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Cosine Fréchet Loss Distribution: Properties, Actuarial Measures and Insurance Applications

John Abonongo[1*], **Ivivi J. Mwaniki**[2], **Jane A. Aduda**[3]

[1] Institute for Basic Sciences, Technology and Innovation. Pan African University, Nairobi, Kenya
[2] School of Mathematics, Statistics and Actuarial Science Division, University of Nairobi, Kenya
[3] Department of Statistics and Actuarial Science, Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya

* **Correspondence:** abonongojohn@gmail.com

Abstract: In this paper, the cosine Fréchet loss (CFrL) distribution is proposed as a modified version of the Fréchet using the cosine F-Loss generator. This distribution is flexible and able to model varying shapes of the hazard rate compared with the traditional two parameter Fréchet distribution. The density exhibits different kinds of decreasing, and right-skewed shapes. The hazard rate function show different kinds of increasing-constant-decreasing, reversed-J, bathtub, and upside down bathtub shapes. The statistical properties including quantile function, generating functions, inequality measures, order statistics, mean and median deviations, moments and incomplete moments are studied. Using numerical integration, the first four moments of the CFrL distribution are obtained. These moments are then used in estimating the standard deviation, coefficient of variation, coefficient of skewness, and coefficient of kurtosis. The skewness is always positive and the kurtosis is increasing. Actuarial measures including value at risk, tail value at risk, and tail variance are derived and studied. The numerical values of the actuarial measures show that increasing confidence levels are associated with increasing value at risk, tail value at risk, and tail variance. The maximum likelihood estimators are studied and simulations carried out to ascertain the behavior of the estimators. It is observed that the estimators are consistent. The usefulness of the CFrL distribution is demonstrated on two insurance loss datasets and its performance compared with other known classical heavy-tailed distributions. The results showed that the proposed distribution provides the best parametric fit to the two insurance loss datasets.

Keywords: cosine F-Loss; Fréchet distribution; Monte Carlo simulation; heavy-tailed distribution; insurance loss.
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1. Introduction

The development of more adaptable distributions has received a lot of attention in recent years. In the recent past, a number of generated families of distributions have been introduced and studied with real-world data in a variety of application areas, including engineering, economics, medical sciences, biological research, environmental studies, and insurance. It is possible to model phenomena having extreme values, such as insurance claims and other related events, using extreme value techniques. From the literature, heavy-tailed distributions have demonstrated to be effective models for extreme-value datasets. Also, researchers have shown a deep concern in the financial sector to study new heavy-tailed distributions. Among the applicability of the statistical distributions in the applied area, heavy-tailed distributions have received much attention for modeling financial phenomena.
One of the fundamental distributions in extreme value theory is the Fréchet distribution (Fréchet [1]). The Fréchet distribution and its applications are used in accelerated life testing, rainfall, earthquakes, floods, horse racing, wind speeds, finance, and sea waves, among other fields. For additional information, see (Kotz and Nadarajah [2]; Mubarak, [3]). The Fréchet distribution has been widely generalized in the literature. For instance, Mahmoud and Mandouh [6] proposed the transmuted Fréchet, Nadarajah and Kotz [4] introduced the exponentiated Fréchet, Nadarajah and Gupta [5] studied the beta Fréchet, Krishna et al. [7] introduced the Marshall-Olkin Fréchet, and Silva et al. [8] introduced the gamma extended Fréchet, respectively. The Kumaraswamy Fréchet was proposed by Mead and Abd-Eltawab [9], and the transmuted Marshall-Olkin Fréchet and the Weibull Fréchet were introduced by Afify et al. [10] and Afify et al. [11]. Ramos et al. [13] developed a novel generalization of the Fréchet distribution with long-term survival, whereas Mead et al. [12] introduced the beta exponential Fréchet. Tablada and Cordeiro [14] studied the modified Fréchet, and Abouelmagd et al. [15] introduced the Burr X Fréchet distribution. Nasiru [16] proposed the extended odd Fréchet family of distributions with the extended odd Fréchet Nadarajah-Haghighi and extended odd Fréchet Weibull as special distributions. Harlow [17] showed that the Fréchet distribution is an important distribution for modeling the statistical behavior of materials properties for a variety of engineering applications.
Moreover, Abonongo et al. [20] introduced the cosine F-Loss (CFL) family of distributions with cumulative distribution function (CDF) given by

$$G(x;\omega) = 1 - \cos\left[\frac{\pi}{2}\left(1 - \frac{\sigma\bar{F}(x;\omega)}{\sigma - \log\left(\bar{F}(x;\omega)\right)}\right)\right],\ \sigma > 0, x \in \mathbb{R}, \tag{1.1}$$

where $\bar{F}(x;\omega) = 1 - F(x;\omega)$ is the survival function of the baseline distribution, ω is a $p \times 1$ vector of parameters, and σ is a shape parameter.
The probability density function (PDF) is given by

$$g(x;\omega) = \frac{\pi}{2}\left[\frac{\sigma f(x;\omega)\left[1 + \sigma - \log(\bar{F}(x;\omega))\right]}{\left[\sigma - \log(\bar{F}(x;\omega))\right]^2}\right]\sin\left[\frac{\pi}{2}\left(1 - \frac{\sigma\bar{F}(x;\omega)}{\sigma - \log\left(\bar{F}(x;\omega)\right)}\right)\right], x \in \mathbb{R}. \tag{1.2}$$

They proposed the cosine Weibull loss, cosine Burr III loss, and cosine Lomax loss as special distributions.
The purpose of this paper is to improve flexibility of the Fréchet distribution by using a trigonometric

transformation via the use of the cosine F-Loss family of distributions by Abonongo et al. [20], and thus create a distribution that can model both monotonic and non-monotonic failure rates. The motivation is that the trigonometric function ensures that the parameters oscillate with changing values thereby improving the shapes of the new proposed distribution. From the literature most distributions have a lot of parameters in the bit to make them flexible but with our proposed distribution, over-parameterization is checked. Also, most distribution extensions are based on algebraic functions with few on the use of trigonometric functions. Thus, this has triggered the need to extend existing classical distributions or develop new ones.

To the best of our knowledge, the Fréchet distribution has not been modified using the cosine F-Loss family of distributions. Hence, using the cosine F-Loss generator developed by Abonongo et al. [20], we propose the cosine Fréchet Loss distribution (CFrL) as a modified version of the Fréchet distribution. In comparison with the traditional Fréchet distribution and other heavy tailed distributions, we explore and demonstrate the flexible of the CFrL distribution.

The rest of the paper is organized as follows: Section 2 presents the cosine Fréchet Loss distribution. The impact of changing parameter values is presented in section 3. Useful series expansion of the PDF of the CFrL distribution is presented in section 4. Some statistical properties are presented in Section 5. In Section 6, we present some actuarial measures, including value at risk, tail value at risk, and tail variance. In Section 7, we present the parameter estimation of the cosine Fréchet Loss distribution using maximum likelihood estimation. The behaviors of the estimators are ascertained in Section 8 using Monte Carlo simulations. In Section 9, the usefulness of the new distribution is illustrated using two insurance loss datasets and the conclusion is presented in Section 10.

2. Cosine Fréchet Loss Distribution

In this section, we introduce a modified version of the traditional Fréchet distribution which is capable of modeling insurance losses. The proposed model is called cosine Fréchet Loss (CFrL) distribution.

Consider the Fréchet distribution as the baseline distribution with CDF and PDF defined as $F(x) = e^{-\alpha x^{-\beta}}$ and $f(x) = \beta\alpha x^{-(\beta+1)}e^{-\alpha x^{-\beta}}$ for $x > 0$ and $\alpha, \beta > 0$, respectively in Equation (1.1), we obtain the CFrL distribution. Thus, the CDF of the CFrL is given by

$$G(x;\alpha,\beta,\sigma) = 1 - \cos\left[\frac{\pi}{2}\left(1 - \frac{\sigma\left(1 - e^{-\alpha x^{-\beta}}\right)}{\sigma - \log\left(1 - e^{-\alpha x^{-\beta}}\right)}\right)\right],\ \alpha,\beta,\sigma > 0, x > 0, \tag{2.1}$$

where α is a scale parameter, β and $\sigma > 0$ are shape parameters.

The related PDF is given by

$$g(x;\alpha,\beta,\sigma) = \frac{\pi\alpha\beta\sigma}{2}\left[\frac{x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\left[1+\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]}{\left[\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]^2}\right]\sin\left[\frac{\pi}{2}\left(1 - \frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right], x > 0. \tag{2.2}$$

The hazard rate function is given by

$$h(x;\alpha,\beta,\sigma) = \frac{\pi\alpha\beta\sigma\left[x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\left[1+\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]\right]}{2\left[\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]^2}\tan\left[\frac{\pi}{2}\left(1 - \frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right],\quad x > 0.$$

3. Impact of Changing Parameter Values

In this section, the impact of changing parameter values on the plots of the PDF and hazard rate function of the CFrL distribution is studied. From Figure 1, the density plots exhibits decreasing and right skewed shapes for different parameter values.

