Computational Journal of Mathematical and Statistical Sciences
3(1), 112–124
DOI:10.21608/CJMSS.2023.246513.1026
https://cjmss.journals.ekb.eg/

Cambanis-type Bivariate Uniform Distribution: Properties and Moment Estimation

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Abstract: The families of distributions are crucial in statistical modeling, offering a versatile foundation for a variety of applications. The development of bivariate distributions with specific marginal distributions and correlation coefficients is of considerable interest due to its wide-ranging relevance in real-world situations. The range of correlation between variables is an important characterization of the family. The variety of methods of construction of bivariate/multivariate distributions are developed in the literature. The Cambanis family is an important class of multivariate distributions with a wide range of correlation than the traditional families. In this paper, we consider a Cambanis-type bivariate uniform distribution and develop key statistical properties of the Cambanis-type bivariate uniform distribution. We obtain moment estimators of parameters for Cambanis-type bivariate uniform distribution. To evaluate the performance of the estimators, we develop an algorithm to simulate samples from the Cambanis-type bivariate uniform distribution and implement it in the R software. We perform a simulation study to present the performance of moment estimator.

Keywords: Cambanis family, Cambanis-type bivariate uniform distribution, moment estimation.
Mathematics Subject Classification: 60E05, 62E10, 62F10.
Received: 5 November 2023; Revised: 11 December 2023; Accepted: 20 December 2023; Online: 23 December 2023.
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1. Introduction

Statistical distributions serve as fundamental tools in modeling and understanding complex real-world phenomena. In the context of modeling bivariate data, it can be advantageous to consider families of bivariate distributions that have specified marginal distributions, especially when prior information is available in the form of these marginal distributions. Morgenstern [21] introduced a flexible family of distributions for such scenario. One noteworthy limitation of the Morgenstern family is that it constrains the correlation coefficient to a relatively narrow range $\left(-\frac{1}{3}, \frac{1}{3}\right)$. Consequently, distributions within the Morgenstern family are best suited for modeling data with low correlation between variables. To overcome this limitation and expand the range of correlation between variables, various modifications to the Morgenstern family have been proposed in the existing literature. Numerous researchers, including Sarmanov [23], Cambanis [5], Huang and Kotz [14, 15], Bairamov et al. [4], Bairamov and Kotz [3] and Veena and Thomas [26] have extended the Morgenstern family to enhance the range of correlation between variables and provide more flexibility. In particular, Cambanis [5] introduced a family of distributions that naturally generalizes the Morgenstern family.

Within the diverse landscape of statistical distributions, the Cambanis family stands out for its flexibility and efficacy in capturing the underlying structures of correlated data. Bivariate distribution belongs to Cambanis family excel in modeling scenarios where two variables are jointly distributed, accounting for their interdependencies. This family not only broadens the scope of correlation among variables but also extends into higher dimensions of association parameter space. Notably, Nair et al. [22] have explored the distributional characteristics, nature of dependence, reliability properties, and applications of the Cambanis family, showcasing its superiority in improving dependence coefficients compared to the Morgenstern family. Koshti and Kamalja [19] obtained the estimator for scale parameter associated with study variable for Cambanis-type bivariate uniform distribution (CTBU) based on different ranked set sampling schemes. Alawady et. al. [2] studied the concomitants of generalized order statistics and dual generalized order statistics from Cambanis family of bivariate distributions with nonzero parameter values. For some other developments for families of bivariate distributions, see, Thomas and Scaria [25], Koshti [18], Kamalja and Koshti [17], Abd Elgawad et al. [1], Husseiny et al. [16], El-Sherpieny et al. [8], Chacko and George [6], George and Chacko [11], Chesneau [7], Koshti and Kamalja [20], Fayomi et al. [9, 10] and Haj et al. [12]. In this paper, we focus on a specific member of the Cambanis family, the CTBU distribution, and thoroughly investigate its statistical properties. The structure of this paper is as follows:

Section 2 provides a comprehensive review of the Cambanis-type bivariate distribution and discusses some key distributional properties of CTBU distribution. In Section 3, we propose moment estimators of the parameters of the CTBU distribution. Section 4 is dedicated to present the validation of moment estimators through simulation studies. Finally, we conclude the paper in Section 5.

2. Distributional properties for CTBU distribution

In this section, we briefly discuss about the family of Cambanis-type bivariate distribution introduced by Cambanis [5]. Further, we explore some distributional properties of CTBU distribution.

