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Bayesian Estimation and Prediction for Exponentiated Inverted Topp-Leone Distribution

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Abstract: This paper focuses on Bayesian estimation for the shape parameters, reliability and hazard rate functions of the exponentiated inverted Topp-Leone distribution. Bayesian estimation is performed under two different loss functions. The Bayes estimators are derived under the squared error loss function as a symmetric loss function and the linear-exponential loss function as an asymmetric loss function, based on Type II censored sample. Credible intervals for the parameters, reliability and hazard rate functions are derived. The Bayesian two-sample prediction (point and interval) for a future observation from independent future sample from the same distribution, exponentiated inverted Topp-Leone distribution, is obtained based on Type II censored sample. Numerical illustration is proposed, and some interesting comparisons are presented to investigate the theoretical results through some measurements of accuracy. Moreover, the results are applied to real data set to ensure the theoretical results and to prove the applicability of the exponentiated inverted Topp-Leone distribution in real life.

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1. Introduction

There are many methods for generalizing a distribution by adding one parameter or more to a baseline distribution for providing great flexibility in modeling and analyzing data in several applied areas such as reliability, engineering, economics, environmental sciences, finance and medical applications.

A simple way of generalization is adding a parameter to a distribution by exponentiation. Several authors focused on the exponentiated distributions and their applications. Gupta and Kundu [11] studied some properties of the distribution and observed that many properties of the new family are similar to those of the Weibull or gamma family. Ali et al. [5] considered some exponentiated distributions. Also, the exponentiated Lomax distribution was proposed by Abdul-Moniem and Abdel-Hameed [2]. Ebraheim [8] introduced exponentiated transmuted Weibull distribution. Pourdarvish et al. [14] presented a generalization of the TL distribution referred to as the exponentiated TL distribution. Rao and Mbwambo [15] derived the exponentiated inverse Rayleigh distribution and presented an application to coating weights of iron sheets data. Tharu et al. [19] studied exponentiated Marshall-Olkin Exponential distribution and using it to predict mortality rate of COVID-19 second wave in Nepal.

Inverted Topp- Leone (ITL) distribution was introduced by Hassan et al. [12] and some of its properties were presented. The *cumulative distribution function* (cdf) and probability density function (pdf) of the ITL distribution are as follows:

$$g_{ITL}(t;\alpha) = 2\alpha t(1+t)^{-2\alpha-1}(1+2t)^{\alpha-1}, \qquad t>0,\ \alpha>0, \tag{1.1}$$

and

$$G_{ITL}(t;\alpha) = 1-\frac{(1+2t)^{\alpha}}{(1+t)^{2\alpha}}, \qquad t>0,\ \alpha>0. \tag{1.2}$$

Exponentiated Inverted Topp-Leone (EITL) distribution was introduced by Ashour et al. [6] using the method suggested by Nadarajah and Kotz [13] as given below:

$$F(t) = 1-[1-G(t)]^{\theta},\ \theta>0.$$

The cdf and pdf of the EITL distribution are

$$F(t;\alpha,\theta) = 1-\left[2(t+1)^{-1}-(t+1)^{-2}\right]^{\alpha\theta}, \qquad t>0, \qquad \alpha,\ \theta>0, \tag{1.3}$$

and

$$f(t;\alpha,\theta) = 2\theta\alpha\left[2(t+1)^{-1}-(t+1)^{-2}\right]^{\alpha\theta-1}\left[(t+1)^{-2}-\left(t+1\right)^{-3}\right],\ \ t>0,\ \alpha,\theta>0. \tag{1.4}$$

The reliability function (rf) and *hazard rate function* (hrf) are given, respectively, by

$$R(t) = \left[2(t+1)^{-1}-(t+1)^{-2}\right]^{\alpha\theta}, \qquad t>0, \qquad \alpha,\ \theta>0, \tag{1.5}$$

and

$$h(t) = \frac{2\theta\alpha t}{(t+1)(2t+1)]}, \qquad t>0, \qquad \alpha,\ \theta>0. \tag{1.6}$$

The Bayesian approach of estimation has received great attention from most statisticians' researchers. For example, Goyal et al. [10] introduced Bayesian estimation for exponentiated inverted Weibull distribution under different loss functions. Aijaz et al. [3] studied Bayesian analysis of inverse Topp-Leone distribution under different loss functions. Eraikhuemen et al. [9] introduced Bayesian and maximum likelihood estimation of the shape parameter of exponential inverse exponential distribution. Abd AL-Fattah et al. [1] introduced Bayesian estimation and prediction for exponentiated generalized inverted Kumaraswamy distribution based on dual generalized order statistics.

Many researchers have considered prediction for future observation from different lifetime distributions. See for example, AL-Hussaini and Hussein [4] studied Bayesian prediction of the future observation from exponentiated populations based on one and two-sample prediction with fixed and random sample size. Also, Singh et al. [18], Sen et al. [16] and Ateya et al. [7] studied the one-sample and two-sample prediction schemes for a future observation from Burr X distribution based on a unified hybrid censoring scheme using the likelihood and the Bayesian prediction methods.

This paper concerns Bayesian estimation for the shape parameters, reliability and hazard rate functions of the EITL distribution. It is performed under two different loss functions; The Bayes estimators are derived under the squared error (SE) loss function as a symmetric loss function and the *linear-exponential* (LINEX) as an asymmetric loss function, based on Type II censored sample. Credible intervals for the parameters, rf and hrf are derived. The Bayesian two-sample prediction (point and interval) for a future observation from independent future sample from the EITL distribution is obtained based on Type II censored sample. Numerical illustration including simulation study and an application is proposed.

The paper is organized as follows: In Section 2, the Bayes estimators of the unknown parameters, rf and hrf using non-informative and informative priors, based on Type II censored samples under SE and LINEX loss functions of the EITL distribution are derived. Bayesian two-sample prediction (point and interval) for a future observation of the EITL distribution are obtained based on Type II censored in Section 3. Also, a numerical illustration is presented in Section 4 to demonstrate the theoretical results developed in this paper.

2. Bayesian Estimation

This section is devoted to estimate the parameters α, θ, rf and hrf for the EITL distribution based on Type II censored sample using the Bayesian approach. The Bayes estimators are derived using two different loss functions: the SE loss function as a symmetric loss function and LINEX loss function as an asymmetric loss function. The non-informative distribution is used as an improper prior distribution and gamma distribution as a conjugate prior distribution.

2.1. Bayesian estimation based on non-informative prior

Suppose that $T_{(1)} \leq T_{(2)} \leq, \ldots, \leq T_{(r)}$ is Type II censored sample of size r obtained from a life-test on n items with lifetimes from the EITL distribution, then the *likelihood function* (LF) can be written as follows:

$$L\left(\alpha, \theta | \underline{t}\right) \propto (\theta\alpha)^r \prod_{i=1}^{r} \left[\left(2(t_i+1)^{-1} - (t_i+1)^{-2}\right)\right]^{\alpha\theta-1} \left[2(t_r+1)^{-1} - (t_r+1)^{-2}\right]^{\alpha\theta(n-r)}. \quad (2.1)$$

Assuming that, before sampling, little or no information about the unknown parameters(α, θ) are available, then the non-informative priors are assumed as prior distributions for the parameters. If the parameters α and θ are unknown and independent distributions, hence the joint prior of the parameters is

$$\pi(\alpha,\theta) \propto \frac{1}{\alpha\theta}, \qquad \alpha,\theta > 0. \tag{2.2}$$

Combining the LF in (2.1) and the joint prior distribution given by (2.2), then the joint posterior distribution for α and θ can be obtained as follows:

$$\begin{aligned}\pi\left(\alpha,\theta \mid \underline{t}\right) &\propto L\left(\alpha,\theta \mid \underline{t}\right)\pi(\alpha,\theta)\\ &= k(\theta\alpha)^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)},\end{aligned} \tag{2.3}$$

where k is the normalizing constant.

$$k^{-1} = \int_0^\infty\int_0^\infty (\theta\alpha)^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\alpha d\theta. \tag{2.4}$$

The marginal posterior distributions of the parameters (α,θ) can be obtained by integrating the joint posterior for another parameter as given below

$$\begin{aligned}\pi\left(\alpha \mid \underline{t}\right) &= \int_0^\infty \pi\left(\alpha,\theta \mid \underline{t}\right)d\theta,\\ &= \int_0^\infty k(\theta\alpha)^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\theta,\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\pi\left(\theta \mid \underline{t}\right) &= \int_0^\infty \pi\left(\alpha,\theta \mid \underline{t}\right)d\alpha,\\ &= \int_0^\infty k(\theta\alpha)^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\alpha,\end{aligned} \tag{2.6}$$

where $\pi\left(\alpha,\theta \mid \underline{t}\right)$ is given by (2.3) and k^{-1} is defined in (2.4).