The plots of the hazard rate function as shown in Figure 2 exhibit reversed-J, increasing-constant-

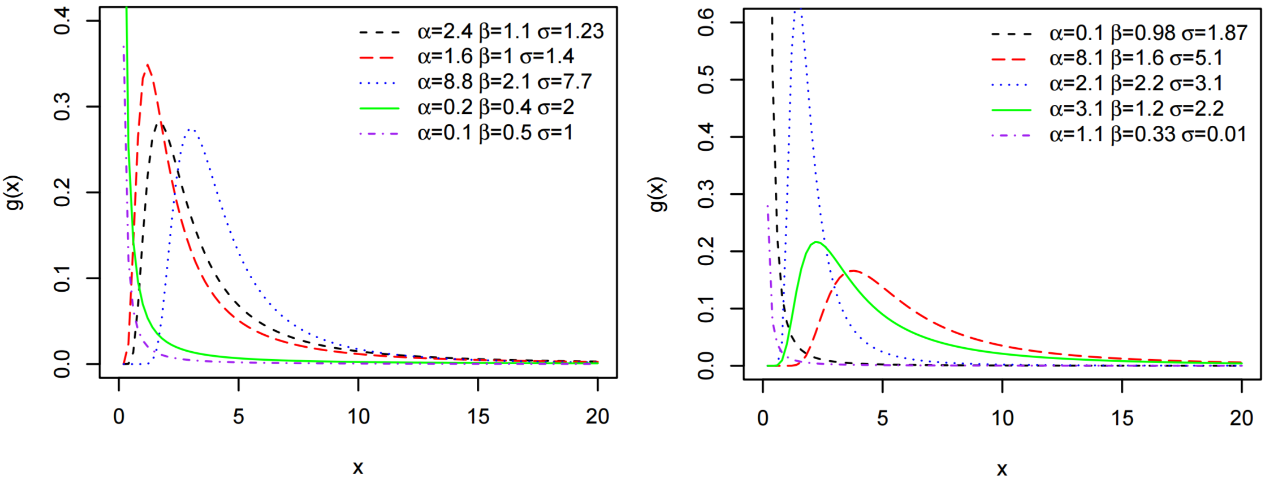

Figure 1. Plots for the density function of the CFrL distribution

decreasing, bathtub, and upside-down-bathtub shapes for different parameter values.

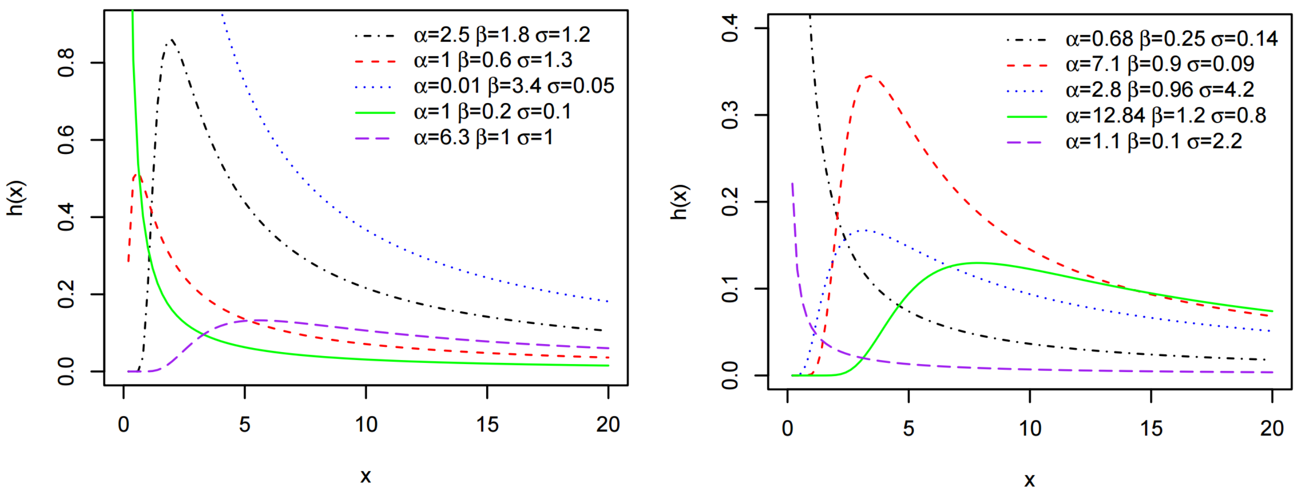

Figure 2. Plots for the hazard rate function of the CFrL distribution

4. Useful Series Expansion

We present useful series expansion of the PDF of the CFrL distribution in this section.

Lemma 1. The PDF of the CFrL distribution has a mixture representation of the form

$$g(x;\alpha,\beta,\sigma) = \alpha\beta \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}(j+m+t)x^{-(\beta+1)}(e^{-\alpha x^{-\beta}})^{j+m+t}, \tag{4.1}$$

where $D_{imkj} = \frac{(-1)^{i+j+k+m}(\frac{\pi}{2})^{2k}\binom{2k}{i}\binom{i+j-1}{j}\binom{i}{m}}{2k!\sigma^{j}}$ and $A_{jtw} = j\sum_{t=0}^{\infty}\binom{t-j}{t}\sum_{w=0}^{t}\frac{(-1)^{t+w}}{j-w}\binom{t}{w}P_{w,t}$.

Proof. Using the Taylor series expansion of cosine function; $\cos(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$, the generalized binomial expansion; $(1-t)^z = \sum_{i=0}^{z}(-1)^i\binom{z}{i}t^i, |t| \le 1$ and $(1+v)^{-z} = \sum_{l=0}^{\infty}(-1)^j\binom{z+j-1}{j}v^j, |v| \le 1$ and the fact that $0 < \frac{\sigma\bar{F}(x;\omega)}{\sigma-\log\bar{F}(x;\omega)} < 1$, the CDF of the CFL in Equation (1) can be rewritten as

$$\begin{aligned}G(x;\omega) &= 1 - \sum_{k=0}^{\infty}\sum_{i=0}^{2k}\frac{(-1)^{k+i}(\frac{\pi}{2})^{2k}}{2k!}\binom{2k}{i}\left[\frac{\sigma\bar{F}(x;\omega)}{\sigma-\log(\bar{F}(x;\omega))}\right]^i\\ &= 1 - \sum_{k=0}^{\infty}\sum_{i=0}^{2k}\frac{(-1)^{k+i}(\frac{\pi}{2})^{2k}}{2k!}\binom{2k}{i}\frac{\bar{F}(x;\omega)^i}{\left[1-\frac{\log(\bar{F}(x;\omega))}{\sigma}\right]^i}.\end{aligned}$$

Letting $v = -\frac{\log(\bar{F}(x;\omega))}{\sigma}$, we have

$$G(x;\omega) = 1 - \sum_{j,k=0}^{\infty}\sum_{i=0}^{2k}\frac{(-1)^{i+j+k}(\frac{\pi}{2})^{2k}}{2k!}\binom{2k}{i}\binom{i+j-1}{j}\bar{F}(x;\omega)^i\left[-\frac{\log(\bar{F}(x;\omega))}{\sigma}\right]^j.$$

Also, using $\bar{F}(x;\omega)^i = \sum_{m=0}^{i}(-1)^m\binom{i}{m}F(x;\omega)^m$, we get

$$G(x;\omega) = 1 - \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k}\frac{(-1)^{i+j+k}(\frac{\pi}{2})^{2k}}{2k!\sigma^j}\binom{2k}{i}\binom{i+j-1}{j}\binom{i}{m}F(x;\omega)^m\left[-\log(\bar{F}(x;\omega))\right]^j.$$

Making use of the expansion $(-\log(1-x))^n = n\sum_{t=0}^{\infty}\binom{t-n}{t}\sum_{w=0}^{t}\frac{(-1)^{t+w}}{n-w}\binom{t}{w}P_{w,t}, x^{n+t}$, where $n > 0$ is any real value. The constants $P_{w,t}$ can be estimated recursively by, $P_{w,t} = \frac{1}{t}\sum_{a=0}^{t}\frac{aw+a-w}{a+1}P_{w,t-a}$ for $t = 1,2,3,...$ and $P_{w,0} = 1$.

$$\begin{aligned}G(x;\omega) = 1-&\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k}\frac{(-1)^{i+j+k}(\frac{\pi}{2})^{2k}}{2k!\sigma^j}\binom{2k}{i}\binom{i+j-1}{j}\binom{i}{m}F(x;\omega)^m\\ &\times j\sum_{t=0}^{\infty}\binom{t-j}{t}\sum_{w=0}^{t}\frac{(-1)^{t+w}}{j-w}\binom{t}{w}P_{w,t}F(x;\omega)^{j+t}.\end{aligned}$$

Then, we can write

$$G(x;\omega) = 1 - \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k}D_{imkj}A_{jtw}F(x;\omega)^{j+m+t}, \tag{4.2}$$

where $D_{imkj} = \frac{(-1)^{i+j+k+m}(\frac{\pi}{2})^{2k}\binom{2k}{i}\binom{i+j-1}{j}\binom{i}{m}}{2k!\sigma^{j}}$ and $A_{jtw} = j\sum_{t=0}^{\infty}\binom{t-j}{t}\sum_{w=0}^{t}\frac{(-1)^{t+w}}{j-w}\binom{t}{w}P_{w,t}$. From Equation (4.2), we have

$$g(x;\omega) = \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}(j+m+t)f(x;\omega)F(x;\omega)^{j+m+t-1}. \tag{4.3}$$

Therefore, substituting the CDF and PDF of the Fréchet distribution into Equation (4.3), we get the mixture representation of the PDF of the CFrL distribution as

$$g(x;\alpha,\beta,\sigma) = \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} \alpha\beta D_{imkj}A_{jtw}(j+m+t)x^{-(\beta+1)}(e^{-\alpha x^{-\beta}})^{j+m+t}.$$

5. Statistical Properties

In this section, some statistical properties of the CFrL distribution including the quantile function, generating functions, inequality measures, order statistics, mean and median deviations, moments and incomplete moments are presented.

5.1. Quantile Function

The quantile function is vital in describing the random variable of a distribution. It helps in simulating random samples which are useful in simulations. It can also be used to compute measures of shape such as skewness and kurtosis.

Lemma 2. The quantile function of the CFrL distribution for $u \in (0,1)$ is defined by

$$x_u = Q(u) = G^{-1}(u) \tag{5.1}$$

which is obtained by the solution of the equation; $\left[1-\left(\frac{2}{\pi}\arccos(1-u)\right)\right]\left[\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right] - \sigma\left(1-e^{-\alpha x^{-\beta}}\right) = 0$. The first quartile, the median, and the upper quartile are obtained by substituting $u = 0.25, 0.5,$ and 0.75 respectively, into Equation (5.1).

5.2. Moments

The moments of a distribution is important in estimating measures of variation like the variance, standard deviation, coefficient of variation, mean deviation, median deviation, kurtosis, skewness amongst others.