The distribution function (df) of Cambanis-type bivariate distribution corresponding to a bivariate random variable (X, Y) with parameters $\alpha_1,\ \alpha_2, \alpha_3$ $(CTB(\alpha_1,\ \alpha_2, \alpha_3))$ as given by Cambanis [5] is,

$$H_{X,Y}(x, y) = F_X(x) F_Y(y) [1 + \alpha_1 \{1 - F_X(x)\} + \alpha_2 \{1 - F_Y(y)\} + \alpha_3 \{1 - F_X(x)\} \{1 - F_Y(y)\}],$$

where the parameters $\alpha_1,\ \alpha_2$ and α_3 are real constants satisfying the following conditions.

$$1 + \alpha_1 + \alpha_2 + \alpha_3 \geq 0,\ \ 1 + \alpha_1 - \alpha_2 - \alpha_3 \geq 0,$$

$$1-\alpha_1+\alpha_2-\alpha_3 \geq 0,\ \ 1-\alpha_1-\alpha_2+\alpha_3 \geq 0.$$

The Cambanis family for bivariate distributions reduces to the Morgenstern family when both α_1 and α_2 are zero. Further the two variables are independent when $\alpha_i = 0$ for $i = 1,\ 2,\ 3$. The spearman's correlation for $CTB\left(\alpha_1,\ \alpha_2, \alpha_3\right)$ distribution given by Nair et al. [22] is $\rho = \frac{\alpha_3-\alpha_1\alpha_2}{3}$ while for Morgenstern family (i.e. $CTB\left(0, 0, \alpha_3\right)$ is $\rho = \frac{\alpha_3}{3}$.

We consider a CTBU distribution with parameters $\left(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2\right)$ and denote it as $CTBU\left(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2\right)$. The df of $CTBU\left(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2\right)$ distribution is,

$$H_{X,Y}\left(x, y\right) = \frac{xy}{\theta_1\theta_2}\left[1+\alpha_1\left(1-\frac{x}{\theta_1}\right)+\alpha_2\left(1-\frac{y}{\theta_2}\right)+\alpha_3\left(1-\frac{x}{\theta_1}\right)\left(1-\frac{y}{\theta_2}\right)\right];0<x<\theta_1,\ 0<y<\theta_2,\ \theta_1,\ \theta_2>0,$$

where $\alpha' s$ are real constants satisfying,

$$1+\alpha_1+\alpha_2+\alpha_3 \geq 0,\ \ 1+\alpha_1-\alpha_2-\alpha_3 \geq 0,$$

$$1-\alpha_1+\alpha_2-\alpha_3 \geq 0,\ \ 1-\alpha_1-\alpha_2+\alpha_3 \geq 0.$$

Here $F_X\left(x\right)$ and $F_Y\left(y\right)$ are $df' s$ of $U\left(0, \theta_1\right)$ and $U\left(0, \theta_2\right)$ respectively. Note that the marginal df of X and Y are $H_X\left(x\right)$ and $H_Y\left(y\right)$ and not $F_X\left(x\right)$ and $F_Y\left(y\right)$ respectively. The df of X and Y are given by,

$$H_X\left(x\right) = \frac{x}{\theta_1}\left(1+\alpha_1\left(1-\frac{x}{\theta_1}\right)\right)\ ;\ 0<x<\theta_1,$$

and

$$H_Y\left(Y\right) = \frac{y}{\theta_2}\left(1+\alpha_2\left(1-\frac{y}{\theta_2}\right)\right)\ ;\ 0<y<\theta_2.$$

The pdf of $CTBU\left(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2\right)$ distribution corresponding to df $H_{X,Y}\left(x, y\right)$ is given by,

$$h\left(x, y\right) = \frac{1}{\theta_1\theta_2}\left[1+\alpha_1\left(1-\frac{2x}{\theta_1}\right)+\alpha_2\left(1-\frac{2y}{\theta_2}\right)+\alpha_3\left(1-\frac{2x}{\theta_1}\right)\left(1-\frac{2y}{\theta_2}\right)\right];0<x<\theta_1,\ 0<y<\theta_2,\ \theta_1,\ \theta_2>0. \tag{2.1}$$

Figures 1, 2 and 3 shows the 3-dimensional plots of the joint pdf of CTBU distribution with $\left(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2\right) = \left(0, 0, 0,\ 1, 1\right)$, $\left(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2\right) = \left(0.1, 0.3, 0.2,\ 1, 1\right)$ and $\left(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2\right) = \left(0.1, 0.3, -0.2,\ 1, 1\right)$ respectively.

