ln x_{(i)})^2, & case2. \\ \sum_{i=1}^{D_2} x_{(i)}^{\beta}(R_i + 1)(\ln x_{(i)})^2 + R_{D_2}^* T_2^{\beta}(\ln T_2)^2, & case3 \end{cases}$$

Also, we obtained the 100(1 − α)% confidence intervals for the parameters β, λ_1 and λ_2 using the normal approximate of the MLEs and asymptotic variance covariance matrix, as

$$\hat{\beta} - Z_{\alpha/2}\sqrt{var(\hat{\beta})}, \hat{\beta} + Z_{\alpha/2}\sqrt{var(\hat{\beta})},$$

$$\hat{\lambda}_1 - Z_{\alpha/2}\sqrt{var(\hat{\lambda}_1)}, \hat{\lambda}_1 + Z_{\alpha/2}\sqrt{var(\hat{\lambda}_1)},$$

$$\hat{\lambda}_2 - Z_{\alpha/2}\sqrt{var(\hat{\lambda}_2)}, \hat{\lambda}_2 + Z_{\alpha/2}\sqrt{var(\hat{\lambda}_2)}.$$

where $Z_{\alpha/2}$ is the percentile of the standard normal distribution with right-tail probability $\alpha/2$.

4. Bayesian Estimation

In this section, we consider Bayesian inference of the unknown parameters when the latent failure times follow Weibull distribution based on competing risks model using GPH type-II censoring scheme with competing risks. BEs and HPD credible intervals were obtained for the unknown parameters.

4.1. Prior distributions

In this subsection, we assumed that the priors of β, λ_1 and λ_2 are independent and follow gamma (a_1, b_1), gamma (a_2, b_2) and gamma (a_3, b_3), respectively. Therefore, the priors for β, λ_1 and λ_2 are of the forms

$$\pi(\beta) = \frac{b_1^{a_1}}{\Gamma a_1}\beta^{a_1-1}e^{-b_1\beta} \quad \beta>0 \quad and \quad a_1, b_1>0.$$

$$\pi(\lambda_1) = \frac{b_2^{a_2}}{\Gamma a_2}\lambda_1^{a_2-1}e^{-b_2\lambda_1} \quad \lambda_1>0 \quad and \quad a_2, b_2>0,$$

and

$$\pi(\lambda_2) = \frac{b_3^{a_3}}{\Gamma a_3}\lambda_2^{a_3-1}e^{-b_3\lambda_2} \quad \lambda_2>0 \quad and \quad a_3, b_3>0,$$

then, the joint prior is given by

$$\pi(\beta, \lambda_1, \lambda_2) = \pi(\beta)\pi(\lambda_1)\pi(\lambda_2). \tag{4.1}$$

4.2. Posterior Distribution

Based on equation(8), the posterior density function of the parameters β, λ_1 and λ_2 can be written as follows:

$$\pi(\beta, \lambda_1, \lambda_2|x) = \frac{L(x_1, x_2, ..., x_n|\lambda)\pi(\beta, \lambda_1, \lambda_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(x_1, x_2, ..., x_n|\lambda)\pi(\beta, \lambda_1, \lambda_2)d\beta d\lambda_1 d\lambda_2}, \tag{4.2}$$

$$\propto \beta^{a_1+j^*-1}\lambda_1^{a_2+j_1-1}\lambda_2^{a_3+j_2-1}e^{-\left(\lambda_1(w+b_1)+\lambda_2(w+b_2)\right)}e^{-\left(\beta\left(b_3-\sum_{i=1}^{j^*}\ln x_{(i)}\right)\right)}.$$

Bayes estimator of any function of β, λ_1 and λ_2, say $\zeta(\beta,\lambda_1,\lambda_2)$ under SEL function is the posterior mean, denoted by $\tilde{\zeta}(\beta,\lambda_1,\lambda_2)$ and can be obtained as follows:

$$\tilde{\zeta}(\beta,\lambda_1,\lambda_2) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \zeta(\beta,\lambda_1,\lambda_2)L(x_1,x_2,......,x_n|\lambda)\pi(\beta,\lambda_1,\lambda_2)d\beta d\lambda_1 d\lambda_2}{\int_0^\infty\int_0^\infty\int_0^\infty L(x_1,x_2,......,x_n|\lambda)\pi(\beta,\lambda_1,\lambda_2)d\beta d\lambda_1 d\lambda_2}, \tag{4.3}$$

Bayes estimator of $\zeta(\beta,\lambda_1,\lambda_2)$ cannot be expressed in closed form, so we need to employ some approximation method to compute the estimate given in (10). We propose to use MCMC method to obtain BEs and HPD credible intervals of the unknown parameters. This method is particularly useful in Bayesian inference as a result of focusing on subsequent distributions that are often difficult to work with through mathematical analysis. MH algorithm starts with simulating a candidate sample θ^* from the proposal distribution $q(.)$. Samples from the proposal distribution are not accepted automatically as posterior samples; they are accepted probabilistically based on the acceptance probability.
MH sampling algorithm, developed by Metropolis et al. [14] , Hastings [11] and David [7], can be described as followss:
Step 1: Start with any initial guess θ which $\theta^{(0)} = (\hat{\beta},\hat{\lambda}_1,\hat{\lambda}_2)$.
Step 2: Set $t = 1$.
Step 3: Generate θ^* with normal proposal distribution $q(\theta) = N(\hat{\theta}, var(\hat{\theta}))$.
Step 4: Given the candidate point θ^*, calculate the acceptance probability
$A(\theta^{(t-1)},\theta^*) = \min\left[1, \frac{\pi(\theta^*|\underline{x})q(\theta^{(t-1)}|\theta^*)}{\pi(\theta^{(t-1)}|\underline{x})q(\theta^*|\theta^{(t-1)})}\right].$
Step 5: Generate a sample from uniform distribution, i.e., $u \sim u(0,1)$.

$$\text{If}\begin{cases} u \le A(\theta^{(t-1)},\theta^*) & Accept \quad \theta^* = \theta^{(t)}, \\ u \ge A(\theta^{(t-1)},\theta^*) & Accept \quad \theta^{(t)} = \theta^{(t-1)}. \end{cases}$$

Step 6: Set $t = t+1$, and repeat steps 2-5 M times until get M samples and obtain $(\beta^{(t)},\lambda_1^{(t)},\lambda_2^{(t)}), t = 1,2,.....M$.
From the random samples of size M drawn from the posterior density, some of the initial samples can be discarded (burn-in), and remaining samples can be carried out to calculate BEs. Then, BEs of θ with respect to SEL function is
$\tilde{\theta} = \sum_{t=\curlyvee+1}^{M} \theta^{(t)}/(M-\curlyvee).$
where $\curlyvee$ represent the number of burn-in samples and $t = \curlyvee+1,.....M$.
Step 7: To obtain the HPD credible intervals of θ, arrange $\theta^t = (\beta^t,\lambda_1^t,\lambda_2^t)$, for $t = 1,2,....M$ in an ascending order, as $\theta^{[1]},\theta^{[2]},...\theta^{[M]}$, after burn in as $\theta^{[\curlyvee+1]},\theta^{[\curlyvee+2]},...\theta^{[M]}$, then for arbitrary $0<\alpha<1$, the $100(1-\alpha)\%$ two-sided HPD credible interval of θ can be obtained as
$(\theta^{[(\alpha/2)(M-\curlyvee)]},\theta^{[(1-(\alpha/2))(M-\curlyvee)]})$.