Proposition 1. The r^{th} non-central moment of the CFrL distribution is given by

$$\mu_r' = \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} \alpha^{r/\beta}D_{imkj}A_{jtw}(j+m+t)^{r/\beta}\Gamma\left(1-\frac{r}{\beta}\right), r<\beta. \tag{5.2}$$

Proof. By definition the r^{th} non-central moment is given by

$$\mu_r' = \int_0^{\infty} x^r g(x)dx.$$

This implies that,

$$\mu_r' = \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}(j+m+t)\int_0^{\infty} \alpha\beta x^r.x^{-\beta-1}\left(e^{-\alpha x^{-\beta}}\right)^{j+m+t} dx.$$

Letting $y = \alpha(j+m+t)x^{-\beta}$. This implies, $x = \left(\frac{y}{\alpha(j+m+t)}\right)^{-1/\beta}$. Also, $dx = -\frac{dy}{\alpha\beta(j+m+t)x^{-\beta-1}}$. Thus,

$$\begin{aligned}\mu_r' &= \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} \alpha^{r/\beta} D_{imkj}A_{jtw}(j+m+t)(j+m+t)^{r/\beta-1}\int_0^{\infty} y^{-r/\beta+1-1}e^{-y}dy \\ &= \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} \alpha^{r/\beta} D_{imkj}A_{jtw}(j+m+t)^{r/\beta}\Gamma\left(1-\frac{r}{\beta}\right), r<\beta,\end{aligned}$$

where $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and $r = 1, 2, 3...,$. The values for the first four moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) of the CFrL distribution for selected values of the parameters are shown in Table 1. The values of the first four moments are obtained by using numerical integration. The standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) are defined as $SD = \sqrt{\mu_2' - (\mu_1')^2}$, $CV = \frac{\sigma}{\mu_1'}$, $CS = \frac{\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3}{\sigma^3}$, and $CK = \frac{\mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4}{\sigma^4}$ respectively.

Table 1. First four moments, SD, CV, CS and CK of the CFrL distribution for some parameter values

μ_r'	$\alpha = 0.1, \beta = 1.7, \sigma = 0.2$	$\alpha = 0.14, \beta = 1.42, \sigma = 0.25$	$\alpha = 0.2, \beta = 1.5, \sigma = 0.12$
μ_1'	0.261	0.293	0.304
μ_2'	1.105	1.561	1.541
μ_3'	7.014	11.233	10.382
μ_4'	34.256	55.338	50.881
SD	1.018	1.2156	1.204
CV	3.901	4.145	3.961
CS	5.862	5.263	5.175
CK	25.459	19.740	18.626

Remark 1. By substituting $r = 1$ into Equation (5.2), we can get the mean of the CFrL distribution.

$$\mu_1' = \sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} \alpha^{1/\beta} D_{imkj}A_{jtw}(j+m+t)^{1/\beta}\Gamma\left(1-\frac{1}{\beta}\right), \beta > 1.$$

Figures 3 and 4 show the mean and variance, skewness and kurtosis plots of the CFrL distribution for $\sigma = 6.1$, and a range of values for α and β respectively. From Figure 3, the mean and variance are increasing. Also, in Figure 4, it can be seen that the skewness is positive, an indication that the CFrL is right skewed and the kurtosis is increasing; meaning CFrL distribution is leptokurtic and heavy-tailed.

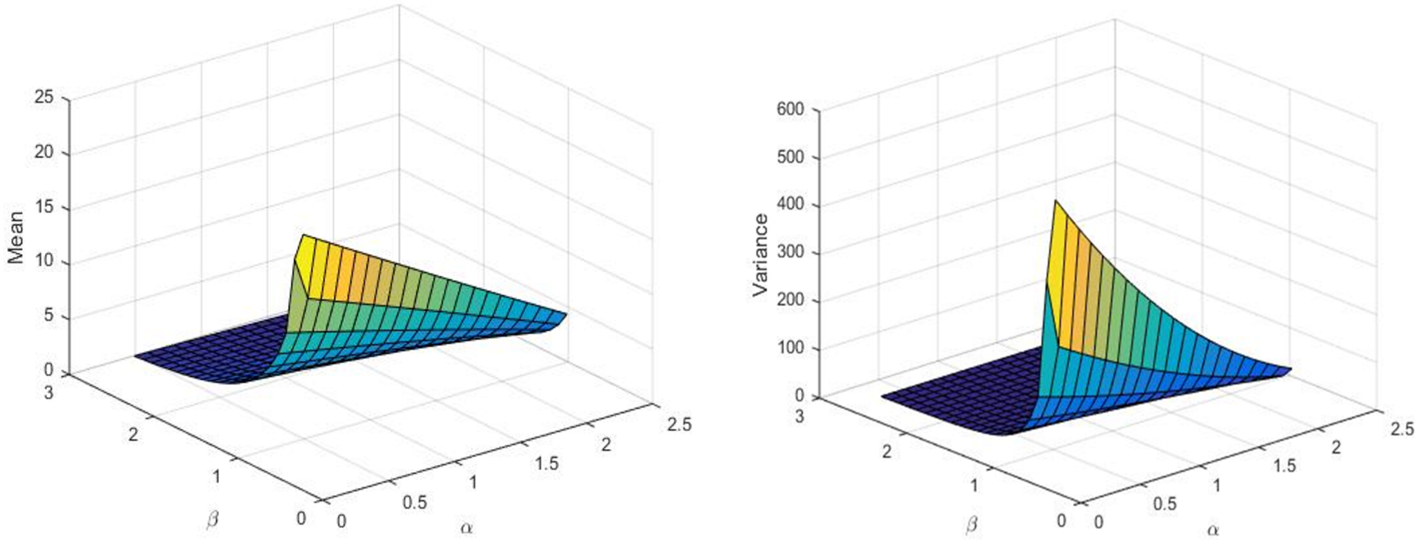

Figure 3. Mean and variance plots of the CFrL distribution

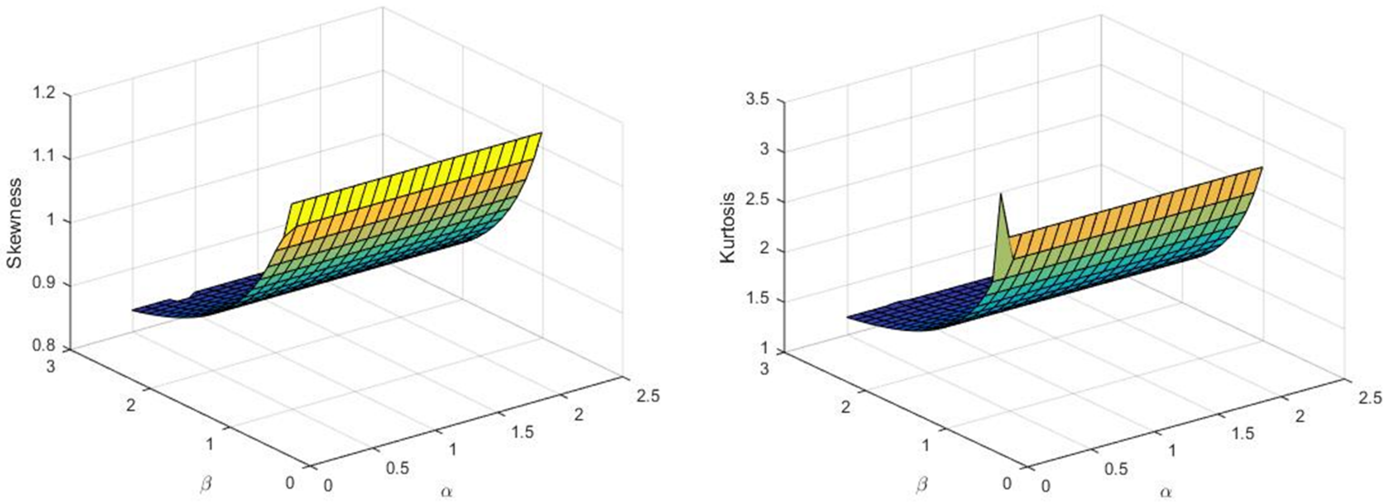

Figure 4. Skewness and kurtosis plots of the CFrL distribution

5.3. Generating Functions

The moment generating function (MGF), characteristic function, and cumulant generating function of the CFrL distribution are derived in this section.

Proposition 2. The MGF of the CFrL distribution is given by

$$\begin{aligned} M_X(z) &= \alpha^{r/\beta}(j+m+t)^{r/\beta} \sum_{j,k,r=0}^{\infty} \sum_{m=0}^{i} \sum_{i=0}^{2k} \frac{Z^r D_{imkj} A_{jtw}}{r!} \\ &\times \Gamma\left(1-\frac{r}{\beta}\right), r<\beta, \end{aligned} \tag{5.3}$$

where $\Gamma(.,.)$ is the upper incomplete gamma function and $r = 1, 2, 3, ...$

Proof. By definition the MGF is given as;

$$M_X(z) = \mathbb{E}(e^{zx}) = \int_0^{\infty} e^{zx} g(x) dx.$$

Using series expansion,

$$M_X(z) = \mathbb{E}\left[\sum_{r=0}^{\infty} \frac{z^r X^r}{r!}\right] = \sum_{r=0}^{\infty} \frac{z^r}{r!} \mathbb{E}(X^r),$$

$$M_X(z) = \sum_{r=0}^{\infty} \frac{z^r}{r!} \mu_r'.$$

Therefore,

$$\begin{aligned} M_X(z) &= \alpha^{r/\beta}(j+m+t)^{r/\beta} \sum_{j,k,r=0}^{\infty} \sum_{m=0}^{i} \sum_{i=0}^{2k} \frac{z^r D_{imkj} A_{jtw}}{r!} \\ &\times \Gamma\left(1-\frac{r}{\beta}\right), r<\beta. \end{aligned}$$