2.1.1. Point estimation based on squared error loss function

Using the SE loss function as a symmetric loss function, hence the Bayes estimators of the parameters, based on Type II censoring, is the mean of the posterior density. Using the joint non-informative prior, the Bayes estimators can be obtained by using the marginal posterior distribution (2.5) and (2.6) as follows:

$$\alpha^*_{SE} = E\left(\alpha \mid \underline{t}\right) = \int_0^\infty\int_0^\infty k\alpha^r\theta^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\theta d\alpha, \tag{2.7}$$

and

$$\theta^*_{SE} = E\left(\theta \mid \underline{t}\right) = \int_0^\infty\int_0^\infty k\theta^r\alpha^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\theta d\alpha. \tag{2.8}$$

The Bayes estimators of the rf and hrf under the SE loss function can be obtained as given below:

$$R^*_{SE}(t) = E\left(R(t) \mid \underline{t}\right) = \int_0^\infty \int_0^\infty k\left[2(t+1)^{-1} - (t+1)^{-2}\right]^{\alpha\theta} (\theta\alpha)^{r-1} \prod_{i=1}^{r} \left[\left(2(t_i+1)^{-1} - (t_i+1)^{-2}\right)\right]^{\alpha\theta-1} \left[2(t_r+1)^{-1} - (t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\theta d\alpha, \tag{2.9}$$

and

$$h^*_{SE}(t) = E\left(h(t) \mid \underline{t}\right) = \int_0^\infty \int_0^\infty k \frac{2(\theta\alpha)^r t}{(t+1)(2t+1)} \prod_{i=1}^{r} \left[\left(2(t_i+1)^{-1} - (t_i+1)^{-2}\right)\right]^{\alpha\theta-1} \left[2(t_r+1)^{-1} - (t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\theta d\alpha. \tag{2.10}$$

To obtain the Bayes estimates of the parameters, rf and hrf based on the SE loss function, (2.7)-(2.10) can be solved numerically using the Metropolis–Hastings algorithm of *Markov chain Monte Carlo* (MCMC) method of simulation.

2.1.2. Point estimation based on linear exponential loss function

The Bayes estimators of the parameters under the LINEX loss function can be derived by using the marginal posterior distribution in (2.5) and (2.6) as follows:

$$\alpha^*_{LINEX} = \left(-\frac{1}{v}\right) \ln\left[\mathrm{E}\left(e^{-v\alpha} \mid \underline{t}\right)\right] = \left(-\frac{1}{v}\right) \ln\left[\int_0^\infty \int_0^\infty e^{-v\alpha} k(\theta\alpha)^{r-1} \prod_{i=1}^{r} \left[\left(2(t_i+1)^{-1} - (t_i+1)^{-2}\right)\right]^{\alpha\theta-1} \left[2(t_r+1)^{-1} - (t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\theta d\alpha\right], \tag{2.11}$$

and

$$\theta^*_{LINEX} = \left(-\frac{1}{v}\right) \ln\left[\mathrm{E}\left(e^{-v\theta} \mid \underline{t}\right)\right] = \left(-\frac{1}{v}\right) \ln[\int_0^\infty \int_0^\infty e^{-v\theta} k(\theta\alpha)^{r-1} \prod_{i=1}^{r} \left[\left(2(t_i+1)^{-1} - (t_i+1)^{-2}\right)\right]^{\alpha\theta-1} \left[2(t_r+1)^{-1} - (t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\alpha d\theta]. \tag{2.12}$$

The Bayes estimators of the rf and hrf under the LINEX loss function are

$$R^*_{LINEX}(t) = \left(-\frac{1}{v}\right) \ln\left[\mathrm{E}\left(e^{-vR(t)} \mid \underline{t}\right)\right] = \left(-\frac{1}{v}\right) \ln[\int_0^\infty \int_0^\infty \exp(-v\left[2(t+1)^{-1} - (t+1)^{-2}\right]^{\alpha\theta}) k(\theta\alpha)^{r-1} \prod_{i=1}^{r} \left[\left(2(t_i+1)^{-1} - (t_i+1)^{-2}\right)\right]^{\alpha\theta-1} \left[2(t_r+1)^{-1} - (t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\theta d\alpha], \tag{2.13}$$

and

$$h^{*}_{(LINEX)}(t) = \left(-\frac{1}{v}\right)\ln\left[\mathrm{E}\left(e^{-vh(t)} \mid \underline{t}\right)\right]$$
$$= \left(-\frac{1}{v}\right)\ln[\int_0^\infty \int_0^\infty \exp((-v)\,\frac{2\theta\alpha t}{(t+1)(2t+1)})k(\theta\alpha)^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}$$
$$\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)}d\theta d\alpha]. \tag{2.14}$$

The Bayes estimates of the parameters, rf and hrf using the LINEX loss function under Type II censoring, can be obtained by solving (2.11) - (2.14) numerically applying the Metropolis–Hastings algorithm of MCMC method of simulation through R programming language.

2.1.3. Credible intervals

The credible intervals for the parameters of θ and α are given for the EITL distribution based on Type II censored data. In general, a two sided $L(\underline{t})$ and $U(\underline{t})$ is a $100(1-\tau)\%$ credible interval for $\underline{\vartheta}$ if

$\mathrm{p}[L(\underline{t}) < \underline{\vartheta} < U(\underline{t})|\underline{t}]=\int_{L(\underline{t})}^{U(\underline{t})}\pi\left(\underline{\vartheta} \mid \underline{t}\right)d\underline{\vartheta} = 1-\tau,$

where $\underline{\vartheta} = (\alpha, \theta)$, $L(\underline{t})$ is the *lower limit* (LL) and $U(\underline{t})$ is the *upper limit* (UL).

Since, the marginal posterior distribution is given by (2.5) and (2.6), then

a $100\,(1-\tau)\%$ credible interval for α is

$$P[\alpha > L\left(\underline{t}\right)|\underline{t}] = \int_{L(\underline{t})}^{\infty}\int_0^\infty k(\theta\alpha)^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)}d\theta d\alpha$$
$$= 1-\frac{\tau}{2}, \tag{2.15}$$

and

$$P[\alpha > U\left(\underline{t}\right)|\underline{t}] = \int_{U(\underline{t})}^{\infty}\int_0^\infty k(\theta\alpha)^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)}d\theta d\alpha$$
$$= \frac{\tau}{2}. \tag{2.16}$$

Also, a $100(1-\tau)\%$ credible interval for θ is

$$P[\theta > L\left(\underline{t}\right)|\underline{t}] = \int_{L(\underline{t})}^{\infty}\int_0^\infty k(\theta\alpha)^{r-1}\prod_{i=1}^{r}\left[\left(2(t_i+1)^{-1}-(t_i+1)^{-2}\right)\right]^{\alpha\theta-1}\left[2(t_r+1)^{-1}-(t_r+1)^{-2}\right]^{\alpha\theta(n-r)}d\alpha d\theta$$
$$= 1-\frac{\tau}{2}, \tag{2.17}$$

and

$$P[\theta > U(\underline{t}) \left| \underline{t} \right.] = \int_{U(\underline{t})}^{\infty} \int_{0}^{\infty} k(\theta\alpha)^{r-1} \prod_{i=1}^{r} \left[\left(2(t_i+1)^{-1} - (t_i+1)^{-2}\right)\right]^{\alpha\theta-1} \left[2(t_r+1)^{-1} - (t_r+1)^{-2}\right]^{\alpha\theta(n-r)} d\alpha d\theta$$
$$= \frac{\tau}{2}. \tag{2.18}$$

The credible intervals can be evaluated numerically by means of R programming language.