Then the HPD credible interval of θ^t is that interval which has the smallest width. **Step 8:** Under LINEX loss function, we obtain BEs of θ as

$$\theta = \frac{1}{c}\sum_{t=\curlyvee+1}^{M} e^{-c\theta^t}/(M-\curlyvee).$$

5. Simulation Study

The study evaluates the performance of theoretical results, including point and interval estimators, using maximum likelihood and Bayesian methods. Monte Carlo simulation study is performed when initial values of the parameters of WD are $(\beta,\lambda_1,\lambda_2) = (1, 0.2, 0.3)$ and $(\beta,\lambda_1,\lambda_2) = (1.5, 0.4, 0.7)$, By considering different combinations of n(sample sizes for each cause of failure), r(effective sample size), R(removal pattern) and $T_i, i = 1, 2$ (threshold points), we generate a large 1000 GPH type-II censored samples from WD. Further, different values of (n, r, T_2) are taken and $T_1 = 0.7$ (fixed). In each setting, the MLEs , BEs and the corresponding ACIs / HPD are evaluated. Three different progressive censoring schemes are

Sc-I: $R_1 = n - r, R_i = 0$ for $i \neq 1$

Sc-II:$R_{r/2} = n - r, R_i = 0$,for $i \neq \frac{r}{2}$

Sc-III: $R_r = n - r, R_i = 0$ for $i \neq r$.

Consider the following different cases in Table 1.

Table 1. Different cases considered for simulation when $T_2 = (.8, 1)$

Sc-I	Sc-II	Sc-III
$n = 30, r = 10, T_1 = 0.7$	$n = 30, r = 10, T_1 = 0.7$	$n = 30, r = 10, T_1 = 0.7$
$n = 30, r = 20, T_1 = 0.7$	$n = 30, r = 20, T_1 = 0.7$	$n = 30, r = 20, T_1 = 0.7$
$n = 50, r = 20, T_1 = 0.7$	$n = 50, r = 20, T_1 = 0.7$	$n = 50, r = 20, T_1 = 0.7$
$n = 50, r = 30, T_1 = 0.7$	$n = 50, r = 30, T_1 = 0.7$	$n = 50, r = 30, T_1 = 0.7$

Note that in the simulation study the performance of proposed point estimates are compared based on root mean squared errors (RMSEs). The performance of the proposed interval estimates are compared with respect to average interval lengths (AILs) and coverage probability (CP).

• RMSE: It is given by $\sqrt{\frac{1}{N}\sum_{\tau=1}^{N}(\tilde{\theta}_i^{\tau} - \theta_i)^2}$ for $i = 1, 2, 3$, where N denotes the number of replications in the Monte carlo simulation and $\theta_1 = \beta, \theta_2 = \lambda_1, \theta_3 = \lambda_2$. The smaller value of RMSE represents an estimator with better accuracy.

• AIL: The smaller length of the interval estimates indicates that the prediction model represents better accuracy with the experimental data.

• CP: The probability that true parameter value is contained in the interval estimates. When the value of CP is nearly about the nominal value, that is, $100(1 - \alpha)\%$, then it provides better result in terms of CP.

Note that in simulation study BEs are obtained under SEL and LINEX function (when c = 1.5). It is clear that the values of hyper-parameters are obtained by using the elicitation method. The method of hyper-parameter will depend on the prior knowledge of the unknown parameters . These informative priors will be obtained from the MLEs for $(\beta,\lambda_1,\lambda_2)$ by equating the mean and variance of $(\hat{\beta}^t, \hat{\lambda}_1^t, \hat{\lambda}_2^t)$ with the mean and variance of the priors under consideration (Gamma priors), where $t = 1, 2..., M$ and M is the number of samples available from Weibull distribution. Thus, on equating mean and variance of $(\hat{\beta}^t, \hat{\lambda}_1^t, \hat{\lambda}_2^t)$ with the mean and variance of Gamma priors, one can obtain (Dey et al. [8])

$$\frac{a_1}{b_1} = \frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t, \qquad \frac{a_1}{b_1^2} = \frac{1}{M-1}\sum_{t=1}^{M}(\hat{\beta}^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t)^2,$$
$$\frac{a_2}{b_2} = \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t, \qquad \frac{a_2}{b_2^2} = \frac{1}{M-1}\sum_{t=1}^{M}(\hat{\lambda}_1{}^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t)^2,$$
$$\frac{a_3}{b_3} = \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t \text{ and } \frac{a_3}{b_3^2} = \frac{1}{M-1}\sum_{t=1}^{M}(\hat{\lambda}_2^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t)^2.$$

By solving the above equations , the estimated hyper-parameters can be written as

$$a_1 = \frac{(\frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t)^2}{\frac{1}{M-1}\sum_{t=1}^{M}(\hat{\beta}^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t)^2},$$
$$b_1 = \frac{(\frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t)}{\frac{1}{M-1}\sum_{t=1}^{M}(\hat{\beta}^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t)^2},$$
$$a_2 = \frac{(\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t)^2}{\frac{1}{M-1}\sum_{t=1}^{M}(\hat{\lambda}_1^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t)^2},$$
$$b_2 = \frac{(\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t)}{\frac{1}{M-1}\sum_{t=1}^{M}(\hat{\lambda}_1^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t)^2},$$
$$a_3 = \frac{(\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t)^2}{\frac{1}{M-1}\sum_{t=1}^{M}(\hat{\lambda}_2^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t)^2}$$

and

$$b_3 = \frac{(\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t)}{\frac{1}{M-1}\sum_{t=1}^{M}(\hat{\lambda}_2^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t)^2}. \tag{5.1}$$

Based on the generated data, MLEs and associated 95% ACIs / HPD are computed. Note that the initial guess values are considered to be the same as the true parameter values while obtaining MLEs and subsequently get the hyper-parameter values from equation (11). These values, hyper-parameters, are then plugged-in to calculate the desired estimates. At the end, using MH algorithm to calculate BEs, 2000 burn-in samples are discarded among the total 10000 MCMC samples generated from the posterior density. From Tables 2, 3, 4 and 5, it is observed that:

• All BEs of parameters are derived with respect to two different loss functions, SEL and LINEX function. Under LINEX function associated estimates are obtained for $c = 1.5$.

• All of the average estimates and related RMSEs for both methods are showed.

• Under fixed censoring schemes, as the effective increases (i.e., n or r or T_2, or their combinations), the AEs and RMSEs of both MLEs and BEs for parameter β decrease. Also, the AEs and RMSEs of both MLEs and BEs for parameter λ_1 and λ_2 increases.