Observe that the joint density of $CTBU\left(0,\ 0,\ 0,\ 1, 1\right)$ distribution in Figure 1 represents a horizontal plane in a 3-dimensional space. While the pdf of $CTBU\left(0.1, 0.3, 0.2,\ 1, 1\right)$ and $CTBU\left(0.1, 0.3, -0.2,\ 1, 1\right)$ in Figure 2 and 3 is represented by inclined planes respectively.

The marginal pdf of X and Y are as follows:

$$h_x\left(x\right) = \frac{1}{\theta_1}+\frac{\alpha_1}{\theta_1}\left(1-\frac{2x}{\theta_1}\right)\ \ ;0<x<\theta_1,$$

and

$$h_Y\left(y\right) = \frac{1}{\theta_2}+\frac{\alpha_2}{\theta_2}\left(1-\frac{2y}{\theta_2}\right)\ \ ;0<y<\theta_2.$$

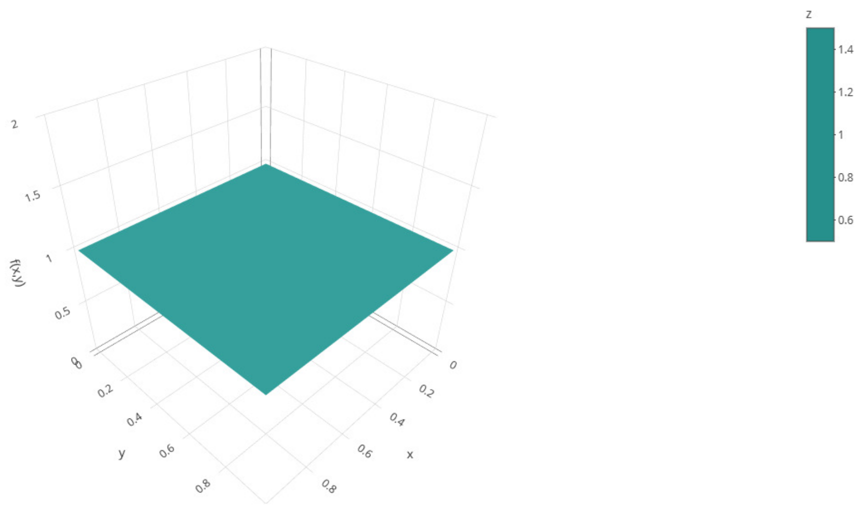

Figure 1. Joint pdf plot for $CTBU$ (0, 0, 0, 1, 1) distribution

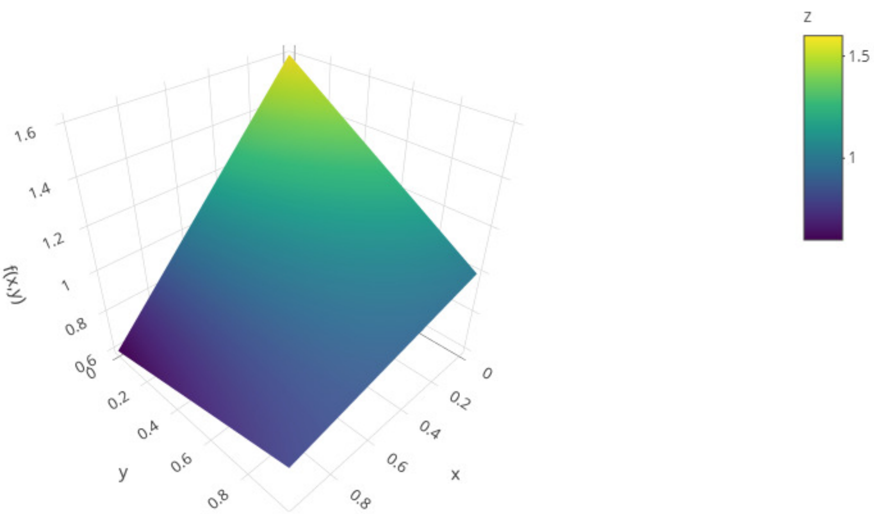

Figure 2. Joint pdf plot for $CTBU$ (0.1, 0.3, 0.2, 1, 1) distribution

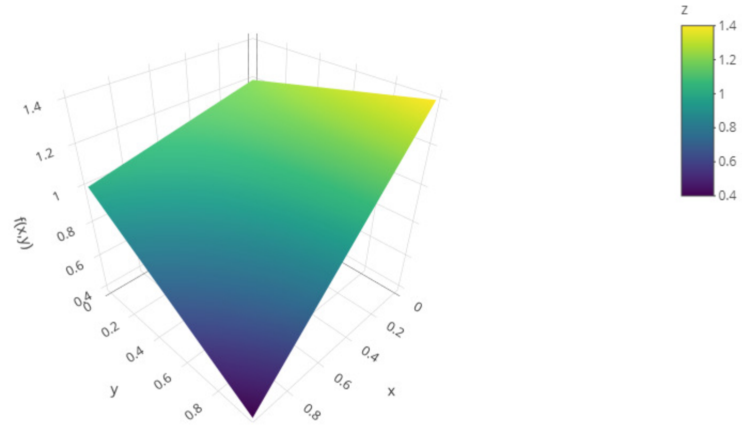

Figure 3. Joint pdf plot for $CTBU\,(0.1, 0.3, -0.2,\ 1, 1)$ distribution

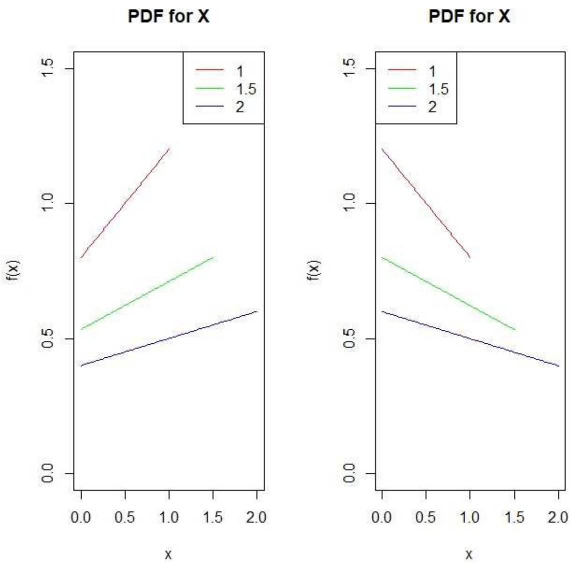