Proposition 3. The characteristic function of the CFrL distribution is given by

$$\begin{aligned} \Theta_X(z) &= \alpha^{r/\beta}(j+m+t)^{r/\beta} \sum_{j,k,r=0}^{\infty} \sum_{m=0}^{i} \sum_{i=0}^{2k} \frac{(iz)^r D_{imkj} A_{jtw}}{r!} \\ &\times \Gamma\left(1-\frac{r}{\beta}\right), r<\beta. \end{aligned} \tag{5.4}$$

Proof. By definition the characteristic function is given as;

$$\Theta_X(z) = \mathbb{E}(e^{izX}) = \sum_{r=0}^{\infty} \frac{(iz)^r}{r!} \mu_r', \quad i = \sqrt{-1}.$$

Therefore,

$$\Theta_X(z) = \alpha^{r/\beta}(j+m+t)^{r/\beta} \sum_{j,k,r=0}^{\infty} \sum_{m=0}^{i} \sum_{i=0}^{2k} \frac{(iz)^r D_{imkj} A_{jtw}}{r!} \times\Gamma\left(1-\frac{r}{\beta}\right), r<\beta.$$

Proposition 4. The cumulant generating function of the CFrL distribution is given by

$$\kappa_X(z) = r/\beta \log(\alpha) + r/\beta \log(j+m+t) + \log \sum_{j,k,r=0}^{\infty} \sum_{m=0}^{i} \sum_{i=0}^{2k} \frac{(iz)^r D_{imkj} A_{jtw}}{r!} \times\Gamma\left(1-\frac{r}{\beta}\right), r<\beta. \tag{5.5}$$

Proof. By definition,

$$\kappa_X(z) = \log(\Theta_X(z)).$$

Therefore,

$$\kappa_X(z) = r/\beta \log(\alpha) + r/\beta \log(j+m+t) + \log \sum_{j,k,r=0}^{\infty} \sum_{m=0}^{i} \sum_{i=0}^{2k} \frac{(iz)^r D_{imkj} A_{jtw}}{r!} \times\Gamma\left(1-\frac{r}{\beta}\right), r<\beta.$$

5.4. Incomplete Moment

The incomplete moment is vital in estimating the mean deviation, median deviation, and measures of inequalities like Bonferroni and Lorenz curves.

Proposition 5. The r^{th} incomplete moment of the CFrL distribution is given by

$$M_r(x) = \alpha^{r/\beta}(j+m+t)^{r/\beta} \sum_{j,k=0}^{\infty} \sum_{m=0}^{i} \sum_{i=0}^{2k} D_{imkj} A_{jtw} \times\Gamma\left(1-r/\beta, \alpha(j+m+t)x^{-\beta}\right), r<\beta, \tag{5.6}$$

where $\Gamma(.,.)$ is the lower incomplete gamma function and $r = 1, 2, 3, ...$

Proof. Using the identity,

$$\Gamma(a,y) = \int_0^y x^{a-1} e^{-x} dx.$$

and the concept in proving the moment, the incomplete moment of the CFrL distribution is

$$\begin{aligned} M_r(y) &= \int_0^y u^r g(u) du \\ &= \sum_{j,k=0}^{\infty} \sum_{m=0}^{i} \sum_{i=0}^{2k} D_{imkj} A_{jtw} (j+m+t) \int_0^y \alpha\beta x^r x^{-(\beta+1)} \left(e^{-\alpha x^{-\beta}}\right)^{j+m+t} dx. \end{aligned}$$

Hence,

$$M_r(y) = \alpha^{r/\beta}(j+m+t)^{r/\beta}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}$$
$$\times\Gamma\left(1-r/\beta,\alpha(j+m+t)y^{-\beta}\right), r<\beta.$$

5.5. Mean and Median Deviations

The totality of the deviations from the mean and median can be used to estimate the variation in a population with some certainty. If the random variable X follows the CFrL distribution, then the mean and median deviations are given by the following propositions.

Proposition 6. The expected value of the absolute deviation of a random variable X having the CFrL distribution from its mean is

$$\begin{aligned}\delta_1(x) &= 2\mu G(\mu) - 2\alpha^{1/\beta}(j+m+t)^{1/\beta-1}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw} \\ &\quad \times\Gamma\left(1-1/\beta,\alpha(j+m+t)x^{-\beta}\right),\beta>1,\end{aligned} \tag{5.7}$$

where $\mu = \mu_1^{'}$ is the mean of X.

Proof. By definition,

$$\begin{aligned}\delta_1(x) &= \int_0^{\infty}|x-\mu|g(x)dx \\ &= \int_0^{\mu}(\mu-x)g(x)dx + \int_{\mu}^{\infty}(\mu-x)g(x)dx \\ &= 2\mu G(\mu) - 2\int_0^{\mu}xg(x)dx \\ &= 2\mu G(\mu) - 2\alpha^{1/\beta}(j+m+t)^{1/\beta}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw} \\ &\quad \times\Gamma\left(1-1/\beta,\alpha(j+m+t)x^{-\beta}\right),\beta>1,\end{aligned}$$

where $\int_0^{\mu} xg(x)dx$ is simplified using the first incomplete moment.

Proposition 7. The expected value of the absolute deviation of a random variable X having the CFrL distribution from its median is

$$\begin{aligned}\delta_2(x) &= \mu - 2\alpha^{1/\beta}(j+m+t)^{1/\beta}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw} \\ &\quad \times\Gamma\left(1-1/\beta,a^*\right),\beta>1,\end{aligned} \tag{5.8}$$

where $a^* = \alpha(j+m+t)M^{-\beta}$, and M is the median of X.

Proof. By definition,

$$\delta_2(x) = \int_0^{\infty}|x-M|g(x)dx$$

$$
\begin{aligned}
&= \int_0^{\mu}(M-x)g(x)dx + \int_M^{\infty}(M-x)g(x)dx \\
&= \mu - 2\int_0^{M} xg(x)dx \\
&= \mu - 2\alpha^{1/\beta}(j+m+t)^{1/\beta}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw} \\
&\quad \times\Gamma\left(1-1/\beta, a^{*}\right), \beta > 1,
\end{aligned}
$$

where $\int_0^M xg(x)dx$ is simplified using the first incomplete moment.

5.6. Inequality Measures

The Lorenz and Bonferroni curves are frequently used to assess the level of economic inequality in a population. The Bonferroni curve, $B_G(x)$, is the scaled conditional mean curve, which is the ratio of the group mean income of the population. The Lorenz curve, $L_G(x)$, indicates the proportion of total income volume accumulated by those units with income lower than or equal to volume x.

Proposition 8. If $X \sim CFrL(\alpha, \beta, \sigma)$, then the Lorenz curve $L_G(x)$ is given by

$$
\begin{aligned}
L_G(x) &= \frac{\alpha^{1/\beta}(j+m+t)^{1/\beta}}{\mu}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw} \\
&\quad \times\Gamma\left(1-1/\beta, \alpha(j+m+t)x^{-\beta}\right), \beta > 1.
\end{aligned} \tag{5.9}
$$

Proof. By definition,

$$
\begin{aligned}
L_G(x) &= \frac{1}{\mu}\int_0^{x} yg(y)dy \\
&= \frac{\alpha^{1/\beta}(j+m+t)^{1/\beta}}{\mu}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw} \\
&\quad \times\Gamma\left(1-1/\beta, \alpha(j+m+t)x^{-\beta}\right), \beta > 1.
\end{aligned}
$$

Proposition 9. If $X \sim CFrL(\alpha, \beta, \sigma)$, then the Bonferroni curve $B_G(x)$ is given by

$$
\begin{aligned}
B_G(x) &= \frac{\alpha^{1/\beta}(j+m+t)^{1/\beta}}{\mu G(x)}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw} \\
&\quad \times\Gamma\left(1-1/\beta, \alpha(j+m+t)x^{-\beta}\right), \beta > 1.
\end{aligned} \tag{5.10}
$$

Proof. By definition,

$$
\begin{aligned}
B_G(x) &= \frac{L_G(x)}{G(x)} \\
&= \frac{\alpha^{1/\beta}(j+m+t)^{1/\beta}}{\mu G(x)}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw} \\
&\quad \times\Gamma\left(1-1/\beta, \alpha(j+m+t)x^{-\beta}\right), \beta > 1.
\end{aligned}
$$

5.7. *Order Statistics*

For estimating summary statistics like a dataset's minimum, maximum, and range, order statistics are crucial. Additionally, they are utilized in reliability and quality control testing to predict future item failure based on a small number of early failures. Let $X_1, X_2, \dots, X_n$ be a sample of size n from the CFrL distribution and $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ denote the order statistics of the sample.
The PDF of the i^{th} order statistics $g_{i:n}(x)$ is defined as

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i}g(x). \tag{5.11}$$

Using the binomial series expansion, we have

$$[1-G(x)]^{n-i} = \sum_{w=0}^{n-i}(-1)^w\binom{n-i}{w}[G(x)]^w.$$

That is, Equation (5.11) becomes

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}g(x)\sum_{w=0}^{n-i}(-1)^w\binom{n-i}{w}[G(x)]^{w+i-1}. \tag{5.12}$$

Substituting the CDF and PDF of the CFrL distribution into Equation (5.12), we get the i^{th} order statistics as

$$g_{i:n}(x) = \frac{\pi\sigma\alpha\beta x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\left[1+\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right]n!}{2[\sigma-\log(1-e^{-\alpha x^{-\beta}})]^2(i-1)!(n-i)!}$$
$$\times\sum_{w=0}^{n-i}(-1)^w\binom{n-i}{w}\left[1-\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right]\right]^{w+i-1}.$$

The first order statistics is defined by

$$g_{1:n}(x) = n[1-G(x)]^{n-1}g(x). \tag{5.13}$$

Substituting the CDF and PDF of the CFrL distribution into Equation (5.13), we get the PDF of the first-order statistics as

$$\begin{aligned} g_{1:n}(x) &= n\left[\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right]\right]^{n-1} \\ &\times\frac{\pi\alpha\beta\sigma}{2}\left[\frac{x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\left[1+\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]}{\left[\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]^2}\right] \\ &\times\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right]. \end{aligned} \tag{5.14}$$

Additionally, the PDF of the n^{th} order statistics is defined as

$$g_{n:n}(x) = n[G(x)]^{n-1} g(x). \tag{5.15}$$

Substituting the CDF and PDF of the CFrL distribution into Equation (5.15), we get the PDF of the n^{th} order statistics as

$$\begin{aligned} g_{n:n}(x) =& n\left[1-\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right]\right]^{n-1} \\ &\times \frac{\pi\alpha\beta\sigma}{2}\left[\frac{x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\left[1+\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]}{\left[\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]^2}\right] \\ &\times \sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right]. \end{aligned} \tag{5.16}$$

6. Actuarial Measures

In this section, actuarial measures of the CFrL distribution such as the value at risk, tail value at risk, and tail variance are derived and studied.