2.2. Bayesian estimation based on informative prior

The LF given by (2.1) can be rewritten as follows:

$$L\left(\alpha, \theta \mid \underline{t}\right) \propto (\theta\alpha)^{r} \exp[(\alpha\theta - 1)\sum_{i=1}^{r} \ln\left[\left(2(t_i+1)^{-1} - (t_i+1)^{-2}\right)\right]] \exp\left[\alpha\theta\,(n-r)\ln\left[2(t_r+1)^{-1} - (t_r+1)^{-2}\right]\right]. \tag{2.19}$$

The conjugate informative priors are assumed as prior distributions for the parameters. Further that, the two parameters α and θ are independent and each has gamma distribution. Then the joint prior distribution of the parameters has a joint pdf given by:

$$\pi^{**}(\alpha, \theta) \propto \theta^{\beta_1-1} \alpha^{b_1-1} \exp\left(-\beta_2\theta - b_2\alpha\right), \qquad \alpha, \theta > 0, \beta_i, b_j > 0, \tag{2.20}$$

where $i, \quad j = 1, 2$.

Combining the LF in (2.19) and the joint prior distribution given by (2.20), then the joint posterior distribution for α and θ can be obtained as:

$$\pi^{**}\left(\alpha, \theta \mid \underline{t}\right) = k^{*} \theta^{\beta_1+r-1} \alpha^{b_1+r-1} \exp[-(\beta_2\theta + b_2\alpha) + (\alpha\theta - 1)\sum_{i=1}^{r} \ln[G_i]] \exp[\alpha\theta\,(n-r)\ln[G_r]], \tag{2.21}$$

and

$$k^{*-1} = \int_{0}^{\infty} \int_{0}^{\infty} \theta^{\beta_1+r-1} \alpha^{b_1+r-1} \exp\left[-(\beta_2\theta + b_2\alpha) + (\alpha\theta - 1)\sum_{i=1}^{r} \ln[G_i]\right] \exp[\alpha\theta\,(n-r)\ln[G_r]]\, d\theta d\alpha, \tag{2.22}$$

where k^{*} is the normalizing constant for informative posterior distribution,

$$G_i = \left(2(t_i+1)^{-1} - (t_i+1)^{-2}\right) \text{and } G_r = 2(t_r+1)^{-1} - (t_r+1)^{-2}. \tag{2.23}$$

The marginal posterior distribution for α and θ can be obtained as given below:

$$\pi^{**}\left(\alpha \mid \underline{t}\right) = \int_{0}^{\infty} \pi^{**}\left(\alpha, \theta \mid \underline{t}\right) d\theta, \tag{2.24}$$

and

$$\pi^{**}\left(\theta \mid \underline{t}\right) = \int_{0}^{\infty} \pi^{**}\left(\alpha, \theta \mid \underline{t}\right) d\alpha, \tag{2.25}$$

where $\pi^{**}\left(\alpha, \theta \mid \underline{t}\right)$ is given by (2.21).

2.2.1. Point estimation based on squared error loss function

The Bayes estimators of the parameters based on informative prior, based on the SE loss function can be obtained by using (2.24) and (2.25) as follows:

$$\alpha^{**}_{SE} = \int_0^{\infty}\int_0^{\infty} k^{*}\theta^{\beta_1+r-1}\alpha^{b_1+r}\exp\left[-(\beta_2\theta+b_2\alpha)+(\alpha\theta-1)\sum_{i=1}^{r}\ln[G_i]\right]\exp[\alpha\theta(n-r)\ln[G_r]]\ d\theta d\alpha, \tag{2.26}$$

and

$$\theta^{**}_{SE} = \int_0^{\infty}\int_0^{\infty} k^{*}\theta^{\beta_1+r}\alpha^{b_1+r-1}\exp[-(\beta_2\theta+b_2\alpha)+(\alpha\theta-1)\sum_{i=1}^{r}\ln[G_i]]\exp[\alpha\theta(n-r)\ln[G_r]]\, d\alpha d\theta, \tag{2.27}$$

where $k^{*}$is given by (2.22), G_i and G_r are defined in (2.23).

The Bayes estimators of the rf and hrf under the SE loss function are

$$R^{**}_{SE}(t) = \int_0^{\infty}\int_0^{\infty} k^{*}\left[2(t+1)^{-1}-(t+1)^{-2}\right]^{\alpha\theta}\theta^{\beta_1+r-1}\alpha^{b_1+r-1}\exp[-(\beta_2\theta+b_2\alpha)+(\alpha\theta-1)\sum_{i=1}^{r}\ln[G_i]] \\ \times\exp[\alpha\theta(n-r)\ln[G_r]]\ d\theta d\alpha, \tag{2.28}$$

and

$$h^{**}_{SE}(t) = \int_0^{\infty}\int_0^{\infty} k^{*}\frac{2\theta^{\beta_1+r}\alpha^{b_1+r}t}{(t+1)(2t+1)}\exp[-(\beta_2\theta+b_2\alpha)+(\alpha\theta-1)\sum_{i=1}^{r}\ln[G_i]] \\ \times\exp[\alpha\theta(n-r)\ln[G_r]]\ d\theta d\alpha. \tag{2.29}$$

To obtain the Bayes estimates of the parameters, rf and hrf based on the SE loss function, (2.26)-(2.29) should be solved numerically utilizing the Metropolis–Hastings algorithm of MCMC method of simulation in R programming language.

2.2.2. Point estimation based on linear exponential loss function

The Bayes estimators of the parameters under the LINEX loss function can be derived by using the posterior distribution in (2.21) as given below:

$$\alpha^{**}_{LINEX} = \left(-\frac{1}{v}\right)\ln\int_0^{\infty}\int_0^{\infty} k^{*}\theta^{\beta_1+r-1}\alpha^{b_1+r-1}\exp[-(\beta_2\theta+\alpha(b_2+v))] \\ \times\exp\left[(\alpha\theta-1)\sum_{i=1}^{r}\ln[G_i]+[(\alpha\theta(n-r)\ln[G_r])]\right]\ d\theta d\alpha, \tag{2.30}$$

and

$$\theta^{**}_{LINEX} = \left(-\frac{1}{v}\right)\ln\left[\int_0^{\infty}\int_0^{\infty} k^{*}\theta^{\beta+r-1}\alpha^{b+r-1}\exp[-(\theta(\beta_2+v)+b_2\alpha)] \\ \times\exp\left[(\alpha\theta-1)\sum_{i=1}^{r}\ln[G_i]+[(\alpha\theta(n-r)\ln[G_r])]\right] d\alpha d\theta\right], \tag{2.31}$$

where k^* is defined in (2.22), $G_r$and G_i are given by (2.23).

The Bayes estimators of the rf and hrf under LINEX loss function are

$$R^{**}_{LINEX}(t) = \left(-\frac{1}{v}\right)\ln\left[\int_0^\infty \int_0^\infty k\theta^{\beta_1+r-1}\alpha^{b_1+r-1}\exp\left[-(\beta_2\theta + b_2\alpha)\right]\right.$$
$$\left.\times\exp\left[v\left[2(t+1)^{-1} - (t+1)^{-2}\right]^{\alpha\theta}\right]\exp\left[(\alpha\theta - 1)\sum_{i=1}^{r}\ln\left[G_i\right] + \left[(\alpha\theta(n-r)\ln\left[G_r\right])\right]\right] d\theta d\alpha\right], \tag{2.32}$$

and

$$h^{**}_{LINEX}(t) = -\frac{1}{v}\ln\left[\int_0^\infty \int_0^\infty k^*\theta^{\beta_1+r-1}\alpha^{b_1+r-1}\exp(-v\frac{2\theta\alpha t}{(t+1)(2t+1)})\right.$$
$$\left.\times\exp\left[-(\beta_2\theta + b_2\alpha) + (\alpha\theta(n-r)\ln\left[G_r\right])\right]\exp[(\alpha\theta - 1)\sum_{i=1}^{r}\ln\left[G_i\right]]d\theta d\alpha\right]. \tag{2.33}$$

To obtain the Bayes estimates of the parameters, rf and hrf based on the LINEX loss function, (2.30)-(2.33) can be solved numerically applying the Metropolis–Hastings algorithm of MCMC method of simulation by R programming language.

2.2.3 Credible intervals

The credible intervals for the parameters of θ and α are given from EITL distribution based on Type II censored data.