• For fixed value of n, when effective sample size r increases, the simulated RMSEs decreases for most cases.

• Under fixed censoring schemes, the RMSE for parameter β increase of MLEs and RMSEs for parameters λ_1 and λ_2 decrease of MLEs when T_2 increase.

• Based on AEs and RMSEs, the Bayes estimates under SEL and LINEX provide better results than other estimates for MLEs.

In case of interval estimates, from Tables 6, 7, 8 and 9, some observations are made, which are presented below:

• When sample size n and effective sample size r increase, AILs of intervals for parameters β, λ_1 and λ_2 decrease.

• Based on AIL and CP, the HPD credible intervals perform better than confidence interval of the MLEs.

• Further, the corresponding AILs and CPs for all the proposed confidence intervals, namely; ACIs and HPD interval are presented when $T_2 = 0.8$ and $T_2 = 1$, respectively.

• When the values of time thresholds T_2 increase, the length of intervals decrease.

•The performance of classical and Bayesian estimates are quite satisfactory.

• The performance of BEs relative to LINEX loss function have perform better than SEL function.

•It has been observed that censoring Sc-III provides the smallest RMSE and AIL among three censoring schemes for most cases.

Moreover, as further illustration, the trace and density plots for all parameters in an MCMC trace with their histograms for each parameter and the convergence of MCMC estimation for β, λ_1 and λ_2 of GPHT-II using MH algorithm are showed in Figure 2.

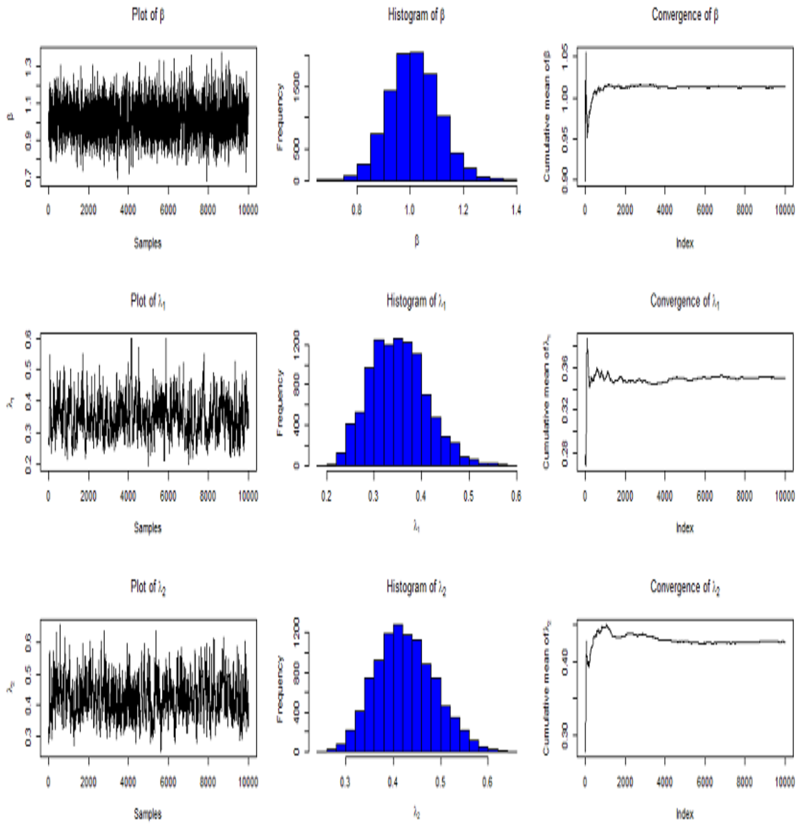

Figure 2. MCMC trace plots , histograms and convergence for β, λ_1 and λ_2 using MH algorithm.

6. Real Data

A real data set is analyzed in this section to illustrate the proposed competing risk model. Lawless [13] provides real data set that concerns failure times for electrical applications subjected to automatic life tests. Although there were 18 different causes of failure in original data, most of causes of failure appeared once and only modes 6 and 9 appeared more than twice. In this example, we are mainly focusing on failure mode 9 viewed it as cause one, and treated all other failure modes as cause two. Each failure time point in the original data set has been divided by one thousand for computational convenience. Transformed failure times of the electronic applications are reported in Table 10. Hence, the total number of observed failures due to causes 1 and 2 from the complete failure times are 19 and 17, respectively. Before further proceeding, we investigate whether Weibull distribution can be employed or not to analyze these data. Since the Kolmogorov–Smirnov distances and the corresponding p-values (within bracket) for cause one and cause two are 0.1596(0.6607) and 0.2359(0.2570), respectively, it is seen that Weibull distribution provides a reasonable model for these data. Moreover, as further illustration, the empirical cumulative distributions plot and the fitted densities plot with a histogram of probability are graphed in Figure 3, which also imply that Weibull distribution provides a reasonable fit for these data.

From Table 10 , we took $n = 36$, we have $r = 24, R_1 = 6, R_{24} = 6$ and $R_i = 0$ for $i = 2, ..., 23$. Based

Table 2. AEs , RMSEs of the MLEs and BEs at $(\beta, \lambda_1, \lambda_2) = (1, 0.2, 0.3)$ and $T_2 = .8$

n	r	T_1	Scheme	par	MLEs		BEs			
							SEL		LINEX (c=1.5)	
					AE	RMSE	AE	RMSE	AE	RMSE
30	10	.7	I	β	1.1632	0.5703	1.0986	0.1146	1.0904	0.1072
				λ_1	0.1172	0.2917	0.2821	0.1219	0.2801	0.1238
				λ_2	0.1587	0.3513	0.3721	0.1319	0.3696	0.1343
			II	β	1.0814	0.6002	1.0973	0.1117	1.0898	0.1048
				λ_1	0.0966	0.3116	0.2666	0.1358	0.2649	0.1374
				λ_2	0.1291	0.3784	0.3400	0.1626	0.3376	0.1649
			III	β	0.9867	0.5061	1.1003	0.1161	1.0926	0.1091
				λ_1	0.0784	0.3236	0.2526	0.1492	0.2511	0.1507
				λ_2	0.1040	0.3983	0.3205	0.1813	0.3184	0.1833
30	20	.7	I	β	1.1603	0.5033	1.0841	0.1036	1.0762	0.0967
				λ_1	0.1364	0.2778	0.2657	0.1397	0.2641	0.1411
				λ_2	0.1650	0.3474	0.3449	0.1602	0.3429	0.1621
			II	β	1.1553	0.4805	1.0835	0.1045	1.0757	0.0977
				λ_1	0.1427	0.2699	0.2649	0.1414	0.2633	0.1428
				λ_2	0.1719	0.3409	0.3431	0.1635	0.3412	0.1653
			III	β	1.1451	0.4788	1.0842	0.1039	1.0765	0.0973
				λ_1	0.1451	0.2750	0.2617	0.1447	0.2601	0.1461
				λ_2	0.1753	0.3436	0.3393	0.1678	0.3374	0.1695
50	20	.7	I	β	1.1242	0.3871	1.1092	0.1284	1.1016	0.1216
				λ_1	0.1346	0.2754	0.2537	0.1519	0.2522	0.1531
				λ_2	0.1680	0.3404	0.3316	0.1744	0.3298	0.1760
			II	β	1.0073	0.3277	1.1048	0.1264	1.0973	0.1196
				λ_1	0.1082	0.2945	0.2358	0.1668	0.2346	0.1679
				λ_2	0.1382	0.3651	0.3095	0.1938	0.3079	0.1952
			III	β	0.9372	0.3155	1.1097	0.1322	1.1021	0.1253
				λ_1	0.0899	0.3109	0.2193	0.1821	0.2182	0.1831
				λ_2	0.1139	0.3871	0.2883	0.2132	0.2869	0.2148
50	30	.7	I	β	1.0836	0.3348	1.0881	0.1148	1.0809	0.1087
				λ_1	0.1349	0.2746	0.2386	0.1674	0.2375	0.1685
				λ_2	0.1646	0.3451	0.3091	0.1974	0.3076	0.1987
			II	β	1.0903	0.3229	1.0899	0.1139	1.0827	0.1077
				λ_1	0.1449	0.2661	0.2406	0.1670	0.2395	0.1679
				λ_2	0.1759	0.3354	0.3109	0.1977	0.3095	0.1989
			III	β	1.0657	0.3145	1.0899	0.1163	1.0827	0.1103
				λ_1	0.1373	0.2699	0.2339	0.1722	0.2328	0.1732
				λ_2	0.1669	0.3410	0.3022	0.2052	0.3008	0.2064