6.1. Value at Risk

For any insurance company, risk exposure is an inevitable occurrence. The Value at Risk (VaR) evaluates the amount a set of investments could lose and establishes the risk of a potential loss for the insurance company with a given likelihood. That is, VaR represents the percentage of loss in a portfolio value that will be equaled or exceeded only X percent of the time.

Proposition 10. The VaR of the CFrL distribution is defined by

$$x_q = G^{-1}(q) \tag{6.1}$$

which is obtained by the solution of the equation; $\left[1-\left(\frac{2}{\pi}\arccos(1-q)\right)\right]\left[\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right] - \sigma\left(1-e^{-\alpha x^{-\beta}}\right) = 0,\ q \in (0,1).$

6.2. Tail Value at Risk

The tail value at risk (TVaR) is also called the tail conditional expectation (TCE) or conditional tail expectation (CTE) and is used for determining the average loss beyond a given probability level.

Proposition 11. If $X \sim CFrL(\alpha,\beta,\sigma)$, then, the $TVaR_q(x)$ of the CFrL distribution is given by

$$TVaR_q(x) = \frac{1}{1-q}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}\frac{\Gamma\left(1-1/\beta, \alpha(j+m+t)VaR_q^{-\beta}\right)}{\alpha^{-1/\beta}(j+m+t)^{-1/\beta}}, \beta > 1, \tag{6.2}$$

where $\Gamma(.,.)$ is the upper incomplete gamma function and $r = 1, 2, 3, ...$
Proof. By definition,

$$TVaR_q(x) = \frac{1}{1-q}\int_{VaR_q}^{\infty} xg(x)dx.$$

This implies that,

$$TVaR_q(x) \quad = \quad \frac{1}{1-q}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}(j+m+t)\int_{VaR_q}^{\infty} \alpha\beta x.x^{-\beta-1}e^{-\alpha(j+m+t)x^{-\beta}}dx.$$

Letting $z = \alpha(j+m+t)x^{-\beta}$, implies that if $x \to VaR_q$, $z = \alpha(j+m+t)VaR_q^{-\beta}$, $z \to 0$, $x \to 0$, $x = \left(\frac{z}{\alpha(j+m+t)}\right)^{-1/\beta}$, and $dx = -\frac{dz}{\alpha\beta(j+m+t)x^{-\beta-1}}$. After some algebraic manipulations, and making use of the incomplete gamma function of the form $\Gamma(a,q) = \int_0^q x^{s-1}e^{-x}dx$, we have,

$$TVaR_q(x) = \frac{1}{1-q}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}\frac{\Gamma\left(1-1/\beta, \alpha(j+m+t)VaR_q^{-\beta}\right)}{\alpha^{-1/\beta}(j+m+t)^{-1/\beta}}, \beta > 1.$$

6.3. Tail Variance

Tail variance (TV) is an important risk measure in insurance sciences. It is vital in determining the risk level at the tails.
Proposition 12. If $X \sim CFrL(\alpha, \beta, \sigma)$, the $TV_q(x)$ of the CFrL distribution is given by

$$\begin{aligned} TV_q(x) \quad = \quad & \frac{1}{1-q}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}\frac{\Gamma\left(1-2/\beta, \alpha(j+m+t)VaR_q^{-\beta}\right)}{\alpha^{-2/\beta}(j+m+t)^{-2/\beta}} \\ & -\left[\frac{1}{1-q}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}\frac{\Gamma\left(1-1/\beta, \alpha(j+m+t)VaR_q^{-\beta}\right)}{\alpha^{-1/\beta}(j+m+t)^{-1/\beta}}\right]^2. \end{aligned} \tag{6.3}$$

Proof. By definition,

$$TV_q(x) = \mathbb{E}(X^2|X > x_q) - (TVaR_q)^2.$$

Considering,

$$\mathbb{E}(X^2|X > x_q) = \frac{1}{1-q}\int_{VaR_q}^{\infty} x^2 g(x)dx.$$

This implies that,

$$\mathbb{E}(X^2|X > x_q) = \frac{1}{1-q}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}(j+m+t)\int_{VaR_q}^{\infty} \alpha\beta x^2.x^{-\beta-1}e^{-\alpha(j+m+t)x^{-\beta}}dx.$$

On solving,

$$\mathbb{E}(X^2|X > x_q) = \frac{1}{1-q}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}\frac{\Gamma\left(1-2/\beta, \alpha(j+m+t)VaR_q^{-\beta}\right)}{\alpha^{-2/\beta}(j+m+t)^{-2/\beta}}, \beta > 2.$$

Therefore,

$$\begin{aligned} TV_q(x) &= \frac{1}{1-q}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}\frac{\Gamma\left(1-2/\beta, \alpha(j+m+t)VaR_q^{-\beta}\right)}{\alpha^{-2/\beta}(j+m+t)^{-2/\beta}} \\ &\quad -\left[\frac{1}{1-q}\sum_{j,k=0}^{\infty}\sum_{m=0}^{i}\sum_{i=0}^{2k} D_{imkj}A_{jtw}\frac{\Gamma\left(1-1/\beta, \alpha(j+m+t)VaR_q^{-\beta}\right)}{\alpha^{-1/\beta}(j+m+t)^{-1/\beta}}\right]^2. \end{aligned}$$

Table 2 shows the numerical results of the actuarial measures for set I : $\alpha = 0.3, \beta = 1.74, \sigma = 1.8$ and set II: $\alpha = 0.5, \beta = 1.65, \sigma = 1.3$ and a range of confidence levels for the CFrL distribution. This is displayed graphically in Figures 5 to 7. It is evident that increasing confidence levels are linked to increasing VaR, TVaR and TV. In the insurance business, if more funds are channeled towards managing risk, then a company is likely to remain solvent or operational. Also, increasing values of the actuarial measures are indications that the CFrL distribution is a heavy-tailed distribution. The simulation steps are as follows:

1. Random sample of size $n = 150$ is generated from the CFrL distribution and parameters estimated via maximum likelihood method.
2. 1000 repetitions are made to calculate the VaR, TVaR and TV for the CFrL distribution.

Table 2. Simulation results of the actuarial measures for CFrL distribution

Parameters	Confidence Level	VaR	TVaR	TV
	0.710	0.573	6.821	43.013
	0.750	0.576	7.417	46.274
$\alpha = 0.3$	0.790	0.580	7.545	47.296
$\beta = 1.74$	0.820	0.582	7.625	48.159
$\sigma = 1.8$	0.860	0.585	7.690	48.371
	0.900	0.591	7.783	48.864
	0.950	0.592	8.544	53.446
	0.970	0.595	9.591	60.781
	0.710	0.585	6.694	54.784
	0.750	0.588	6.856	55.888
$\alpha = 0.5$	0.790	0.592	7.267	59.346
$\beta = 1.65$	0.820	0.597	7.415	60.388
$\sigma = 1.3$	0.860	0.598	8.306	68.106
	0.900	0.600	8.550	70.385
	0.950	0.605	9.077	74.368
	0.970	0.609	9.273	76.237