Figure 4. Plots of pdf of X for $\alpha_1 = -0.2$ (left panel) and $\alpha_1 = 0.2$ (right panel)

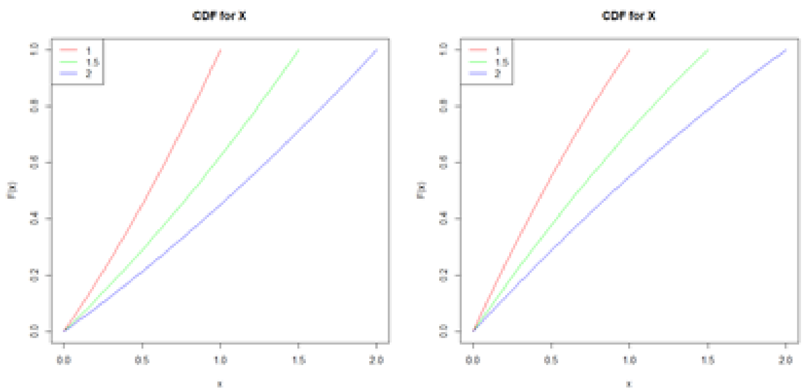

Figure 5. Plots of df of X for $\alpha_1 = -0.2$ (left panel) and $\alpha_1 = 0.2$ (right panel)

Observe that the marginal distribution of X and Y is not uniform. Figures 4 and 5 shows the plots of pdf and corresponding df of marginal distribution of X for CTBU distribution for $\theta = 1,\ 1.5$ and 2 when $\alpha_1 = -0.2$ and 0.2 respectively.

The marginal pdf of X for CTBU distribution for $\theta_1 = 1,\ 1.5$ and 2 when $\alpha_1 = -0.2$ and 0.2 are presented in Figure 4 and forms a trapezoid which can be seen as a generalization of a rectangular shape of general uniform distribution. Observe that the trapezoidal $pdf's$ of X have exactly opposite inclinations when $\alpha_1 > 0$ and $\alpha_1 < 0$. Further, it can be seen that as θ_1 increases, the height of the trapezoidal *pdf* decreases. Figure 5 shows the *df* of X for CTBU distribution for $\alpha_1 = -0.2$ and 0.2.