Table 3. AEs , RMSEs of the MLEs and BEs at $(\beta, \lambda_1, \lambda_2) = (1, 0.2, 0.3)$ and $T_2 = 1$

n	r	T_1	Scheme	par	MLEs		BEs			
							SEL		LINEX (c=1.5)	
					AE	RMSE	AE	RMSE	AE	RMSE
30	10	.7	I	β	1.1949	0.5989	1.0903	0.1081	1.0823	0.1010
				λ_1	0.1328	0.2815	0.2897	0.1143	0.2876	0.1162
				λ_2	0.1783	0.3345	0.3812	0.1231	0.3785	0.1256
			II	β	1.0517	0.5798	1.0957	0.1128	1.0878	0.1058
				λ_1	0.1061	0.3514	0.2696	0.1327	0.2677	0.1345
				λ_2	0.1376	0.3977	0.3552	0.1470	0.3530	0.1492
			III	β	0.9429	0.4644	1.1027	0.1197	1.0946	0.1123
				λ_1	0.0781	0.3235	0.2534	0.1482	0.2518	0.1497
				λ_2	0.1031	0.3984	0.3357	0.1655	0.3337	0.1675
30	20	.7	I	β	1.1549	0.4476	1.0482	0.0769	1.0410	0.0719
				λ_1	0.1643	0.2595	0.2714	0.1401	0.2697	0.1414
				λ_2	0.1995	0.3234	0.3511	0.1624	0.3490	0.1640
			II	β	1.1649	0.4771	1.0517	0.0812	1.0444	0.0762
				λ_1	0.1761	0.2476	0.2764	0.1362	0.2746	0.1375
				λ_2	0.2131	0.3108	0.3571	0.1577	0.3549	0.1593
			III	β	1.1642	0.4693	1.0578	0.0849	1.0504	0.0795
				λ_1	0.1797	0.2480	0.2787	0.1326	0.2769	0.1340
				λ_2	0.2178	0.3079	0.3596	0.1540	0.3574	0.1557
50	20	.7	I	β	1.0618	0.2719	1.0398	0.0801	1.0334	0.0765
				λ_1	0.1509	0.2644	0.2351	0.1759	0.2339	0.1768
				λ_2	0.1825	0.3339	0.3021	0.2108	0.3007	0.2119
			II	β	1.0973	0.2955	1.0501	0.0849	1.0437	0.0806
				λ_1	0.1678	0.2485	0.2461	0.1676	0.2449	0.1685
				λ_2	0.2009	0.3161	0.3144	0.2015	0.3130	0.2026
			III	β	1.0718	0.2840	1.0459	0.0859	1.0396	0.0819
				λ_1	0.1713	0.2414	0.2503	0.1623	0.2491	0.1632
				λ_2	0.2078	0.3064	0.3205	0.1948	0.3189	0.1959
50	30	.7	I	β	1.0979	0.3240	1.0512	0.0896	1.0446	0.0853
				λ_1	0.1698	0.2478	0.2545	0.1587	0.2532	0.1597
				λ_2	0.2065	0.3113	0.3275	0.1881	0.3259	0.1893
			II	β	1.1109	0.3206	1.0618	0.0947	1.0551	0.0899
				λ_1	0.1834	0.2303	0.2663	0.1453	0.2649	0.1464
				λ_2	0.2234	0.2909	0.3422	0.1718	0.3404	0.1732
			III	β	1.0766	0.3064	1.0684	0.1014	1.0616	0.0963
				λ_1	0.1631	0.2444	0.2547	0.1528	0.2534	0.1539
				λ_2	0.2005	0.3078	0.3286	0.1806	0.3270	0.1819

Table 4. AEs , RMSEs of the MLEs and BEs at $(\beta, \lambda_1, \lambda_2) = (1.5, 0.4, 0.7)$ and $T_2 = .8$