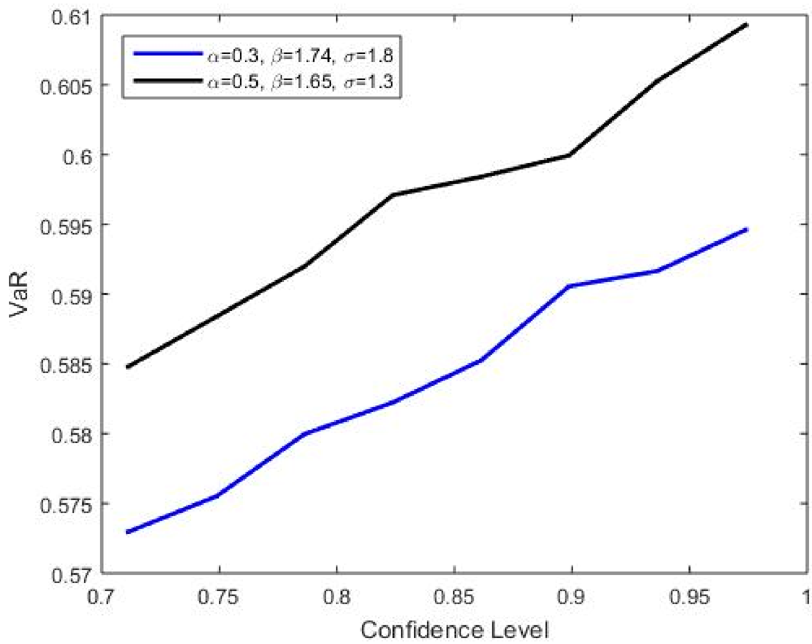

Figure 5. Plot for VaR of the CFrL distribution

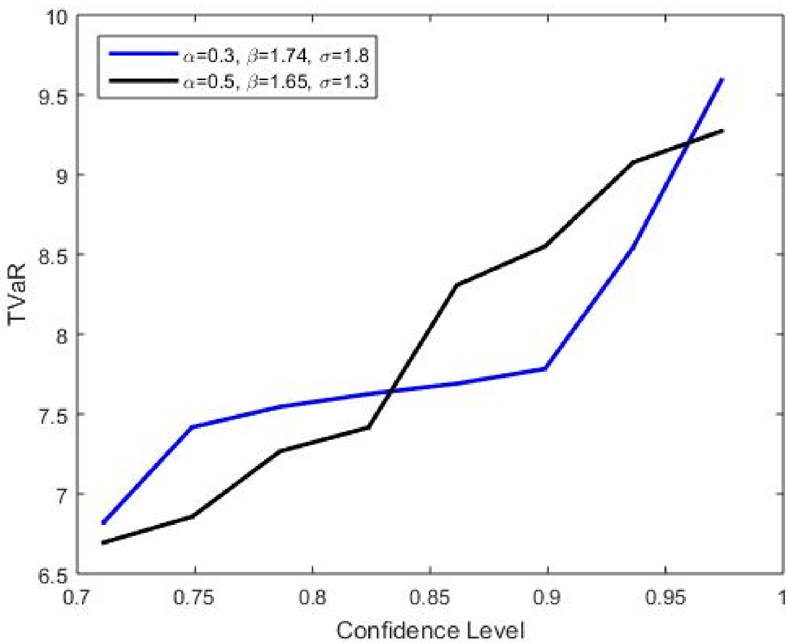

Figure 6. Plot for TVaR of the CFrL distribution

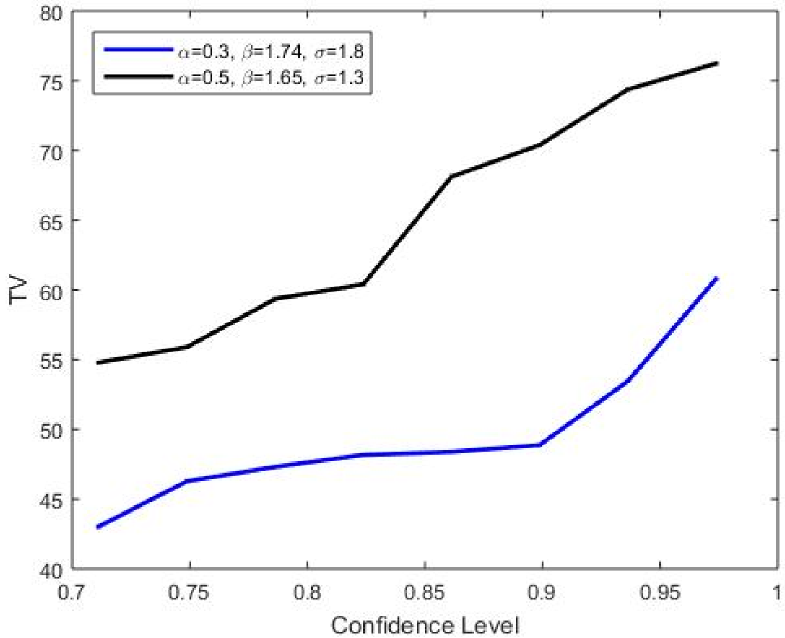

Figure 7. Plot for TV of the CFrL distribution

7. Parameter Estimation

In this section the unknown parameters of the CFrL are estimated using the maximum likelihood estimation (MLE) technique.

7.1. Maximum Likelihood Estimation

The MLE is used in estimating the parameters of the CFrL distribution. If $X_1, X_2, ..., X_n$ are n random sample from the CFrL distribution and $\theta = (\sigma, \alpha, \beta)^T$, then the log-likelihood function, $\ell = \ell(\theta)$, is given by

$$\begin{aligned}\ell \;=\; & n\left(\log\left(\frac{\pi\alpha\beta\sigma}{2}\right)\right) - (\beta+1)\sum_{i=1}^{n}\log(x_i) - \alpha\sum_{i=1}^{n}x_i^{-\beta} \\ & + \sum_{i=1}^{n}\log\left[1+\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)\right] - 2\sum_{i=1}^{n}\log\left[\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)\right] \\ & + \sum_{i=1}^{n}\log\left[\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x_i^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)}\right)\right]\right]. \end{aligned} \tag{7.1}$$

The log-likelihood function in Equation (7.1) is differentiated with respect to each parameter to obtain the score function, $U(\theta) = \left(\frac{\partial\ell}{\partial\sigma}, \frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}\right)^T$. The elements of the score function are given by

$$\frac{\partial\ell}{\partial\sigma} = \frac{n}{\sigma} - 2\sum_{i=1}^{n}\frac{1}{\sigma-\log\left[1-e^{-\alpha x_i^{-\beta}}\right]} + \sum_{i=1}^{n}\frac{1}{1+\sigma-\log\left[1-e^{-\alpha x_i^{-\beta}}\right]}$$

$$
\begin{aligned}
&+\frac{\pi}{2}\sum_{i=1}^{n}\cot\left(\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x_i^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)}\right)\right) \\
&\times\left[\frac{\sigma\left(1-e^{-\alpha x_i^{-\beta}}\right)}{\left(\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)\right)^2}-\frac{1-e^{-\alpha x_i^{-\beta}}}{\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)}\right],
\end{aligned}
\tag{7.2}
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \alpha}=&\frac{n}{\alpha}-\sum_{i=1}^{n}x_i^{-\beta}+2\sum_{i=1}^{n}\frac{e^{-\alpha x_i^{-\beta}}x_i^{-\beta}}{\left(1-e^{-\alpha x_i^{-\beta}}\right)\left(\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)\right)} \\
&-\sum_{i=1}^{n}\frac{e^{-\alpha x_i^{-\beta}}x_i^{-\beta}}{\left(e^{-\alpha x_i^{-\beta}}x_i^{-\beta}\right)\left(1+\sigma-\log\left(e^{-\alpha x_i^{-\beta}}x_i^{-\beta}\right)\right)} \\
&+\frac{\pi}{2}\sum_{i=1}^{n}\cot\left(\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x_i^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)}\right)\right) \\
&\times\left[\frac{-\sigma e^{-\alpha x_i^{-\beta}}x_i^{-\beta}}{\left(\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)\right)^2}-\frac{\sigma e^{-\alpha x_i^{-\beta}}x_i^{-\beta}}{\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)}\right]
\end{aligned}
\tag{7.3}
$$

and

$$
\begin{aligned}
\frac{\partial \ell}{\partial \beta}=&\frac{n}{\beta}-\sum_{i=1}^{n}\log x_i+\alpha\sum_{i=1}^{n}x_i^{-\beta}\log x_i-2\sum_{i=1}^{n}\frac{e^{-\alpha x_i^{-\beta}}x_i^{-\beta}\alpha\log x_i}{\left(1-e^{-\alpha x_i^{-\beta}}\right)\left(\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)\right)} \\
&+\sum_{i=1}^{n}\frac{e^{-\alpha x_i^{-\beta}}\alpha\log x_i}{\left(1-e^{-\alpha x_i^{-\beta}}\right)\left(1+\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)\right)} \\
&+\frac{\pi}{2}\sum_{i=1}^{n}\cot\left(\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x_i^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)}\right)\right) \\
&\times\left[\frac{e^{-\alpha x_i^{-\beta}}x_i^{-\beta}\alpha\sigma\log x_i}{\left(\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)\right)^2}+\frac{e^{-\alpha x_i^{-\beta}}x_i^{-\beta}\alpha\sigma\log x_i}{\sigma-\log\left(1-e^{-\alpha x_i^{-\beta}}\right)}\right].
\end{aligned}
\tag{7.4}
$$

The estimates for the parameters α, β and σ are obtained by equating the score functions to zero and solving the system of non-linear equations numerically.

8. Monte Carlo Simulation

In this section, the simulation results are presented in examining the properties of the maximum likelihood estimators for the parameters of the CFrL distribution. The nlminb function in the R program is used in the simulation. The function uses the L-BFGS-B optimization method. Table 3 shows the simulation results for the CFrL distribution. It can be observed that, as the sample size increases, the AB and RMSE for the estimators of the parameters decrease . This shows that the estimators are consistent. The simulation steps are as follows:

1. Generate $N = 1000$ sample of size $n = 50, 100, 150, 200, 400$ from the quantile function of the CFrL distribution.
2. Find the maximum likelihood estimates for the parameters.
3. Repeat steps i-ii for 1000 times.
4. For each parameter estimate, calculate the average baise (AB) and root mean square error (RMSE) defined as

$$AB = \frac{1}{N}\sum_{i=1}^{N}(\hat{\omega}_i - \omega_i)$$

and

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\omega}_i - \omega_i)^2}$$

for $\omega = (\alpha, \beta, \sigma)$, respectively.

Table 3. Monte Carlo Simulation Results: AB and RMSE for the Parameters of the CFrL distribution

n	Parameter value			AB			RMSE		
	α	β	σ	α	β	σ	α	β	σ
50	2.8	2.2	1.1	0.800	0.488	0.900	0.640	0.254	0.810
100	2.8	2.2	1.1	0.510	0.481	0.813	0.640	0.246	0.810
150	2.8	2.2	1.1	0.328	0.404	0.730	0.582	0.243	0.708
200	2.8	2.2	1.1	0.019	0.062	0.211	0.240	0.102	0.380
400	2.8	2.2	1.1	0.011	0.030	0.092	0.099	0.042	0.141
50	1.6	1.5	1.2	0.237	0.165	0.721	0.076	0.042	0.551
100	1.6	1.5	1.2	0.234	0.147	0.714	0.075	0.030	0.534
150	1.6	1.5	1.2	0.231	0.134	0.688	0.073	0.025	0.507
200	1.6	1.5	1.2	0.070	0.033	0.084	0.072	0.024	0.201
400	1.6	1.5	1.2	0.022	0.016	0.051	0.034	0.016	0.117
50	3.3	2.1	1.5	1.308	0.699	0.520	1.691	0.501	0.250
100	3.3	2.1	1.5	1.300	0.698	0.511	1.672	0.493	0.246
150	3.3	2.1	1.5	0.284	0.697	0.502	1.640	0.490	0.218
200	3.3	2.1	1.5	0.251	0.694	0.428	0.859	0.489	0.197
400	3.3	2.1	1.5	0.160	0.217	0.183	0.181	0.130	0.082
50	1.2	2	2.1	0.339	0.242	1.096	0.218	0.119	1.768
100	1.2	2	2.1	0.285	0.218	1.096	0.175	0.102	1.558
150	1.2	2	2.1	0.262	0.208	0.966	0.154	0.093	1.452
200	1.2	2	2.1	0.245	0.205	0.910	0.145	0.090	1.355
400	1.2	2	2.1	0.099	0.162	0.280	0.064	0.029	0.125
50	4.9	1.8	5.1	2.970	1.008	3.100	1.410	1.018	0.610
100	4.9	1.8	5.1	2.901	1.005	3.060	1.402	1.013	0.388
150	4.9	1.8	5.1	2.660	1.003	3.001	1.296	1.011	0.047
200	4.9	1.8	5.1	1.583	1.001	1.894	1.013	1.008	0.005
400	4.9	1.8	5.1	0.923	0.081	0.537	1.009	1.002	0.001
50	0.01	0.9	0.02	0.047	0.404	0.259	0.069	0.275	0.355
100	0.01	0.9	0.02	0.020	0.343	0.184	0.013	0.206	0.217
150	0.01	0.9	0.02	0.015	0.298	0.138	0.002	0.158	0.159
200	0.01	0.9	0.02	0.012	0.297	0.126	0.001	0.156	0.122
400	0.01	0.9	0.02	0.009	0.172	0.116	0.001	0.045	0.091