Since, the marginal posterior distributions are given by (2.24) and (2.25), then a 100 $(1-\tau)\%$ credible interval for α is

$$P[\alpha^{**} > U(\underline{t})\,|\underline{t}] = \int_{L(\underline{t})}^{\infty}\int_0^\infty k^*\theta^{\beta_1+r-1}\alpha^{b_1+r-1}\exp[-(\beta_2\theta + b_2\alpha)$$
$$\times\, exp[(\alpha\theta - 1)\sum_{i=1}^{r}\ln\left[G_i\right]] + (\alpha\theta(n-r)\ln\left[G_r\right])]d\theta d\alpha = 1 - \frac{\tau}{2}, \tag{2.34}$$

and

$$P[\alpha^{**} > U(\underline{t})\,|\underline{t}] = \int_{U(\underline{t})}^{\infty}\int_0^\infty k^*\theta^{\beta_1+r-1}\alpha^{b_1+r-1}\exp[-(\beta_2\theta + b_2\alpha)$$
$$\times\exp[(\alpha\theta - 1)\sum_{i=1}^{r}\ln\left[G_i\right]] + (\alpha\theta(n-r)\ln\left[G_r\right])]d\theta d\alpha = \frac{\tau}{2}. \tag{2.35}$$

Also, a $100(1-\tau)\%$ credible interval for θ is

$$P[\theta^{**} > L(\underline{t})\,|\underline{t}] = \int_{L(\underline{t})}^{\infty}\int_0^\infty k^*\theta^{\beta_1+r-1}\alpha^{b_1+r-1}\exp[-(\beta_2\theta + b_2\alpha)$$

$$\times\exp[(\alpha\theta - 1)\sum_{i=1}^{r}\ln\left[G_i\right]] + (\alpha\theta(n-r)\ln\left[G_r\right])]d\alpha d\theta = 1 - \frac{\tau}{2}, \tag{2.36}$$

and

$$P[\theta^{**} > U(\underset{\sim}{t}) \left| \underset{\sim}{t} \right] = \int_{U(\underset{\sim}{t})}^{\infty} \int_{0}^{\infty} k^{*} \theta^{\beta_1 + r - 1} \alpha^{b_1 + r - 1} \exp[-(\beta_2 \theta + b_2 \alpha)$$
$$\times \exp[(\alpha\theta - 1) \sum_{i=1}^{r} \ln [G_i]] + (\alpha\theta (n - r) \ln [G_r])] d\alpha d\theta \tag{2.37}$$
$$= \frac{\tau}{2}.$$

The credible intervals of the parameters can be derived from solving the previous equations numerically via R programming language.

3. Bayesian Prediction

In this section, Bayesian two-sample prediction (point and interval) for a future observation from EITL distribution based on Type II censored scheme is discussed based on non-informative and informative priors using the SE and the LINEX loss functions.

Suppose that $T_1 \le T_2 \le \cdots \le T_r$ are the first r ordered lifetimes in a random sample of n components Type II censoring whose failures are independent and identically distributed as a random variable T having EITL (α, θ) distribution. Informative sample is given by (1.4), $Y_{(1)}, Y_{(2)}, \ldots, Y_{(m)}$ is a future independent random sample of size m from the same distribution and we want to predict a statistic based on the informative sample.

For the future sample of size m, let $Y_{(s)}$ denotes the s^{th} order statistic, $s = 1, 2, \ldots, m$. The pdf of $Y_{(s)}$ can be obtained as given below:

$$f_{s:m}(y_{(s)}) = \frac{m!}{(s-1)!\,(m-s)!} [F(y_{(s)})]^{s-1} [1 - F(y_{(s)})]^{m-s} f(y_{(s)}), 1 < s < m, \quad y_{(s)} > 0. \tag{3.1}$$

Substituting (1.3), and (1.4) in (3.1), then the pdf of the s^{th} order is as follows:

$$h(y_{(s)} | \alpha, \theta) = D(S) 2\alpha\theta [1 - \left[2(y_{(s)} + 1)^{-1} - (y_{(s)} + 1)^{-2}\right]^{\alpha\theta}]^{s-1} \left[2(y_{(s)} + 1)^{-1} - (y_{(s)} + 1)^{-2}\right]^{\alpha\theta(m-s+1)-1}$$
$$\times \left[(y_{(s)} + 1)^{-2} - (y_{(s)} + 1)^{-3}\right], \qquad y_{(s)} > 0 \textit{ and } \alpha, \theta > 0, \tag{3.2}$$

where

$$D(s) = \frac{m!}{(s-1)!\,(m-s)!} = \frac{1}{B(s, m-s+1)}, \quad s = 1, 2, \ldots, m. \tag{3.3}$$

Assuming that the parameters α and θ are unknown and independent, then the *Bayesian predictive density* (BPD) of Y_s is given by

$$h_1\left(y_{(s)} \left| \underset{\sim}{t} \right.\right) = \int_{\alpha} \int_{\theta} \pi(\alpha, \theta \mid t_r)\, h(y_{(s)} | \alpha, \theta)\, d\theta d\alpha, \tag{3.4}$$

where $h(y_{(s)} | \alpha, \theta)$ is in (3.2) and $\pi(\alpha, \theta \mid t_r)$ is given by (2.3) or (2.21).

3.1. Bayesian prediction based on non-informative prior

The BPD of Y_s given $\underline{t}$ based on non-informative prior can be obtained by substituting (3.2) and (2.3) in (3.4) as given below:

$$h_1\left(y_{(s)}\left|\underline{t}\right.\right) = \int_0^\infty \int_0^\infty \pi(\alpha,\theta\mid t_r)D(S)2\alpha\theta[1-\left[2(y_{(s)}+1)^{-1}-(y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1} \times\left[2(y_{(s)}+1)^{-1}-(y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1}\times\left[(y_{(s)}+1)^{-2}-(y_{(s)}+1)^{-3}\right]d\theta d\alpha. \tag{3.5}$$

where $\pi(\alpha,\theta\mid t_r)$ is given by (2.3) and $D(S)$ is in (3.3).

3.1.1. Point prediction based on non-informative prior

Bayesian two-sample prediction is considered under two types of loss functions the SE loss function as a symmetric loss function and the LINEX loss function as an asymmetric loss function, based on Type II censoring.

I. Squared error loss function

The *Bayes predictor* (BP) for the future observation Y_s, under the SE loss function can be derived as follows:

$$\hat{y}_{(s)(SE)} = E\left(y_{(s)}\left|\underline{t}\right.\right) = \int_{y_{(s)}} y_{(s)}h_1\left(y_{(s)}\left|\underline{t}\right.\right)dy_{(s)}$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha,\theta\mid t_r)D(S)2\alpha\theta y_{(s)}[1-\left[2(y_{(s)}+1)^{-1}-(y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1} \times\left[2(y_{(s)}+1)^{-1}-(y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1} \times\left[(y_{(s)}+1)^{-2}-(y_{(s)}+1)^{-3}\right]d\theta d\alpha dy_{(s)}. \tag{3.6}$$

where $\pi(\alpha,\theta\mid t_r)$ is given by (2.3) and $D(S)$ is defined in (3.3).

Special cases can be obtained from (3.6); such as the minimum observable, $Y_{(1)}$, the maximum observable, $Y_{(m)}$ and the median observable, $Y_{(\frac{m}{2})}$ by putting $s = 1, m\ or\ \frac{m}{2}$.

II. Linear exponential loss function

The BP for the future observation Y_s, under the LINEX loss function can be derived as follows:

$$\hat{y}_{(s)(LINEX)} = \left(\frac{-1}{v}\right)lnE\left(\exp\left(-vy_{(s)}\left|\underline{t}\right.\right)\right)$$

$$= \left(\frac{-1}{v}\right)ln\int_{y_{(s)}}\exp(-vy_{(s)})h_1\left(y_{(s)}\left|\underline{t}\right.\right)dy_{(s)},$$

$$= \left(\frac{-1}{v}\right)ln\int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha,\theta\mid t_r)D(S)2\alpha\theta\exp\left(-vy_{(s)}\right)\left[(y_{(s)}+1)^{-2}-(y_{(s)}+1)^{-3}\right] \times[1-\left[2(y_{(s)}+1)^{-1}-(y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1} \times\left[2(y_{(s)}+1)^{-1}-(y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1}d\theta d\alpha dy_{(s)}, \tag{3.7}$$

where $\pi(\alpha,\theta\mid t_r)$ is given by (2.3), $D(S)$ is defined in (3.3).

Special cases for the BPs under the LINEX loss function can be obtained in a similar manner to those obtained under the SE loss function by using (3.7).