					MLEs		BEs			
n	r	T_1	Scheme	par			SEL		LINEX (c=1.5)	
					AE	RMSE	AE	RMSE	AE	RMSE
30	10	.7	I	β	1.4101	0.5446	1.7651	0.2889	1.7470	0.2714
				λ_1	0.1375	0.5770	0.5150	0.1877	0.51021	0.1923
				λ_2	0.1924	0.7170	0.6905	0.2119	0.6843	0.2179
			II	β	1.2204	0.5599	1.7791	0.3028	1.7608	0.2851
				λ_1	0.1018	0.5994	0.4875	0.2141	0.4832	0.2183
				λ_2	0.1403	0.7613	0.6519	0.2494	0.6462	0.2550
			III	β	1.0916	0.5728	1.7896	0.3146	1.7715	0.2969
				λ_1	0.0818	0.6185	0.4631	0.2379	0.4592	0.2418
				λ_2	0.1119	0.7884	0.6212	0.2795	0.6162	0.2846
30	20	.7	I	β	1.6384	0.5716	1.7112	0.2388	1.6941	0.2228
				λ_1	0.2653	0.4559	0.5168	0.1934	0.5124	0.1973
				λ_2	0.3330	0.5913	0.6811	0.2320	0.6754	0.2370
			II	β	1.5905	0.5635	1.7232	0.2494	1.7060	0.2331
				λ_1	0.2477	0.4733	0.5104	0.1975	0.5061	0.2014
				λ_2	0.3130	0.6077	0.6739	0.2357	0.6683	0.2408
			III	β	1.5182	0.5078	1.7411	0.2660	1.7238	0.2494
				λ_1	0.2161	0.4977	0.4949	0.2104	0.4909	0.2142
				λ_2	0.2725	0.6405	0.6539	0.2525	0.6486	0.2575
50	20	.7	I	β	1.4196	0.4018	1.8034	0.3278	1.7865	0.3113
				λ_1	0.1601	0.5419	0.4626	0.2397	0.4591	0.2432
				λ_2	0.2106	0.6918	0.6166	0.2860	0.6118	0.2906
			II	β	1.2285	0.4228	1.8092	0.3357	1.7923	0.3192
				λ_1	0.1199	0.5805	0.4276	0.2735	0.4245	0.2766
				λ_2	0.157	0.7429	0.5712	0.3300	0.5671	0.3340
			III	β	1.1221	0.4805	1.8155	0.3451	1.7988	0.3287
				λ_1	0.0990	0.6011	0.4010	0.2997	0.3983	0.3024
				λ_2	0.1292	0.7710	0.5363	0.3643	0.5327	0.3679
50	30	.7	I	β	1.5472	0.3977	1.7391	0.2710	1.7235	0.2564
				λ_1	0.2526	0.4574	0.4812	0.2277	0.4776	0.2309
				λ_2	0.3151	0.5955	0.6318	0.2787	0.6272	0.2829
			II	β	1.4716	0.3931	1.7734	0.3004	1.7574	0.2849
				λ_1	0.2078	0.4989	0.4513	0.2524	0.4482	0.2554
				λ_2	0.2605	0.6470	0.5946	0.3102	0.5905	0.3141
			III	β	1.3836	0.3698	1.7874	0.3162	1.7713	0.3006
				λ_1	0.1733	0.5287	0.4262	0.2763	0.2328	0.2791
				λ_2	0.2190	0.6835	0.5626	0.3403	0.5589	0.3439

Table 5. AEs , RMSEs of the MLEs and BEs at $(\beta, \lambda_1, \lambda_2) = (1.5, 0.4, 0.7)$ and $T_2 = 1$

n	r	T_1	Scheme	par	MLEs		BEs			
							SEL		LINEX (c=1.5)	
					AE	RMSE	AE	RMSE	AE	RMSE
30	10	.7	I	β	1.3119	0.4448	1.4634	0.2873	1.4454	0.2699
				λ_1	0.2384	0.4761	0.4158	0.1867	0.4110	0.1914
				λ_2	0.3937	0.5156	0.6916	0.2106	0.6853	0.2167
			II	β	1.2210	0.5607	1.5790	0.3027	1.5607	0.2850
				λ_1	0.3019	0.4993	0.4876	0.2139	0.4833	0.2181
				λ_2	0.24046	0.5794	0.6520	0.2493	0.6464	0.2549
			III	β	1.3916	0.4728	1.5896	0.3146	1.5715	0.2963
				λ_1	0.2818	0.3825	0.4631	0.2379	0.4592	0.2418
				λ_2	0.3119	0.5216	0.6212	0.2795	0.6161	0.2846
30	20	.7	I	β	1.6807	0.5991	1.6778	0.2113	1.6615	0.1967
				λ_1	0.3076	0.4143	0.5429	0.1642	0.5381	0.1686
				λ_2	0.4863	0.5358	0.7145	0.1945	0.7084	0.2001
			II	β	1.6128	0.5777	1.7043	0.2340	1.6876	0.2186
				λ_1	0.3663	0.4540	0.4229	0.1808	0.4198	0.1850
				λ_2	0.5384	0.5808	0.6923	0.2137	0.6865	0.2192
			III	β	1.6310	0.5134	1.5292	0.2562	1.5122	0.2401
				λ_1	0.2280	0.4858	0.4050	0.1983	0.4008	0.2023
				λ_2	0.4884	0.5243	0.6671	0.2364	0.6617	0.2417
50	20	.7	I	β	1.4219	0.4030	1.8018	0.3261	1.7849	0.3097
				λ_1	0.1613	0.5406	0.4641	0.2378	0.4605	0.2414
				λ_2	0.2126	0.6898	0.6186	0.2834	0.6139	0.2881
			II	β	1.2285	0.4228	1.8091	0.3355	1.7922	0.3190
				λ_1	0.2120	0.5804	0.4277	0.2734	0.4246	0.2764
				λ_2	0.2578	0.5517	0.5713	0.3298	0.5672	0.3339
			III	β	1.3221	0.4805	1.7155	0.3451	1.6988	0.3287
				λ_1	0.2990	0.3613	0.4010	0.2997	0.3983	0.3024
				λ_2	0.4292	0.5945	0.5363	0.3643	0.5327	0.3679
50	30	.7	I	β	1.5643	0.4053	1.7238	0.2585	1.7086	0.2445
				λ_1	0.2702	0.4381	0.4972	0.2077	0.4934	0.2113
				λ_2	0.3392	0.5695	0.6529	0.2527	0.6480	0.2573
			II	β	1.4763	0.3958	1.7681	0.2960	1.7522	0.2808
				λ_1	0.2132	0.4933	0.4570	0.2458	0.4537	0.2489
				λ_2	0.2679	0.6395	0.6020	0.3014	0.5978	0.3055
			III	β	1.3859	0.3704	1.7855	0.3146	1.7694	0.2991
				λ_1	0.1753	0.5267	0.4286	0.2734	0.4258	0.2762
				λ_2	0.2218	0.6807	0.5658	0.3365	0.5621	0.3402

Table 6. AILs and CPs of the MLEs and Bayes estimates at $(\beta, \lambda_1, \lambda_2) = (1, 0.2, 0.3)$ and $T_2 = 0.8$