The s^{th} moment of X is given by,

$$E\left(X^s\right) = \frac{\theta_1^s}{(s+1)}\left(1 - \frac{s\alpha_1}{s+2}\right).$$

Hence,

$$E(X) = \frac{\theta_1}{2}\left(1 - \frac{\alpha_1}{3}\right), E\left(X^2\right) = \frac{\theta_1^2}{3}\left(1 - \frac{\alpha_1}{2}\right).$$

The variance of X is,

$$V(X) = \frac{\theta_1^2}{12}\left(1 - \frac{\alpha_1^2}{3}\right).$$

Further, the bivariate $(r, s)^{th}$ product moment of the distribution in (2.1) is given by,

$$E\left(X^r Y^s\right) = \frac{\theta_1^r \theta_2^s}{(r+1)(s+1)}\left[1 - \frac{r\alpha_1}{r+2} - \frac{s\alpha_2}{s+2} + \frac{rs\alpha_3}{(r+2)(s+2)}\right];\ r, s = 1,\ 2,\ 3, \ldots$$

To obtain covariance between X and Y, we use the Hoeffding's formula (Hoeffding [13]) as,

$$Cov(X, Y) = \iint \left[H_{X,Y}(x, y) - H_X(x) H_Y(y)\right] dxdy.$$

The $Cov(X, Y)$ for $CTBU(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2)$ distribution is given by,

$$Cov(X, Y) = \left(\frac{\alpha_3 - \alpha_1\alpha_2}{36}\right)\theta_1\theta_2.$$

Hence the correlation coefficient (ρ) between X and Y is given by,

$$\rho = \frac{(\alpha_3 - \alpha_1\alpha_2)}{\sqrt{\left(3 - \alpha_1^2\right)(3 - \alpha_2^2)}}.$$

The variation in correlation with respect to $\alpha_2,\ \alpha_3$ for $CTBU(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2)$ distribution is shown in Figures 6, 7 and 8 for $\alpha_1 = -0.8,\ 0.8$ and $\alpha_1 = 0$ respectively. These figures, help to observe the marginal and joint effects of α_2 and α_3 on ρ for given α_1.

Further, observe that ρ is monotonic with respect to α_2 and α_3 when $\alpha_1 < 0$ while for $\alpha_1 > 0$, the correlation coefficient has exactly opposite behaviour with respect to α_2 and α_3. Further these graphs will also help to choose the feasible α-parameters and gives an idea about the correlation between the variables.

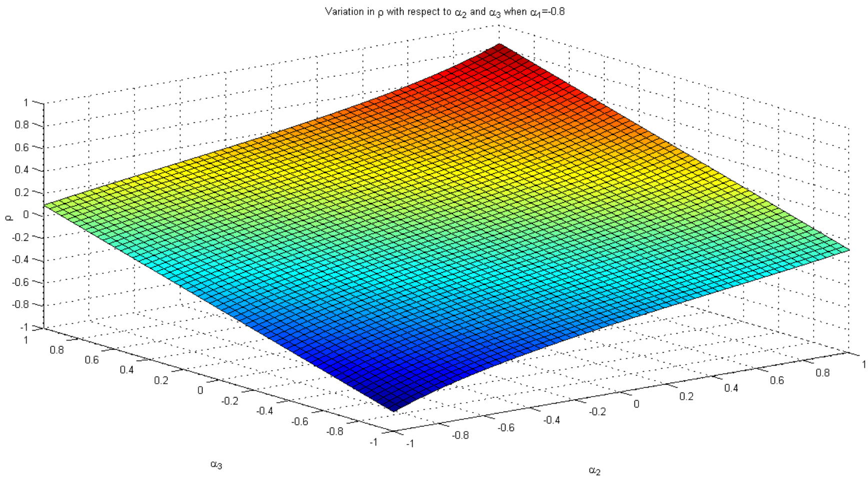

Figure 6. Variation in ρ with respect to α_2, α_3 for $CTBU\,(\alpha_1 = -0.8, \alpha_2, \alpha_3,\ \theta_1, \theta_2)$