3.1.2. Bayesian predictive bounds based on non-informative prior

A $100(1-\tau)\%$ *Bayesian predictive bounds* (BPB) for a future observation $Y_{(s)}$, such that $\mathrm{P}\left(L_{(s)}\left(\underline{t}\right) < Y_{(s)} < U_{(s)}\left(\underline{t}\right) \middle| \underline{t}\right) = 1-\tau$, can be obtained by using (3.5) as given below:

$$\begin{aligned} P\left(Y_s > L_{(s)}\left(\underline{t}\right) \middle| \underline{t}\right) = & \int_{L_{(s)}(\underline{t})}^{\infty} \int_0^{\infty} \int_0^{\infty} \pi\left(\alpha, \theta \mid t_r\right) D(S) 2\alpha\theta [1 - \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1} \\ & \times \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1} \left[(y_{(s)}+1)^{-2} - (y_{(s)}+1)^{-3}\right] d\theta d\alpha dy_{(s)} \\ & = 1 - \frac{\tau}{2}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} P\left(Y_s > U_{(s)}\left(\underline{t}\right) \middle| \underline{t}\right) = & \int_{U_{(s)}(\underline{t})}^{\infty} \int_0^{\infty} \int_0^{\infty} \pi\left(\alpha, \theta \mid t_r\right) D(S) 2\alpha\theta [1 - \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1} \\ & \times \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1} \left[(y_{(s)}+1)^{-2} - (y_{(s)}+1)^{-3}\right] d\theta d\alpha dy_{(s)} \\ & = \frac{\tau}{2}, \end{aligned} \tag{3.9}$$

where $s = 1, 2, 3, \ldots, m$.

3.2. Bayesian prediction based on informative prior

The BP of Y_s given $\underline{t}$ based on informative prior can be obtained by substituting (3.2) and (2.21) in (3.4) as given below:

$$\begin{aligned} h_2\left(y_{(s)} \middle| \underline{t}\right) = & \int_0^{\infty} \int_0^{\infty} \pi^{**}\left(\alpha, \theta \middle| \underline{t}\right) D(S) 2\alpha\theta \times [1 - \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1} \\ & \times \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1} \times \left[(y_{(s)}+1)^{-2} - (y_{(s)}+1)^{-3}\right] d\theta d\alpha, \end{aligned} \tag{3.10}$$

where $\pi^{**}\left(\alpha, \theta \middle| \underline{t}\right)$ is given by (2.21), $D(S)$ is defined in (3.3)

3.2.1. Point prediction based on informative prior

Bayesian prediction based on informative prior is considered under two types of loss functions the SE loss function as a symmetric loss function and the LINEX loss function as an asymmetric loss function, based on Type II censoring.

I. Squared error loss function

The BP for the future observation Y_s, under the SE loss function can be derived using (3.10) as follows:

$$\begin{aligned} \hat{y}_{(s)(SE)} = & \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \pi^{**}\left(\alpha, \theta \middle| \underline{t}\right) D(S) 2\alpha\theta y_{(s)} [1 - \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1} \\ & \times \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1} \times \left[(y_{(s)}+1)^{-2} - (y_{(s)}+1)^{-3}\right] d\theta d\alpha dy_{(s)}. \end{aligned} \tag{3.11}$$

II. Linear exponential loss function

The BP for the future observation Y_s, under the LINEX loss function can be obtained using (3.10) as given below:

$$\hat{y}_{(s)(LINEX)} = \frac{-1}{v} ln \int_0^\infty \int_0^\infty \int_0^\infty \pi^{**}\left(\alpha, \theta \mid \underline{t}\right) D(S) 2\alpha\theta \exp\left(-v y_{(s)}\right)$$
$$\times [1 - \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1}$$
$$\times \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1}$$
$$\times \left[(y_{(s)}+1)^{-2} - (y_{(s)}+1)^{-3}\right] d\theta d\alpha dy_{(s)}, \tag{3.12}$$

where $\pi^{**}\left(\alpha, \theta \mid \underline{t}\right)$ is given by (2.21) and $D(S)$ is defined in (3.3).

Special cases:

Special cases can be obtained for the minimum observable, $Y_{(1)}$,the maximum observable, $Y_{(m)}$ and the median (in the case of m is odd), $Y_{(\frac{m+1}{2})}$, under the SE and LINEX loss function by using (3.11) and (3.12).

3.2.2. Bayesian predictive bounds based on informative prior

A $100(1-\tau)\%$ BPB for the future observation $Y_{(s)}$, such that $\mathrm{P}\left(L_{(s)}\left(\underline{t}\right) < Y_{(s)} < U_{(s)}\left(\underline{t}\right) \middle| \underline{t}\right) = 1-\tau$, is given as:

$$P\left(Y_s > L_{(s)}\left(\underline{t}\right) \middle| \underline{t}\right) = \int_{L_{(s)}(\underline{t})}^\infty \int_0^\infty \int_0^\infty \pi^{**}\left(\alpha, \theta \mid \underline{t}\right) D(S) 2\alpha\theta [1 - \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1}$$
$$\times \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1} \left[(y_{(s)}+1)^{-2} - (y_{(s)}+1)^{-3}\right] d\theta d\alpha dy_{(s)}$$
$$= 1 - \frac{\tau}{2}, \tag{3.13}$$

and

$$P\left(Y_s > U_{(s)}\left(\underline{t}\right) \middle| \underline{t}\right) = \int_{U_{(s)}(\underline{t})}^\infty \int_0^\infty \int_0^\infty \pi^{**}\left(\alpha, \theta \mid \underline{t}\right) D(S) 2\alpha\theta [1 - \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta}]^{s-1}$$
$$\times \left[2(y_{(s)}+1)^{-1} - (y_{(s)}+1)^{-2}\right]^{\alpha\theta(m-s+1)-1} \left[(y_{(s)}+1)^{-2} - (y_{(s)}+1)^{-3}\right] d\theta d\alpha dy_{(s)}$$
$$= \frac{\tau}{2}, \tag{3.14}$$

where $s = 1, 2, 3, \ldots, m$, $\pi^{**}\left(\alpha, \theta \mid \underline{t}\right)$ is given by (2.21) and $D\left(\frac{n+1}{2}\right) = \frac{m!}{\left(\frac{n-1}{2}\right)!\left(m-\frac{n+1}{2}\right)!}$.

4. Numerical illustration

This section aims to investigate the precision of the theoretical results of the Bayes estimates and Bayes predictors using the SE and the LINEX loss functions under Type II censored sample. Numerical results are presented based on simulated and real data.

4.1. Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the Bayes estimates and predictors for different sample sizes.

Bayesian estimation

1. Different random samples of size (n = 30, 50 and 100) from EITL distribution are generated using *number of replications* (N) = 10000 for each sample size with censoring level 10% and 30%. The computations are performed applying the Metropolis–Hastings algorithm of MCMC method of simulation via R programming language.
2. Improper prior is used as non-informative prior and the gamma distribution as an informative distribution to obtain the Bayes estimates.
3. Tables 1 and 2 presented the Bayes averages, *posterior risks* (PR) which are estimates that minimize the expected loss function. Also, 95% credible intervals for the parameters are given, using non-informative prior and informative gamma prior under SE and LINEX loss functions based on Type II censoring.
4. Tables 3 and 4 displayed the Bayes averages, PRs and 95% credible intervals for reliability and hazard rate functions, using non-informative prior under the SE and LINEX loss functions based on Type II censoring at different values of t_0.
5. Tables 5 and 6 proposed the Bayes averages, PRs and 95% credible intervals for reliability and hazard rate functions, using informative prior under SE and LINEX loss functions based on Type II censoring at different values for t_0.

Bayesian prediction

1. The samples are generated from EITL distribution at N = 1000 for sample of size n = 100 with level of censoring 30%, and future sample size m=30. The computations are performed using R programming language.
2. The Bayes two-sample predictors are obtained using non-informative and informative gamma priors and the results are given in Table 7 for the simulated data and Table 9 for the real data set .