n	r	T_1	Scheme	parameter	ACI		HPD			
							SEL		LINEX (c=1.5)	
					AIL	CP	AIL	CP	AIL	CP
30	10	.7	I	β	1.3486	90.5	0.2174	98.6	0.2134	98.6
				λ_1	0.2412	95.8	0.1130	97.1	0.1115	97.1
				λ_2	0.2963	96.5	0.1229	98.5	0.1214	98.4
			II	β	1.2712	89.9	0.2025	98.8	0.1994	98.7
				λ_1	0.1983	97.9	0.0957	97.6	0.0945	97.5
				λ_2	0.2404	97.3	0.1207	99.5	0.1189	99.1
			III	β	1.1723	89.6	0.2199	98.6	0.2169	98.6
				λ_1	0.1587	98.4	0.0846	97.5	0.0835	97.5
				λ_2	0.1893	97.7	0.1015	98.0	0.1003	98.0
30	20	.7	I	β	1.1528	90.7	0.2260	97.1	0.2221	97.8
				λ_1	0.2109	92.2	0.1575	97.6	0.1556	97.6
				λ_2	0.2344	93.0	0.1752	97.2	0.1731	97.2
			II	β	1.1091	90.1	0.2339	98.5	0.2303	98.5
				λ_1	0.2120	90.3	0.1566	96.4	0.1554	96.6
				λ_2	0.2349	88.9	0.1691	98.2	0.1675	98.0
			III	β	1.0759	89.7	0.2266	98.3	0.2239	98.4
				λ_1	0.2141	91.6	0.1580	96.9	0.1568	97.0
				λ_2	0.2375	89.4	0.1723	97.0	0.1701	96.9
50	20	.7	I	β	0.9398	89.5	0.2585	98.8	0.2545	98.9
				λ_1	0.1926	92.3	0.1498	97.9	0.1480	97.9
				λ_2	0.2177	91.7	0.1629	96.9	0.1611	96.9
			II	β	0.8459	89.9	0.2686	98.3	0.2651	96.9
				λ_1	0.1523	95.7	0.1176	97.1	0.1165	97.1
				λ_2	0.1746	93.7	0.1440	97.8	0.1427	97.7
			III	β	0.7955	89.9	0.2771	97.6	0.2729	97.7
				λ_1	0.1256	97.8	0.0899	98.5	0.0891	98.6
				λ_2	0.1433	96.9	0.1142	98.0	0.1129	98.0
50	30	.7	I	β	0.8297	90.1	0.2811	97.9	0.2775	98.2
				λ_1	0.1629	88.4	0.1763	98.2	0.1744	97.5
				λ_2	0.1815	88.6	0.1964	98.8	0.1944	98.1
			II	β	0.8062	91.3	0.2622	98.2	0.2578	98.2
				λ_1	0.1686	83.6	0.1699	96.9	0.1687	96.9
				λ_2	0.1874	82.4	0.1891	96.8	0.1873	96.8
			III	β	0.7795	89.5	0.2815	96.7	0.2772	96.7
				λ_1	0.1575	88.0	0.1558	96.8	0.1542	96.8
				λ_2	0.1751	83.7	0.1831	96.3	0.1815	96.3

Table 7. AILs and CPs of the MLEs and Bayes estimates at $(\beta, \lambda_1, \lambda_2) = (1, 0.2, 0.3)$ and $T_2 = 1$

n	r	T_1	Scheme	parameter	ACI		HPD			
							SEL		LINEX (c=1.5)	
					AIL	CP	AIL	CP	AIL	CP
30	10	.7	I	β	1.3328	89.6	0.2209	98.7	0.2183	98.7
				λ_1	0.2712	95.6	0.1204	98.5	0.1189	98.3
				λ_2	0.3277	96.3	0.1383	98.0	0.1370	98.0
			II	β	1.2113	90.0	0.2308	99.3	0.2283	99.4
				λ_1	0.2199	98.4	0.0868	98.5	0.0863	98.8
				λ_2	0.2609	97.3	0.0874	98.9	0.0878	98.8
			III	β	1.1088	89.3	0.2272	98.5	0.2237	98.5
				λ_1	0.1572	98.0	0.0757	97.0	0.0749	96.9
				λ_2	0.1853	97.8	0.0746	97.5	0.0738	97.3
30	20	.7	I	β	1.0339	89.2	0.2289	97.9	0.2262	97.8
				λ_1	0.2315	85.7	0.1782	96.9	0.1761	96.8
				λ_2	0.2561	82.7	0.2018	97.8	0.1993	98.0
			II	β	1.0265	90.5	0.2462	97.9	0.2437	97.9
				λ_1	0.2458	90.0	0.1791	98.1	0.1771	98.2
				λ_2	0.2720	89.7	0.1907	98.9	0.1891	97.9
			III	β	1.0189	89.5	0.2389	97.7	0.2369	97.8
				λ_1	0.2526	92.2	0.1769	96.9	0.1749	96.9
				λ_2	0.2796	93.0	0.1933	98.8	0.1912	98.8
50	20	.7	I	β	0.7233	90.4	0.2673	98.2	0.2639	98.2
				λ_1	0.1596	78.6	0.2009	96.3	0.1989	96.3
				λ_2	0.1760	76.5	0.2341	96.1	0.2318	96.1
			II	β	0.7332	90.6	0.2711	97.4	0.2678	97.4
				λ_1	0.1735	78.6	0.1962	96.8	0.1948	96.8
				λ_2	0.1905	76.6	0.2289	98.1	0.2270	98.1
			III	β	0.7085	90.1	0.2832	97.0	0.2801	97.1
				λ_1	0.1759	87.5	0.1890	97.7	0.1872	97.6
				λ_2	0.1945	85.4	0.2152	97.6	0.2134	97.7
50	30	.7	I	β	0.7675	90.1	0.2930	97.3	0.2897	97.0
				λ_1	0.1906	84.5	0.1941	96.9	0.1923	96.5
				λ_2	0.2111	81.5	0.2178	98.4	0.2161	98.4
			II	β	0.7681	90.4	0.2799	97.8	0.2765	97.8
				λ_1	0.2041	92.4	0.1849	96.4	0.1832	96.4
				λ_2	0.2266	92.0	0.2079	98.0	0.2061	97.5
			III	β	0.7530	88.9	0.2949	97.0	0.2918	97.0
				λ_1	0.1819	94.6	0.1719	97.1	0.1703	97.1
				λ_2	0.2029	93.7	0.2007	97.7	0.1988	97.6

Table 8. AILs and CPs of the MLEs and Bayes estimates at $(\beta, \lambda_1, \lambda_2) = (1.5, 0.3, 0.5)$ and $T_2 = 0.8$

n	r	T_1	Scheme	parameter	ACI		HPD			
							SEL		LINEX (c=1.5)	
					AIL	CP	AIL	CP	AIL	CP
30	10	.7	I	β	1.5577	92.0	0.4475	97.9	0.4383	97.9
				λ_1	0.2881	99.0	0.1146	98.1	0.1124	98.2
				λ_2	0.3688	97.8	0.1112	99.3	0.1089	99.1
			II	β	1.3866	92.2	0.4501	97.7	0.4402	97.7
				λ_1	0.2088	92.2	0.0972	98.6	0.0959	98.5
				λ_2	0.2611	98.3	0.0989	98.7	0.0977	98.7
			III	β	1.2606	92.7	0.4596	98.3	0.4471	98.0
				λ_1	0.1661	99.8	0.0870	98.7	0.0854	98.6
				λ_2	0.2033	99.9	0.0874	98.2	0.0864	98.9
30	20	.7	I	β	1.3711	90.0	0.4452	97.8	0.4358	97.8
				λ_1	0.3940	93.2	0.2155	98.2	0.2123	98.2
				λ_2	0.4517	91.1	0.2605	98.3	0.2565	98.7
			II	β	1.3364	90.1	0.4400	96.8	0.4310	97.7
				λ_1	0.3693	94.8	0.2143	96.8	0.2117	96.8
				λ_2	0.4242	94.9	0.2500	97.6	0.2461	97.6
			III	β	1.2838	89.8	0.4159	99.1	0.4093	9.0
				λ_1	0.3204	95.6	0.1962	97.7	0.1924	97.7
				λ_2	0.3674	94.5	0.2361	98.0	0.2331	98.1
50	20	.7	I	β	1.1267	91.8	0.4728	98.0	0.4634	98.0
				λ_1	0.2353	97.8	0.1232	98.1	0.1215	98.0
				λ_2	0.2779	95.9	0.1524	97.9	0.1497	97.9
			II	β	1.0029	92.6	0.5039	97.6	0.4930	97.4
				λ_1	0.1739	99.4	0.0939	98.6	0.0932	98.4
				λ_2	0.2047	99.1	0.1080	98.6	0.1063	98.7
			III	β	0.9316	92.1	0.5206	98.2	0.5103	98.2
				λ_1	0.1423	99.7	0.0844	98.3	0.0835	98.4
				λ_2	0.1664	99.7	0.0934	99.5	0.0921	99.5
50	30	.7	I	β	1.0340	92.2	0.4874	98.6	0.4796	98.6
				λ_1	0.2978	93.3	0.2404	97.2	0.2367	97.1
				λ_2	0.3393	91.8	0.2773	97.7	0.2741	97.7
			II	β	0.9997	91.0	0.4790	97.6	0.4709	97.8
				λ_1	0.2464	96.3	0.1908	98.4	0.1893	98.4
				λ_2	0.2817	93.3	0.2393	99.9	0.2361	99.9
			III	β	0.9528	90.3	0.4979	98.1	0.4887	97.3
				λ_1	0.2040	96.4	0.1430	97.8	0.1414	98.0
				λ_2	0.2342	94.5	0.1816	97.9	0.1790	97.9