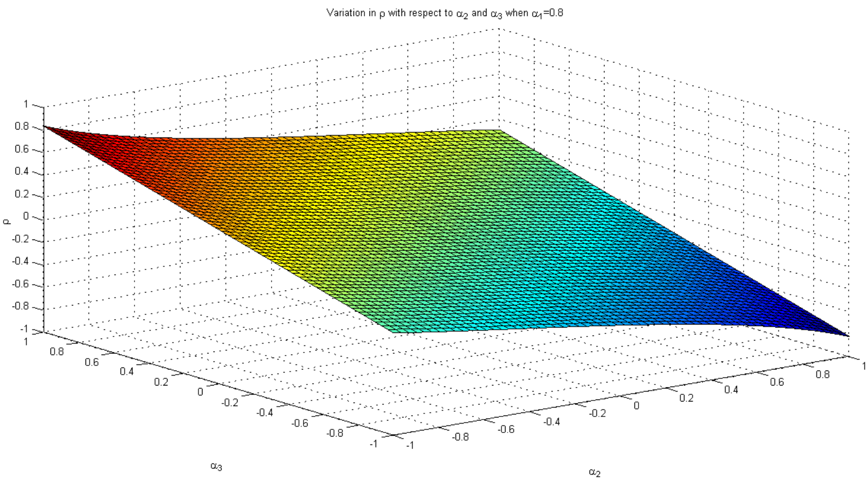

Figure 7. Variation in ρ with respect to α_2, α_3 for $CTBU\,(\alpha_1 = 0.8, \alpha_2, \alpha_3,\ \theta_1, \theta_2)$

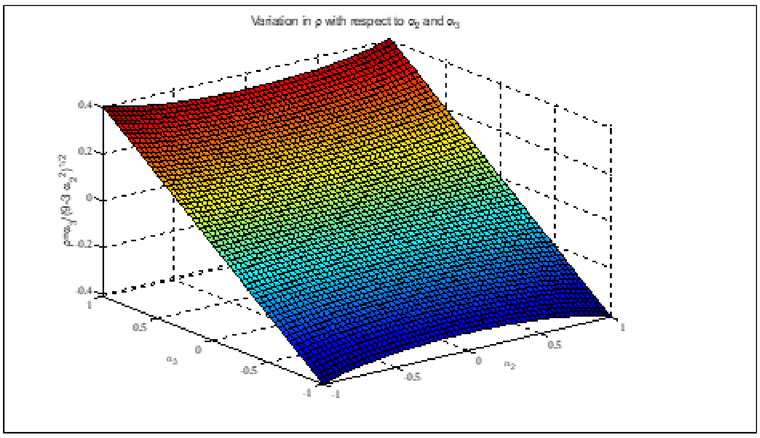

Figure 8. Variation in ρ with respect to α_2, α_3 for $CTBU\,(0, \alpha_2, \alpha_3,\ \theta_1, \theta_2)$

3. Moment estimation of parameters of CTBU distribution

Moment estimation is the simplest approach to estimate the parameters of a given probability distribution. In this section, we focus on the problem of estimation of parameters for the CTBU distribution using the method of moments.

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the simple random sample from $CTBU\,(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2)$ distribution. Let m_1' and m_2' be the first and second sample moments respectively based on X-observations and $\hat{\rho}$ be the sample correlation. The steps to obtain moment estimators of $\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2$ are described systematically in the following.

I. To obtain moment estimator of α_1, we consider ratio of sample moments which lead to following moment equation.

$$\frac{4\left(1-\frac{\alpha_1}{2}\right)}{3\left(1-\frac{\alpha_1}{3}\right)^2} = \frac{m_2'}{{m_1'}^2},$$

This leads to the following quadratic moment equation in α_1.

$$\frac{m_2'}{3}\alpha_1^2 + 2\left({m_1'}^2 - m_2'\right)\alpha_1 + \left(3m_2' - 4{m_1'}^2\right) = 0. \tag{3.1}$$

Among two solutions of (3.1), let $\hat{\alpha}_1$ be the feasible one. Using similar moment equation based on moments of Y-observations the moment estimator $\hat{\alpha}_2$ of α_2 can be obtained.

II. To obtain moment estimator of α_3, we use the moment equation based on correlation between (X, Y) and replace α_1 and α_2 by their respective moment estimates $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

$$\hat{\rho} = \frac{\alpha_3 - \hat{\alpha}_1\hat{\alpha}_2}{\sqrt{\left(3-\hat{\alpha}_1^2\right)\left(3-\hat{\alpha}_2^2\right)}},$$

The moment estimator $\hat{\alpha}_3$ of α_3 along with feasibility condition is given by,

$$\hat{\alpha}_3 = \begin{cases} max\,(-1-\hat{\alpha}_1-\hat{\alpha}_2,\ -1+\hat{\alpha}_1+\hat{\alpha}_2) & ;\text{if }\ \hat{\rho} < a \\ \hat{\rho}\sqrt{\left(3-\hat{\alpha}_1^2\right)\left(3-\hat{\alpha}_2^2\right)} + \hat{\alpha}_1\hat{\alpha}_2 & ;\text{if }\ a \le \hat{\rho} \le b \\ min\,(1+\hat{\alpha}_1-\hat{\alpha}_2, 1-\ \hat{\alpha}_1+\ \hat{\alpha}_2) & ;\text{if }\ \hat{\rho} > b \end{cases},$$