9. Applications

This section illustrates the usefulness and flexibility of the CFrL distribution using two insurance loss datasets. This is done using the R software. The performance of the CFrL distribution is compared with other loss distributions. The performance of the distributions about providing proper parametric fit to the dataset was compared using the AIC, BIC, HQIC, and K-S statistics. The distribution with the least of these measures provides a reasonable fit to the dataset. The fit for CFrL is compared with other heavy-tailed distributions, including the 2-parameter Weibull, Weibull-Loss (W-Loss), Fréchet, Lomax, Power-Lomax, exponentiated Weibull (EW), Dagum, 2-parameter Burr XII (BXII), and sine inverse Lomax Fréchet (SILF). The distribution functions of the W-Loss, Fréchet, Weibull, Dagum, Power-Lomax, SILF, BXII, Lomax, and EW are :

$$F(x;\sigma,\alpha,\gamma) = 1 - \frac{\sigma e^{-\gamma x^{\alpha}}}{\sigma + \gamma x^{\alpha}}, \quad x \geq 0, \sigma, \alpha, \gamma > 0,$$

$$F(x;\alpha,\beta) = e^{-\alpha x^{-\beta}}, \quad x \geq 0, \alpha, \beta > 0,$$

$$F(x;\alpha,\gamma) = 1 - e^{-\gamma x^{\alpha}}, \quad x \geq 0, \alpha, \gamma > 0,$$

$$F(x;\alpha,\beta,\lambda) = (1 + \lambda x^{-\alpha})^{-\beta}, \quad x \geq 0, \alpha, \beta, \lambda > 0,$$

$$F(x;\alpha,\beta,\lambda) = 1 - \lambda^{\alpha}\left(x^{\beta} + \lambda\right)^{-\alpha}, \quad x \geq 0, \alpha, \beta, \lambda > 0,$$

$$F(x;\alpha,\gamma,\beta) = \sin\left(\frac{\pi}{2} e^{-\alpha\left(\frac{\gamma}{x}\right)^{\beta}}\right), \quad x \geq 0, \alpha, \gamma, \beta > 0,$$

$$F(x;c,k) = 1 - (1 + x^{c})^{-k}, \quad x \geq 0, c, k > 0,$$

$$F(x;\alpha,\lambda) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \quad x \geq 0, \alpha, \lambda > 0$$

and

$$F(x;\lambda,\alpha,\gamma) = \left(1 - e^{-\gamma x^{\alpha}}\right)^{\lambda}, \quad x \geq 0, \lambda, \alpha, \gamma > 0.$$

respectively.

9.1. *Belgium Fire Loss*

The first dataset consists of 1,823 fire losses in thousand of Danish krones (DKK). This data is available in CASdataset of R package (Dutang and Charpentier [19]).
Table 4 shows the descriptive statistics of the Belgium fire losses. It can be seen that the losses are right skewed and leptokurtic, with a long right tail.
Figure 8 shows the TTT-transform plot for the Belgium Fire Loss dataset. The data exhibits a decreasing hazard rate since the curve is convex below the 45 degree line. Table 5 shows the maximum likelihood estimates for the parameters of the fitted distributions with their corresponding errors in brackets. The parameters of all the distributions fitted are significant at the 5% with the exception of Dagum distribution which had β and λ to be significant at 10%.

Table 4. Descriptive Statistics of the Belgium Fire Losses

No. of Claims	Mean	Std.	Skewness	Kurtosis	Min.	Max.
1,823	363.460	4868.259	33.903	1289.322	10.150	90541.700

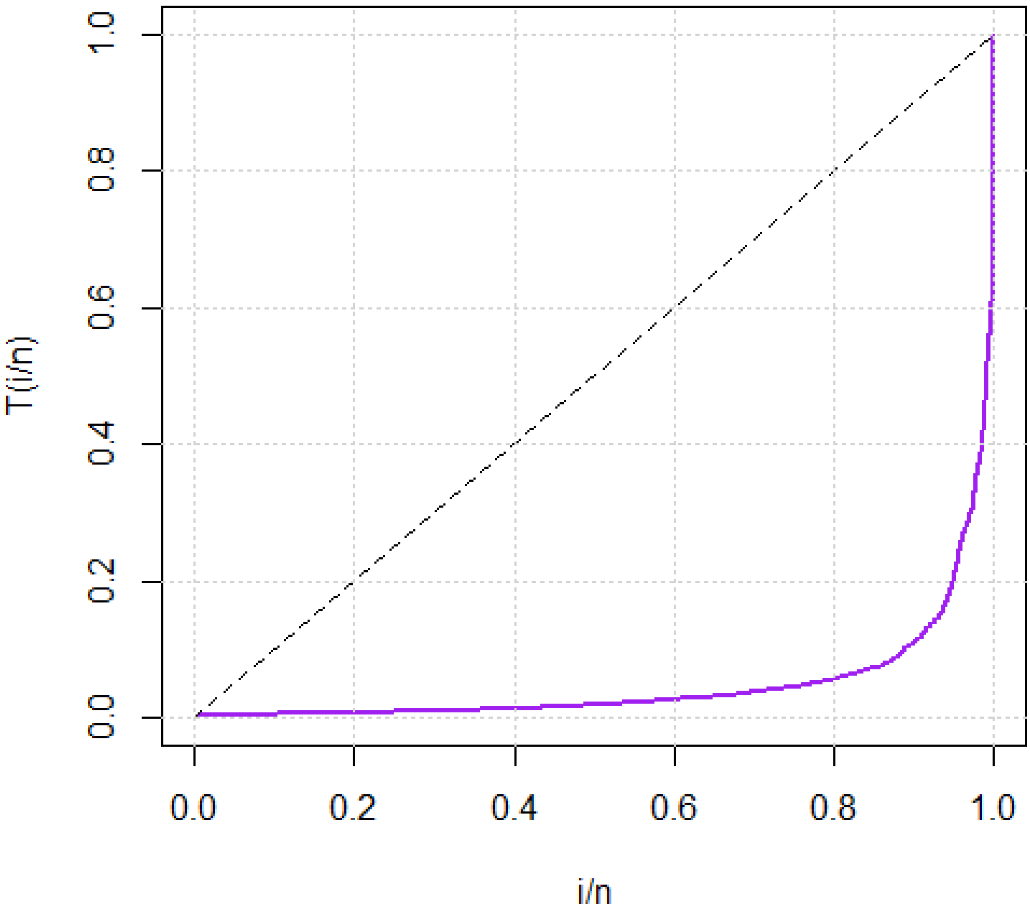

Figure 8. TTT-transform plot for Belgium fire losses

Table 6 shows the information criteria and goodness-of-fit of the fitted distributions. It can be seen that the CFrL distribution is the best distribution providing a reasonable fit to the dataset among the other heavy tailed distributions fitted since it has the least AIC, BIC, HQIC, K-S, and $-2l$ values compared with all the competing distributions.

Table 5. Maximum likelihood estimates of the parameters and standard errors for Belgium fire Loss dataset

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{c}$	$\hat{k}$	$\hat{\lambda}$
CFrL	2.882	0.569	4.730				
	(0.364)	(0.025)	(2.554)				
BXII					0.097	3.919	
					(0.007)	(0.264)	
Dagum	0.681	18.960					0.176
	(0.014)	(5.852)					(0.058)
EW	0.094			5.389			5.952
	(0.002)			(0.026)			(0.004)
Fréchet	4.193	0.461					
	(0.145)	(0.108)					
Lomax	0.680						6.098
	(0.025)						(0.435)
Power-Lomax	0.220	2.262					5.282
	(0.186)	(0.144)					(0.484)
SILF	2.988	1.238		0.458			
	(0.029)	(0.032)		(0.008)			
Weibull	0.411			0.221			
	(0.006)			(0.009)			
W-Loss	0.733		0.015	0.002			
	(0.430)		(0.003)	(0.001)			

Figures 9 and 10 show the plots of the empirical density, the fitted density, the empirical CDF, and the PDF of the fitted distributions respectively. It is evident that the CFrL distribution is also among the distributions that provide reasonable fit to the data.