Table 9. AILs and CPs of the MLEs and Bayes estimates at $(\beta, \lambda_1, \lambda_2) = (1.5, 0.3, 0.5)$ and $T_2 = 1$

n	r	T_1	Scheme	parameter	ACI		HPD			
							SEL		LINEX (c=1.5)	
					AIL	CP	AIL	CP	AIL	CP
30	10	.7	I	β	1.3328	89.6	0.2209	98.7	0.2183	98.7
				λ_1	0.2712	95.6	0.1204	98.5	0.1189	98.3
				λ_2	0.3277	96.3	0.1383	98.0	0.1370	98.0
			II	β	1.2113	90.0	0.2308	99.3	0.2283	99.4
				λ_1	0.2199	98.4	0.0868	98.5	0.0863	98.8
				λ_2	0.2609	97.3	0.0874	98.9	0.0878	98.8
			III	β	1.1088	89.3	0.2272	98.5	0.2237	98.5
				λ_1	0.1572	98.0	0.0757	97.0	0.0749	96.9
				λ_2	0.1853	97.8	0.0746	97.5	0.0738	97.3
30	20	.7	I	β	1.0339	89.2	0.2289	97.9	0.2262	97.8
				λ_1	0.2315	85.7	0.1782	96.9	0.1761	96.8
				λ_2	0.2561	82.7	0.2018	97.8	0.1993	98.0
			II	β	1.0265	90.5	0.2462	97.9	0.2437	97.9
				λ_1	0.2458	90.0	0.1791	98.1	0.1771	98.2
				λ_2	0.2720	89.7	0.1907	98.9	0.1891	97.9
			III	β	1.0189	89.5	0.2389	97.7	0.2369	97.8
				λ_1	0.2526	92.2	0.1769	96.9	0.1749	96.9
				λ_2	0.2796	93.0	0.1933	98.8	0.1912	98.8
50	20	.7	I	β	0.7233	90.4	0.2673	98.2	0.2639	98.2
				λ_1	0.1596	78.6	0.2009	96.3	0.1989	96.3
				λ_2	0.1760	76.5	0.2341	96.1	0.2318	96.1
			II	β	0.7332	90.6	0.2711	97.4	0.2678	97.4
				λ_1	0.1735	78.6	0.1962	96.8	0.1948	96.8
				λ_2	0.1905	76.6	0.2289	98.1	0.2270	98.1
			III	β	0.7085	90.1	0.2832	97.0	0.2801	97.1
				λ_1	0.1759	87.5	0.1890	97.7	0.1872	97.6
				λ_2	0.1945	85.4	0.2152	97.6	0.2134	97.7
50	30	.7	I	β	0.7675	90.1	0.2930	97.3	0.2897	97.0
				λ_1	0.1906	84.5	0.1941	96.9	0.1923	96.5
				λ_2	0.2111	81.5	0.2178	98.4	0.2161	98.4
			II	β	0.7681	90.4	0.2799	97.8	0.2765	97.8
				λ_1	0.2041	92.4	0.1849	96.4	0.1832	96.4
				λ_2	0.2266	92.0	0.2079	98.0	0.2061	97.5
			III	β	0.7530	88.9	0.2949	97.0	0.2918	97.0
				λ_1	0.1819	94.6	0.1719	97.1	0.1703	97.1
				λ_2	0.2029	93.7	0.2007	97.7	0.1988	97.6