where $a = \frac{\max(-1-\hat{\alpha}_1-\hat{\alpha}_2,\ -1+\hat{\alpha}_1+\hat{\alpha}_2)-\hat{\alpha}_1\hat{\alpha}_2}{\sqrt{(3-\hat{\alpha}_1^2)(3-\hat{\alpha}_2^2)}}$ and

$$b = \frac{\min\,(1+\hat{\alpha}_1-\hat{\alpha}_2, 1-\ \hat{\alpha}_1+\ \hat{\alpha}_2)-\hat{\alpha}_1\hat{\alpha}_2}{\sqrt{\left(3-\hat{\alpha}_1^2\right)\left(3-\hat{\alpha}_2^2\right)}}.$$

Note that if $\alpha_1 = 0$ and $\alpha_2 = 0$ (i.e. $(X, Y) \sim MTBU(\alpha_3, \theta_1, \theta_2)$), the above moment estimator of α_3 reduces to estimator given by Tahmasebi and Jafari [24] and is given as follows.

$$\hat{\alpha}_3 = \begin{cases} -1 & ;\text{if }\ \hat{\rho} < -\frac{1}{3} \\ 3\hat{\rho} & ;\text{if }\ -\frac{1}{3} \le \hat{\rho} \le \frac{1}{3} \\ 1 & ;\text{if }\ \hat{\rho} > \frac{1}{3} \end{cases}.$$

Table 1. Sample and population quantities associated with the simulated data

Population quantities		Sample Quantities	$n = 500$	$n = 1000$
$CTBU\,(0.1,\ 0.1, 0.8, 1, 1)$				
$E\,(X)$	0.4833	$\overline{X}$	0.4769	0.4791
$E\,(Y)$	0.4833	$\overline{Y}$	0.4208	0.4174
$Var\,(X)$	0.0831	S_X^2	0.0797	0.0800
$Var\,(Y)$	0.0831	S_Y^2	0.0775	0.0761
ρ	0.2642	r	0.1371	0.1668
$CTBU\,(-0.2,\ -0.1, -0.6, 1, 1.5)$				
$E\,(X)$	0.5333	$\overline{X}$	0.5376	0.5588
$E\,(Y)$	0.7750	$\overline{Y}$	0.8638	0.8869
$Var\,(X)$	0.0822	S_X^2	0.0872	0.0657
$Var\,(Y)$	0.1869	S_Y^2	0.1807	0.1694
ρ	-0.2084	r	-0.1306	-0.2669

III. The moment estimator $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ_1 and θ_2 respectively are obtained as,

$$\hat{\theta}_1 = \frac{2\overline{X}}{\left(1 - \frac{\hat{\alpha}_1}{3}\right)}, \hat{\theta}_2 = \frac{2\overline{Y}}{\left(1 - \frac{\hat{\alpha}_2}{3}\right)}.$$

The moment estimators of $\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2$ can also be used as the initial guess values of the respective parameters while obtaining maximum likelihood estimators.

4. Simulation from $CTBU\,(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution

Koshti and Kamalja [19] developed a Matlab function for simulating random pairs of observations from the $CTBU(\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2)$ distribution when α_1 is set to 0. Following their approach, we have developed a R function, *rctbu*$(\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2,\ n)$, to generate random samples of size n from the $CTBU(\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2)$ distribution. The R code of *rctbu*$(\alpha_!,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2,\ n)$ function to simulate data from $CTBU(\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2)$ distribution is given in Appendix. To assess the validity of the data generated by the *rctbu*() function, we have conducted simulations using sample sizes of 500 and 1000. Subsequently, we have compared various sample statistics with their corresponding population values. The results are summarized in Table 1.

The Figure 9 shows sample mean of X when $(X, Y) \sim CTBU\,(0.1,\ 0.1, 0.8, 1, 1)$ distribution across different sample sizes. It is evident from the plot that the sample mean of X-variable approaches to the population mean (highlighted by red line) as the sample size increases.

We use the *rctbu*$(\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2,\ n)$ function to generate random samples from the $CTBU(\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2)$ distribution, with the aim of assessing the feasibility of moment estimates for $\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1$, and θ_2. The moment estimates for all parameters of the $CTBU(\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2)$ distribution, as proposed in Section 3, have been computed for different sample sizes, using specific values of $\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1$, and θ_2. The outcomes are summarized in Table 2.