Table 1. Bayes averages, posterior risks and 95% credible intervals of the parameters, using non-informative prior based on Type II censoring (N=10000)

loss function	n	r		Average	PR	Upper	Lower	Length
SE	30	21	α	0.2986	1.40×10^{-6}	0.3001	0.2966	0.0035
			θ	0.4989	2.80×10^{-2}	0.501	0.4964	0.0046
		27	α	0.2999	9.51×10^{-7}	0.3016	0.2983	0.0034
			θ	0.5009	2.80×10^{-7}	0.5018	0.4999	0.0019
	50	35	α	0.2989	9.18×10^{-7}	0.3005	0.2973	0.0032
			θ	0.4993	3.04×10^{-7}	0.5004	0.4984	0.0019
		45	α	0.2991	2.70×10^{-7}	0.3001	0.2982	0.0019
			θ	0.4991	6.11×10^{-7}	0.5006	0.498	0.0026
	100	70	α	0.2993	4.60×10^{-7}	0.3005	0.2983	0.0023
			θ	0.5003	9.40×10^{-8}	0.5009	0.4998	0.0011
		90	α	0.3	7.24×10^{-7}	0.3012	0.2984	0.0027
			θ	0.4998	3.81×10^{-7}	0.5008	0.4986	0.0022
LINEX	30	21	α	0.3007	1.48×10^{-6}	0.3031	0.2989	0.0042
			θ	0.5002	4.97×10^{-6}	0.5011	0.4988	0.0024
		27	α	0.3001	5.49×10^{-7}	0.3016	0.2991	0.0025
			θ	0.4993	2.95×10^{-7}	0.5001	0.4984	0.0018
	50	35	α	0.2987	3.56×10^{-7}	0.3002	0.298	0.0022
			θ	0.4996	2.08×10^{-7}	0.5004	0.4998	0.0016
		45	α	0.3011	3.99×10^{-7}	0.302	0.2998	0.0022
			θ	0.5005	4.72×10^{-7}	0.5017	0.4995	0.0022
	100	70	α	0.2994	3.63×10^{-7}	0.3004	0.2984	0.002
			θ	0.5	1.53×10^{-7}	0.5008	0.4992	0.0016
		90	α	0.3005	1.59×10^{-7}	0.3013	0.2999	0.0014
			θ	0.4989	3.46×10^{-7}	0.4999	0.4979	0.002

Table 2. Bayes averages, posterior risks and 95% credible intervals of the parameters, using informative prior based on Type II censoring
$(N = 10000, \beta_1 = 0.6, b_1 = 0.3; \beta_2 = 0.9, b_2 = 0.9)$

Loss function	n	r		Average	PR	Upper	Lower	Length
SE	30	21	α	0.2987	1.40×10^{-6}	0.3002	0.2955	0.0035
			θ	0.4988	2.80×10^{-6}	0.501	0.4964	0.0046
		27	α	0.301	1.43×10^{-6}	0.3025	0.2991	0.0034
			θ	0.4096	9.71×10^{-7}	0.5002	0.4969	0.0033
	50	35	α	0.3005	9.33×10^{-7}	0.3021	0.2988	0.0033
			θ	0.4986	8.79×10^{-7}	0.5003	0.4973	0.003
		45	α	0.3004	4.24×10^{-3}	0.3014	0.2992	0.0022
			θ	0.4998	2.80E+07	0.5009	0.4989	0.002
	100	70	α	0.3015	3.95×10^{-7}	0.3025	0.3003	0.0022
			θ	0.5007	3.50×10^{-7}	0.5017	0.4998	0.0019
		90	α	0.3003	4.51×10^{-7}	0.3014	0.2994	0.0021
			θ	0.5	2.28×10^{-7}	0.5008	0.4991	0.0018
LINEX	*30*	*21*	α	0.299	2.34×10^{-6}	0.3011	0.2969	0.0042
			θ	0.5021	9.70×10^{-7}	0.5039	0.5005	0.0034
		27	α	0.2998	4.41×10^{-6}	0.3009	0.2972	0.0037
			θ	0.5002	9.85×10^{-7}	0.5019	0.4989	0.0031
	50	35	α	0.2986	1.57×10^{-6}	0.3004	0.2967	0.0037
			θ	0.4986	6.39×10^{-7}	0.5002	0.4975	0.0028
		45	α	0.3009	7.32×10^{-7}	0.3023	0.2993	0.003
			θ	0.5012	4.28×10^{-7}	0.5022	0.4999	0.0024
	100	70	α	0.3003	3.46×10^{-7}	0.3014	0.2993	0.0021
			θ	0.5007	3.12×10^{-7}	0.5016	0.4996	0.002
		90	α	0.2987	3.07×10^{-7}	0.2999	0.2978	0.0021
			θ	0.5	2.82×10^{-7}	0.5009	0.4991	0.0018

Table 3. Bayes averages, posterior risks and 95% credible intervals of the reliability function, using non- informative prior based on Type II censoring (N=10000)

Loss function	n	r	t_0	Average	PR	Upper	Lower	Length
SE	30	21	0.7	0.9739	3.50×10^{-7}	0.9748	0.9723	0.0024
			1.4	0.9398	6.43×10^{-7}	0.941	0.9383	0.0028
			2	0.9155	6.88×10^{-7}	0.9171	0.914	0.0031
		27	0.7	0.9731	3.40×10^{-7}	0.9742	0.972	0.0022
			1.4	0.9408	4.72×10^{-7}	0.942	0.9393	0.0027
			2	0.9149	7.03×10^{-7}	0.916	0.9133	0.0027
	50	35	0.7	0.9716	2.77×10^{-7}	0.9726	0.97054	0.002
			1.4	0.9312	4.47×10^{-7}	0.9421	0.9394	0.0027
			2	0.9147	8.88×10^{-7}	0.9159	0.9131	0.0028
		45	0.7	0.9721	2.39×10^{-7}	0.9728	0.9708	0.002
			1.4	0.9405	4.17×10^{-7}	0.9416	0.9389	0.0026
			2	0.9145	4.38×10^{-7}	0.9156	0.9129	0.0027
	100	70	0.7	0.9724	1.92×10^{-7}	0.9735	0.9716	0.0019
			1.4	0.9395	3.46×10^{-7}	0.9403	0.938	0.0023
			2	0.9149	3.77×10^{-7}	0.916	0.9136	0.0024
		90	0.7	0.9732	2.16×10^{-7}	0.9739	0.9722	0.0017
			1.4	0.9391	2.42×10^{-7}	0.94	0.9381	0.0019
			2	0.9158	2.79×10^{-7}	0.9167	0.9144	0.0023
LINEX	30	21	0.7	0.9726	2.70×10^{-7}	0.9736	0.9717	0.0019
			1.4	0.9395	3.93×10^{-7}	0.9405	0.9378	0.0027
			2	0.916	4.40×10^{-7}	0.9171	0.9143	0.0028
		27	0.7	0.9721	2.14×10^{-7}	0.9727	0.9707	0.0019
			1.4	0.9394	4.35×10^{-7}	0.9405	0.9381	0.0024
			2	0.9167	7.17×10^{-7}	0.918	0.9152	0.0028
	50	35	0.7	0.9728	1.39×10^{-7}	0.9735	0.9718	0.0017
			1.4	0.9388	4.48×10^{-7}	0.9398	0.9375	0.0023
			2	0.9156	3.45×10^{-7}	0.9167	0.9143	0.0025
		45	0.7	0.9729	1.68×10^{-7}	0.9735	0.9718	0.0018
			1.4	0.9396	2.53×10^{-7}	0.9404	0.9382	0.0022
			2	0.9164	2.94×10^{-7}	0.9175	0.9152	0.0022
	100	70	0.7	0.973	1.50×10^{-7}	0.9735	0.9719	0.0016
			1.4	0.9387	2.74×10^{-7}	0.9396	0.9375	0.0021
			2	0.9151	4.12×10^{-7}	0.9161	0.9139	0.0022
		90	0.7	0.9726	1.64×10^{-7}	0.9733	0.9718	0.0015
			1.4	0.9387	3.37×10^{-7}	0.9395	0.9375	0.002
			2	0.9147	2.30×10^{-7}	0.9155	0.9134	0.0021

Table 4. Bayes averages, posterior risks and 95% credible intervals of the hazard function, using non- informative prior based on Type II censoring
$(N = 10000, \beta_1 = 0.6, b_1 = 0.3; \beta_2 = 0.9, b_2 = 0.9)$