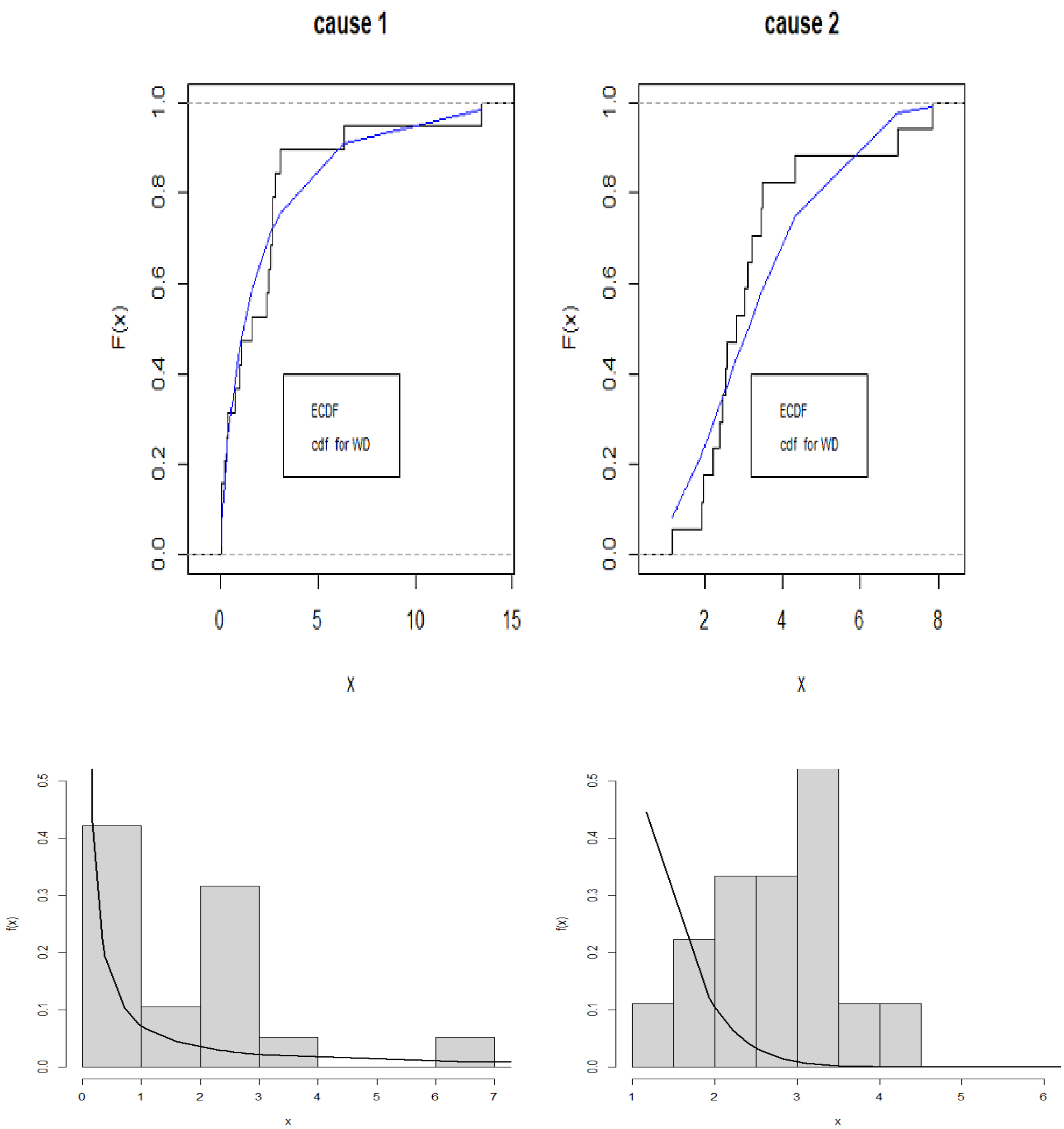

Figure 3. Fitted the cdf and pdf of WD for data

Table 10. Competing risks data.

cause1	0.011	0.035	0.049	0.170	0.329	0.381	0.708	0.958	1.062	1.594
	2.327	2.451	2.565	2.694	2.702	2.761	3.059	6.367	13.403	
cause 2	1.167	1.925	1.990	2.223	2.400	2.471	2.551	2.831	2.568	3.034
	3.112	3.214	3.478	3.504	4.329	6.976	7.846			

on considered real data, we obtain three groups of GPH type-II censored competing risks are generated which are described as follows:

- case 1: $T_1 = 3$, $T_2 = 4$, $j_1 = 16$, $j_2 = 9$, $R^* = 12$ and we have the following data: 0.011, 0.035, 0.049, 0.170, 0.329, 0.381, 0.708, 0.958, 1.062, 1.594,1.167, 1.925, 1.990, 2.223, 2.327 , 2.400, 2.451 ,2.471,2.551, 2.568, 2.565, 2.694, 2.702, 2.761.
- case 2: $T_1 = 2$, $T_2 = 3$, $j_1 = 16$, $j_2 = 8$, $R^* = 12$ and the data is 0.011, 0.035, 0.049, 0.170, 0.329, 0.381, 0.708, 0.958, 1.062, 1.594,1.167, 1.925, 1.990, 2.223, 2.327 , 2.400, 2.451 ,2.471,2.551, 2.568, 2.565, 2.694, 2.702, 2.761,2.831.
- case 3: $T_1 = 2$, $T_2 = 2.5$, $j_1 = 12$, $j_2 = 6$, $R^* = 12$ and we have the following data: 0.011, 0.035, 0.049, 0.170, 0.329, 0.381, 0.708, 0.958, 1.062, 1.594,1.167, 1.925, 1.990, 2.223, 2.327 ,2.400, 2.451 ,2.471.

Since there is no information about the unknown parameters, the non-informative priors (NIPs) with $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$ are adopted in this illustrative example. The MLEs and BEs (with their standard errors) based on both case 1, 2 and 3 are calculated and reported in Table 11. It is observed from this table that the point estimates obtained by maximum likelihood and Bayesian methods of the unknown parameters β, λ_1 and λ_2 are quite close to each other. The results of Table 12 indicate that the HPDs are slightly shorter than the other confidence intervals. ACI: approximate confidence interval.

Table 11. Point estimates of β, λ_1 and λ_2 for real data

case	T_1	T_2	j_1	j_2	par	MLEs Estimate(sd.error)	BEs Estimate(sd.error)
1	3	4	16	9	β	1.148180 (0.199160)	1.198069 (0.197818)
					λ_1	0.361760 (0.106470)	0.331251 (0.090652)
					λ_2	0.103620 (0.038470)	0.080728 (0.024678)
2	2	3	16	8	β	1.260069 (0.213645)	1.201188 (0.221954)
					λ_1	0.585662 (0.166728)	0.513402 (0.162401)
					λ_2	0.112424 (0.043515)	0.097619 (0.031083)
3	2	2.5	12	6	β	0.856925 (0.179229)	0.831438 (0.184133)
					λ_1	0.321732 (0.100115)	0.297584 (0.085035)
					λ_2	0.090818 (0.038997)	0.074931 (0.032886)

7. Conclusions

In this paper, we considered making statistical inference for GPH type-II in presence of competing risks. We obtained both point and interval estimates of the parameters using MLE and Bayesian ap-

Table 12. interval estimates for MLEs and HPD credible interval for real data based on GPH type-II censored

		ACI			HPD		
case	par	lower	upper	AIL	lower	upper	AIL
1	β	0.757829	1.538524	0.780695	0.828564	1.589844	0.761279
	λ_1	0.153074	0.570442	0.417368	0.163726	0.503012	0.339287
	λ_2	0.028221	0.179021	0.150800	0.039056	0.126305	0.087249
2	β	0.841331	1.678806	0.837474	0.887954	1.719689	0.831735
	λ_1	0.258881	0.912443	0.653562	0.225999	0.835509	0.609511
	λ_2	0.027135	0.197712	0.170576	0.022346	0.131416	0.109069
3	β	0.505644	1.208207	0.702563	0.489997	1.192697	0.702701
	λ_1	0.125509	0.517954	0.392445	0.155732	0.462009	0.306277
	λ_2	0.014384	0.167251	0.152867	0.026905	0.140539	0.113635

proaches when latent failure times follow Weibull distribution with the same shape and different scale parameters. The performance of proposed methods are studied using simulations and we observed that Bayes method provides better estimation results compared to the MLE method. Since the likelihood function was obtained in complex form, the posterior density function was obtained in nonlinear form. Therefore, using the MH algorithm, BEs and the associated HPD intervals were developed under the assumption of independent Gamma priors. A real data is also discussed in support of the proposed competing risks model. In most instances, the HPDs have shorter interval lengths than those of associated ACIs. The performance of BEs relative to LINEX loss function have perform better than SEL function. It has been observed that censoring Sc-III provides the smallest RMSE and AIL among three censoring schemes In the literature, progressive hybrid type-II guarantees a specified number of failures, it might take a long time to observe *r* failures. As an extension of the current work is the inference of unknown parameters based on data from GPH type-II in the presence of competing risks when latent failure times follow Weibull distribution with different shape and common scale parameters. The proposed results and methodologies aim to be beneficial to reliability practitioners and extend to other censoring plans.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

Funding: No Funding.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

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