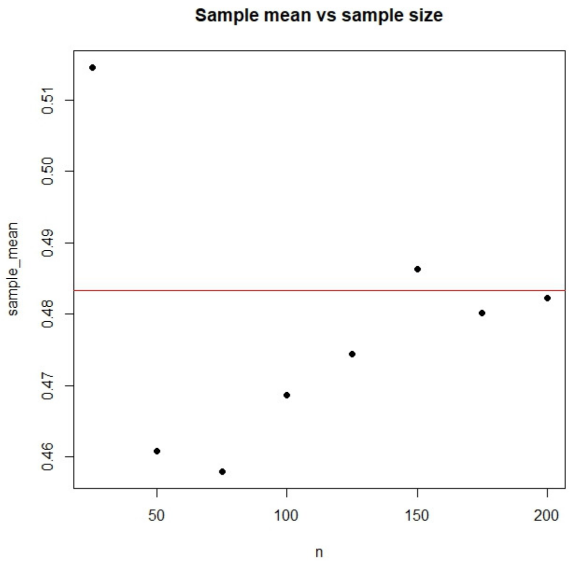

Figure 9. Sample mean of X across different sample sizes for $CTBU\,(0.1,\ 0.1, 0.8, 1, 1)$

Table 2. Moment estimates based on simulated data for $CTBU\,(\alpha_1, \alpha_2, \alpha_3,\ \theta_1, \theta_2)$ distribution

	Population parameter values					Moment estimates of parameters				
n	α_1	α_2	α_3	θ_1	θ_2	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\theta}_1$	$\hat{\theta}_2$
50	0.1	0.1	0.8	1	1	0.1286	0.1978	0.3952	0.8526	0.9216
75	0.1	0.1	0.8	1	1	0.1311	0.1308	0.4878	1.0800	0.9076

From Table 2, it is evident that the moment estimators are imprecise but tend to provide estimates that approximate the true values. The estimators show a reasonable level of accuracy, though there may be slight deviations.

5. Conclusions

In this paper, we study the Cambanis-type bivariate uniform (CTBU) distribution, as an important member of the Cambanis family. This paper has explored its statistical properties, moment estimation, and the validation of moment estimators through simulation studies. The obtained moment estimators can be used as estimates or can be serve as an initial solution for obtaining maximum likelihood estimate for the parameters. We have designed a simulation algorithm to generate samples from the CTBU distribution and implemented in R programming language.

Acknowledgments We would like to thank the Referees for their constructive comments and suggestions, which improved the manuscript.

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Appendix: The R function $'rctbu(\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2,\ n)'$ to generate random sample of size n from $CTBU(\alpha_1,\ \alpha_2,\ \alpha_3,\ \theta_1,\ \theta_2)$ distribution.

```
rctbug <- function(a1, a2, a3, t1, t2, n) {
# DF of X
F1 <- function(x) (1 + a1 * (1 - x / t1)) * x / t1
# DF of Y
F2 <- function(y) (1 + a2 * (1 - y / t2)) * y / t2
# DF of (X,Y)
F12 <- function(x, y) (x * y) * (1 + a1 * (1 - x / t1) +
a2 * (1 - y/t2) + a3 * (1 - x / t1) * (1 - y / t2)) / (t1 * t2)
# DF of conditional distribution of Y|X=x
F2_1 <- function(y, x) F12(x, y) / F1(x)
# Simulation using F(y|x)
sF1 <- runif(n)
sF2_1 <- runif(n)
u <- numeric(n)
v <- numeric(n)
for (i in 1:n) {
uu <- uniroot(function(x) F1(x) - sF1[i], interval = c(0, t1))$root
u[i] <- uu
vv <- uniroot(function(y) F2_1(y, uu) - sF2_1[i], interval = c(0,
t2))$root
v[i] <- vv }
z <- cbind(u, v)
return(z)
}
# Example
set.seed(123)
z <- rctbug(-0.2, -0.1, -0.6, 1, 1.5, 100)
# Computes the sample and population means, variances and correlation
z_p.mean=c(t1*(1-a1/3)/2, t2*(1-a2/3)/2)
z_s.var=apply(z,2,var)
z_p.var=c(t1*t1*(1-a1*a1/3)/12, t2*t2*(1-a2*a2/3)/12)
s.corr=cor(z)
z_s.corr=s.corr[1,2]
z_p.corr=(a3-a1*a2)/sqrt((3-a1*a1)*(3-a2*a2))
round(c(z_s.mean,z_p.mean,z_s.var,z_p.var,z_s.corr,z_p.corr),4)
```

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