Loss function	n	r	t_0	Average	PR	Upper	Lower	Length
SE	30	21	0.7	0.0548	4.67×10^{-6}	0.0586	0.0514	0.0072
			1.4	0.0451	1.05×10^{-6}	0.0465	0.0434	0.0031
			2	0.0389	1.15×10^{-6}	0.0406	0.0368	0.0038
		27	0.7	0.0539	1.68×10^{-6}	0.0563	0.0513	0.005
			1.4	0.0468	5.03×10^{-7}	0.048	0.0455	0.0024
			2	0.0402	2.44×10^{-7}	0.0412	0.0392	0.002
	50	35	0.7	0.0503	6.23×10^{-7}	0.0518	0.0488	0.003
			1.4	0.0457	6.75×10^{-7}	0.0469	0.0443	0.0027
			2	0.0423	3.48×10^{-7}	0.0429	0.04	0.0029
		45	0.7	0.0501	3.51×10^{-7}	0.051	0.0487	0.0023
			1.4	0.0466	3.64×10^{-7}	0.0478	0.0452	0.0026
			2	0.0414	5.17×10^{-7}	0.0423	0.0396	0.0027
	100	70	0.7	0.0511	2.34×10^{-7}	0.052	0.0499	0.0021
			1.4	0.0468	3.09×10^{-7}	0.0477	0.0455	0.0021
			2	0.0399	2.25×10^{-7}	0.0406	0.0384	0.0022
		90	0.7	0.0516	2.38×10^{-7}	0.0525	0.0507	0.0019
			1.4	0.0464	2.76×10^{-7}	0.0473	0.0453	0.002
			2	0.0397	3.15×10^{-7}	0.0403	0.0384	0.002
LINEX	30	21	0.7	0.0518	5.41×10^{-7}	0.05296	0.0504	0.0025
			1.4	0.046	3.91×10^{-7}	0.0474	0.045	0.0024
			2	0.041	1.06×10^{-6}	0.0428	0.0393	0.0035
		27	0.7	0.0516	4.02×10^{-7}	0.0524	0.0499	0.0025
			1.4	0.0469	4.68×10^{-7}	0.0477	0.045	0.0027
			2	0.0395	5.12×10^{-7}	0.0402	0.0378	0.0024
	50	35	0.7	0.0513	3.52×10^{-7}	0.0525	0.0503	0.0022
			1.4	0.0471	2.43×10^{-7}	0.0479	0.0456	0.0023
			2	0.0416	4.64×10^{-7}	0.0427	0.04	0.0027
		45	0.7	0.0508	2.67×10^{-7}	0.0516	0.0499	0.0017
			1.4	0.0454	2.99×10^{-7}	0.0462	0.0442	0.002
			2	0.0403	3.59×10^{-7}	0.0413	0.0391	0.0022
	100	70	0.7	0.0507	1.81×10^{-7}	0.0516	0.0497	0.0019
			1.4	0.0466	2.78×10^{-7}	0.0476	0.0457	0.002
			2	0.0403	2.38×10^{-7}	0.0412	0.0392	0.0019
		90	0.7	0.0521	1.33×10^{-7}	0.0526	0.0509	0.0017
			1.4	0.0452	1.83×10^{-7}	0.0458	0.0439	0.0019
			2	0.0402	2.92×10^{-7}	0.0411	0.0391	0.002

Table 5. Bayes averages, posterior risks and 95% credible intervals for the reliability function, using informative gamma prior based on Type II censoring
(N = 10000, hyper parameters = $(0.6, 0.3, 0.5, 0.7)(0.9, 0.8, 0.7, 0.5)$)

Loss function	n	r	t_0	Average	PR	Upper	Lower	Length
SE	30	21	0.7	0.9742	7.35×10^{-7}	0.9755	0.9722	0.0033
			1.4	0.9404	1.16×10^{-6}	0.9423	0.9384	0.0039
			2	0.9178	1.61×10^{-6}	0.9192	0.9152	0.004
		27	0.7	0.9718	5.45×10^{-7}	0.9729	0.9702	0.0027
			1.4	0.9403	1.13×10^{-6}	0.9421	0.9387	0.0033
			2	0.9179	6.72×10^{-7}	0.9187	0.9153	0.0034
	50	35	0.7	0.9719	5.31×10^{-7}	0.973	0.9705	0.0025
			1.4	0.9377	6.74×10^{-7}	0.9394	0.9362	0.0032
			2	0.9146	7.71×10^{-7}	0.9163	0.9127	0.0036
		45	0.7	0.972	3.92×10^{-7}	0.9731	0.9706	0.0025
			1.4	0.9391	6.06×10^{-7}	0.9402	0.9377	0.0025
			2	0.9129	5.86×10^{-7}	0.9146	0.9117	0.0029
	100	70	0.7	0.9725	1.69×10^{-7}	0.9734	0.9717	0.0017
			1.4	0.9395	3.19×10^{-7}	0.9403	0.9381	0.0022
			2	0.9168	5.75×10^{-7}	0.9179	0.9151	0.0028
		90	0.7	0.9725	1.77×10^{-7}	0.9733	0.9716	0.0016
			1.4	0.9397	3.16×10^{-7}	0.9405	0.9385	0.002
			2	0.9149	5.38×10^{-7}	0.9161	0.9135	0.0026
LINEX	30	21	0.7	0.972	7.69×10^{-7}	0.9732	0.9703	0.0029
			1.4	0.9407	9.46×10^{-7}	0.9423	0.939	0.0032
			2	0.9173	1.19×10^{-6}	0.9188	0.9153	0.0035
		27	0.7	0.9726	5.06×10^{-7}	0.9741	0.9714	0.0027
			1.4	0.9398	7.17×10^{-7}	0.9411	0.9382	0.003
			2	0.914	7.96×10^{-7}	0.9154	0.9123	0.003
	50	35	0.7	0.9715	2.62×10^{-7}	0.9724	0.9703	0.002
			1.4	0.938	4.82×10^{-7}	0.9391	0.9365	0.0024
			2	0.9147	4.60×10^{-7}	0.9158	0.9132	0.0026
		45	0.7	0.9734	3.80×10^{-7}	0.9741	0.9721	0.002
			1.4	0.9401	3.21×10^{-7}	0.9411	0.9392	20
			2	0.9159	4.00×10^{-7}	0.9169	0.9145	0.0023
	100	70	0.7	0.9727	2.69×10^{-7}	0.9735	0.9716	0.0019
			1.4	0.9387	4.06×10^{-7}	0.9396	0.9371	0.0024
			2	0.9154	2.18×10^{-7}	0.9161	0.9135	0.0026
		90	0.7	0.9716	2.83×10^{-7}	0.9724	0.9706	0.0018
			1.4	0.9385	3.72×10^{-7}	0.9394	0.9372	0.0021
			2	0.9152	3.27×10^{-7}	0.9161	0.9138	0.0022

Table 6. Bayes averages, posterior risks and 95% credible intervals for the hazard function, using informative gamma prior based on Type II censoring
(N=10000,hyper parameters= 0.6, 0.3, 0.5, 0.7) (0.9, 0.8, 0.7, 0.5))

Loss function	n	r	t_0	Average	PR	Upper	Lower	Length
SE	30	21	0.7	0.0516	5.97×10^{-7}	0.0527	0.05	0.0028
			1.4	0.0469	6.58×10^{-7}	0.0485	0.0456	0.0029
			2	0.0388	8.83×10^{-7}	0.0401	0.0369	0.0032
		27	0.7	0.0513	3.43×10^{-7}	0.0523	0.0499	0.0024
			1.4	0.0468	3.87×10^{-7}	0.0478	0.0454	0.0024
			2	0.0408	5.74×10^{-7}	0.042	0.0393	0.0027
	50	35	0.7	0.051	3.50×10^{-7}	0.0521	0.0497	0.0024
			1.4	0.0464	4.75×10^{-7}	0.0474	0.0448	0.0026
			2	0.0385	5.69×10^{-7}	0.0397	0.0372	0.0025
		45	0.7	0.0505	2.46×10^{-7}	0.0513	0.0494	0.0019
			1.4	0.0451	3.49×10^{-7}	0.0461	0.0438	0.0023
			2	0.0397	3.06×10^{-7}	0.0406	0.0382	0.0024
	100	70	0.7	0.0522	3.57×10^{-7}	0.0533	0.051	0.0023
			1.4	0.0445	3.58×10^{-7}	0.0456	0.0434	0.0022
			2	0.0406	3.88×10^{-7}	0.0417	0.0392	0.0025
		90	0.7	0.0507	1.93×10^{-7}	0.0513	0.0496	0.0017
			1.4	0.0466	2.59×10^{-7}	0.0474	0.0455	0.0019
			2	0.041	3.29×10^{-7}	0.0417	0.0392	0.0024
LINEX	30	21	0.7	0.0519	2.33×10^{-7}	0.0526	0.0509	0.0018
			1.4	0.0453	2.79×10^{-7}	0.046	0.044	0.002
			2	0.0385	6.68×10^{-7}	0.0398	0.0372	0.0027
		27	0.7	0.0511	1.27×10^{-7}	0.0517	0.0502	0.0015
			1.4	0.0451	1.25×10^{-7}	0.0459	0.0441	0.0018
			2	0.0387	4.31×10^{-7}	0.0398	0.0375	0.0023
	50	35	0.7	0.0521	1.15×10^{-7}	0.0527	0.0511	0.0015
			1.4	0.0463	2.51×10^{-7}	0.0471	0.0453	0.0018
			2	0.0385	3.68×10^{-7}	0.0399	0.0376	0.0023
		45	0.7	0.0521	1.23×10^{-7}	0.0528	0.0514	0.0014
			1.4	0.0465	1.55×10^{-7}	0.0424	0.0456	0.0017
			2	0.0399	3.08×10^{-7}	0.0408	0.0386	0.0022
	100	70	0.7	0.9719	1.00×10^{-7}	0.9723	0.971	0.0013
			1.4	0.0462	1.25×10^{-7}	0.0467	0.045	0.0017
			2	0.0403	1.59×10^{-7}	0.0409	0.039	0.0019
		90	0.7	0.0519	8.70×10^{-8}	0.0524	0.0512	0.0012
			1.4	0.0463	1.53×10^{-7}	0.047	0.04545	0.0015
			2	0.04	1.62×10^{-7}	0.0407	0.0391	0.0016

Table 7. Bayes predictors and bounds using non-informative and informative priors of the future observation based on Type II censoring under two-sample prediction ($N = 1000, n = 100, r = 70, m = 30, t_0 = 0.5$)

	s	SE				LINEX ($v = -0.2$)			
		$\hat{y}_{(s)\,(SE)}$	LL	UL	Length	$\hat{y}_{(s)\,(LINEX)}$	LL	UL	Length
non-informative	1	1.901	1.8995	1.902	0.0025	1.9002	1.8992	1.901	0.0018
	15	1.8996	1.8982	1.9008	0.0026	1.9007	1.8996	1.9019	0.0023
	30	1.9	1.8979	1.9027	0.0048	1.9008	1.899	1.9021	0.003
Informative	1	1.9009	1.8995	1.9018	0.0023	1.9002	1.8994	1.9008	0.0015
	15	1.8999	1.8988	1.9012	0.0025	1.9005	1.8991	1.9014	0.0022
	30	1.9018	1.8998	1.903	0.0032	1.9013	1.8997	1.9024	0.0027

Concluding remark

From Tables 1-7 one can observe the following:

1. The PRs of the Bayes averages of the shape parameters decrease when the sample size increases, as expected. This is compatible with the fact that the estimates are consistent and approach the population parameter values as the sample size increases. Also, the intervals of the parameters become narrower as the sample size increases.
2. The PRs of the Bayes averages for the estimates of the reliability and hazard rate functions decrease when the sample size increases, and the lengths of the intervals of the parameters become narrower as the sample size increases.
3. In general, all the results perform better when the level of censoring decreases, which is expected since decreasing the level of censoring means that more information is provided by the sample and hence increases the accuracy of the estimates.
4. When the time t_0 increases, the value of the PR of the reliability and hazard rate function and the lengths of the intervals increase too.
5. In Table 7, the results for the LINEX loss function are better than the results for the corresponding results of SE loss function. Also, the results of using the informative prior is better than the corresponding results when using the non-informative prior and the BP and the lengths of the BPB increase when s increases.

4.2. Application

This subsection aims to demonstrate how the proposed EITL distribution; based on Type-II censoring with level of censoring 10% and 30%, can be used in practice.

Using the data given by Singh and Maddala [17] the EITL distribution is fitted to the real data using through R programming language.

The data represents the strength of 1.5 cm glass fibers for 60 devices. The data are:

0.636, 0.252, 0.157, 0.187, 2.771, 0.209, 0.617, 2.078, 1.013, 0.499, 0.431, 0.642, 0.46, 0.749, 0.205, 0.576, 0.439, 0.471, 0.262, 0.387, 0.324, 0.424, 0.548, 1.794, 1.233, 0.915, 0.702, 0.417, 0.337, 0.435, 0.359, 0.293, 0.147, 0.87, 0.608, 0.153, 0.098, 0.557, 0.415, 0.122, 0.912, 0.341, 0.725, 0.364, 0.24, 0.594, 0.325, 0.416, 0.08, 0.582, 1.257, 1.575, 0.48, 0.909, 0.17, 0.319, 0.09, 0.154, 2.248, 0.292.

1. The Kolmogorov-Smirnov goodness of fit test is applied to check the validity of the EITL distributions. The p-value is given by 0.2671 and it showed that the model fits the data very well.
2. Table 8 displays the Bayes estimates and *estimated risk* (ERs) based on non-informative and informative priors for the parameters, reliability and hazard functions for the real data set based on Type II censoring.
3. The BP and BPB for the future observation from a future sample based on non-informative and informative priors are given in Table 9.

Table 8. Bayes estimates and ERs for the Bayes estimates of the parameters α, θ, reliability and hazard functions, using non-informative and informative priors distribution based on SE and LINEX loss functions for the real data set based on Type II censoring
(r=0.70 n and 0.90 n)

r	Parameter	Loss function	non-informative		informative	
			estimates	ER	estimates	ER
42	α	SE	6.8007	1.43×10^{-4}	6.7981	1.11×10^{-4}
		LINEX	6.7982	1.28×10^{-4}	6.7998	6.83×10^{-5}
	θ	SE	1.5202	8.58×10^{-5}	1.5213	1.39×10^{-4}
		LINEX	1.5205	5.46×10^{-5}	1.5204	4.40×10^{-5}
	$R(t)$	SE	0.9162	5.07×10^{-5}	0.0025	4.16×10^{-5}
		LINEX	0.0022	2.88×10^{-5}	0.0023	4.13×10^{-5}
	$h(t)$	SE	0.0407	7.18×10^{-5}	2.7564	5.85×10^{-5}
		LINEX	2.7572	7.01×10^{-5}	2.7566	5.16×10^{-5}
54	α	SE	6.7994	1.03×10^{-4}	6.8	6.57×10^{-5}
		LINEX	6.8008	6.88×10^{-5}	6.7999	6.50×10^{-5}
	θ	SE	1.518	1.17×10^{-4}	1.5205	8.31×10^{-5}
		LINEX	1.5183	8.29×10^{-5}	1.5204	7.76×10^{-5}
	$R(t)$	SE	0.9162	5.07×10^{-5}	0.002	4.85×10^{-5}
		LINEX	0.0023	5.62×10^{-5}	0.0021	4.61×10^{-5}
	$h(t)$	SE	0.0407	7.18×10^{-5}	2.7551	5.26×10^{-5}
		LINEX	2.7562	7.08×10^{-5}	2.7571	4.65×10^{-5}

Table 9. Bayes predictors and bounds for the future observation for real data based on Type II censoring using non-informative and informative priors under two-sample prediction (n = 60, r = 42, m=30)

	s	SE				LINEX ($\nu = -0.2$)			
		$\hat{y}_{(s)(SE)}$	LL	UL	Length	$\hat{y}_{(s)(LINEX)}$	LL	UL	Length
non-informative	1	1.8989	1.8979	1.8998	0.0019	1.9001	1.8992	1.9011	0.0019
	30	1.9016	1.9	1.9027	0.0027	1.8995	1.8982	1.901	0.0028
	60	1.9021	1.9	1.9033	0.0033	1.8993	1.8977	1.901	0.0033
informative	1	1.9017	1.9	1.9031	0.0031	1.9007	1.8993	1.9021	0.0029
	30	1.9007	1.8983	1.9017	0.0034	1.8996	1.8974	1.9007	0.0033
	60	1.898	1.8945	1.9008	0.0062	1.8978	1.8963	1.8997	0.0034

5. Conclusions and Concluding remark

1. From Table 8 the ERs of the Bayes estimates of the parameters, reliability and hazard decrease when the sample size increases, i.e., as expected the results become better when the censoring level decreases. The results for the LINEX loss function perform better than the corresponding results for SE loss function, also the results obtained of using the informative prior is better than the results of using the non-informative prior.
2. The BP and the lengths of the BPB increase when s increases. Table 9 shows that the results for the LINEX loss function perform better than the corresponding results for SE loss function, the results of using informative prior is better than the results of using non-informative priors.

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