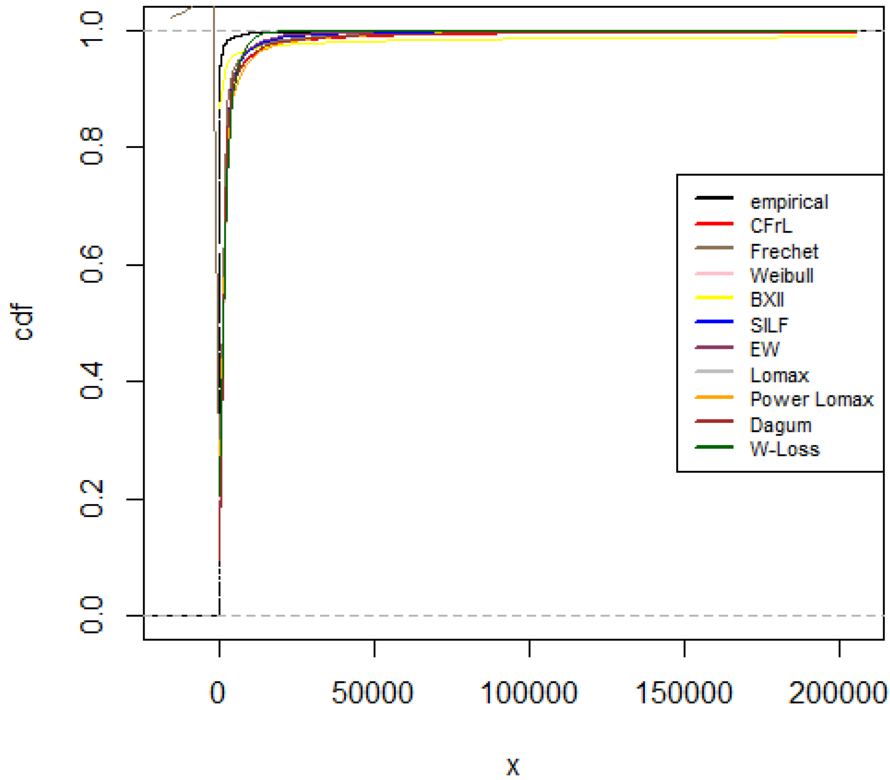

Figure 9. Empirical and CDF plots of Belgium fire loss

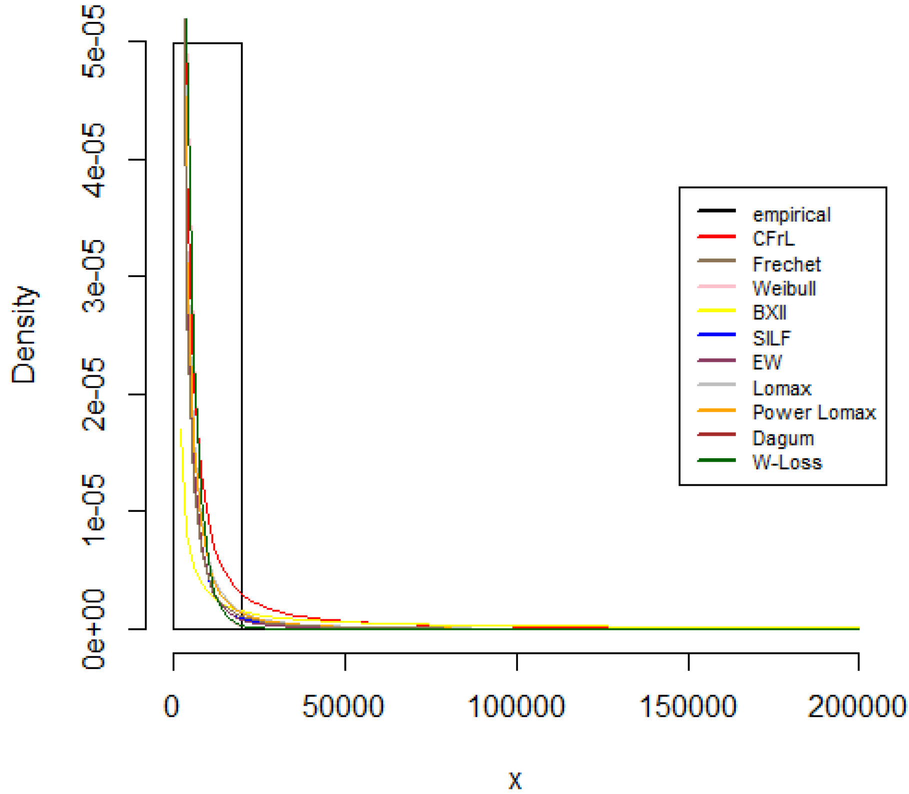

Figure 10. Empirical and PDF plots of Belgium fire loss

Table 6. Information Criteria and Goodness-of-fit of Belgium fire loss dataset

Model	$-2l$	AIC	BIC	HQIC	K-S(p-value)
CFrL	15972.500	15976.500	15987.520	15980.560	0.026(0.682)
BXII	17004.730	28920.660	28931.680	28924.720	0.883(0.184)
Dagum	16801.950	16807.950	16824.470	16884.530	0.044(0.601)
EW	16903.150	16909.150	16925.680	16915.250	0.049(0.508)
Fréchet	16795.820	17395.430	17411.960	17401.530	0.214(0.242)
Lomax	17002.060	17006.060	17017.070	17010.120	0.492(0.287)
Power-Lomax	16806.170	16812.170	16828.700	16818.270	0.042(0.659)
SILF	16876.720	16882.720	16899.240	16888.810	0.055(0.510)
Weibull	17994.820	17998.820	18009.830	18002.880	0.166(0.318)
W-Loss	17271.590	17277.590	17294.110	17283.680	0.105(0.483)

9.2. Vehicle Insurance Loss

The second dataset consists of a one-year vehicle insurance policies. There are 67,856 policies of which 4,624 (6.8%) made at least one claim. This data is available at http://www.businessandeconomics.mq.edu.au. This data was also studied by Ahmad et al. [18]. Table 7 shows the descriptive statistics of the vehicle insurance loss dataset. It can be seen that the data is right-skewed and leptokurtic depicting a typical feature of insurance loss data.

Table 7. Descriptive Statistics of Vehicle Insurance Loss

No. of Claims	Mean	Std.	Skewness	Kurtosis	Min.	Max.
4,624	2014.400	3549.489	5.040	43.205	200.000	55,922.130

Figure 11 shows the TTT-transform plot for the vehicle insurance loss dataset. The data exhibits a decreasing hazard rate since the curve is convex below the 45 degree line.
Table 8 shows the maximum likelihood estimates for the parameters of the fitted distributions with their corresponding errors in brackets. The parameters of all the distributions fitted are significant at the 5%.
Table 9 shows the information criteria and goodness-of-fit of the fitted distributions. It can be seen that, the CFrL distribution is best distribution providing reasonable fit to the dataset among the other heavy tailed distributions fitted since it has the least AIC, BIC, HQIC, K-S, and $-2l$ values compared with all the competing distributions. Figures 12 and **??** shows the plots of the empirical density, the fitted density, the empirical CDF and the PDF of the fitted distributions respectively. It is evident that, the CFrL distribution is also among the distributions that provide reasonable fit to the data.

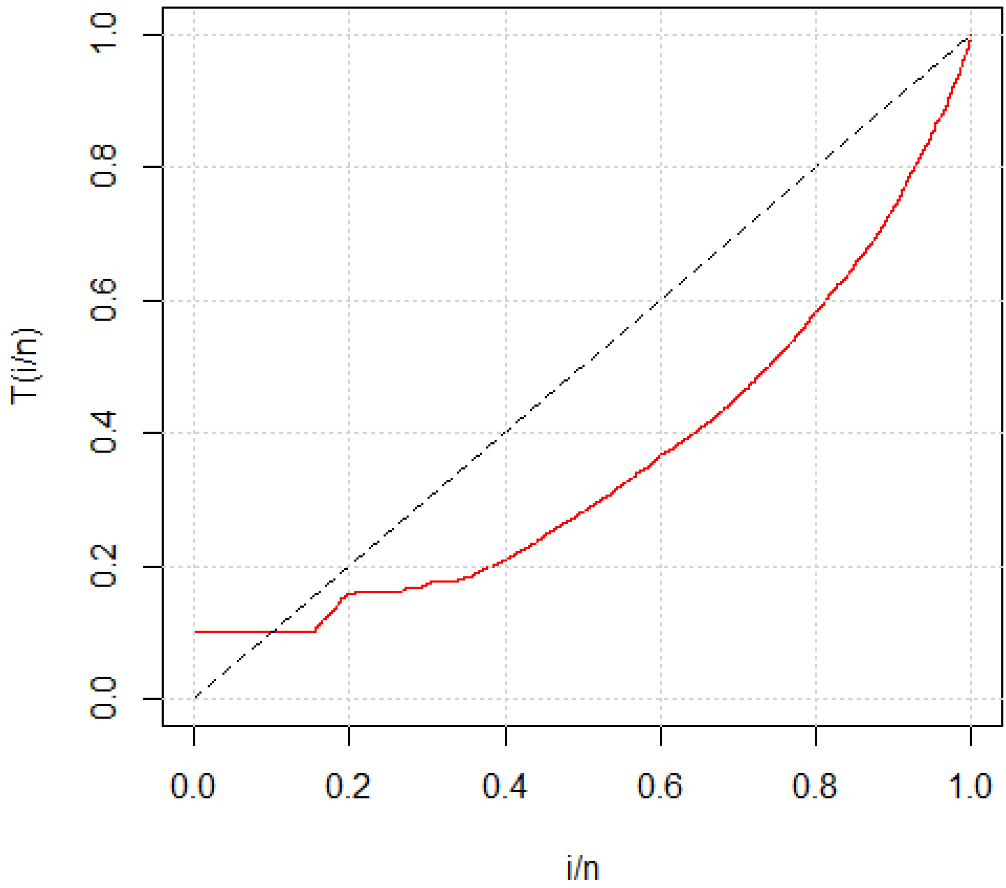

Figure 11. TTT-transform plot for vehicle insurance loss dataset

10. Conclusions, Limitations and Future Research

In this paper, we have proposed a modified version of the Fréchet distribution known as the cosine Fréchet Loss distribution using the cosine F-Loss generator. This distribution is flexible and able to model varying shapes of the hazard rate compared with the traditional two parameter Fréchet distribution. The density exhibits different kinds of decreasing, and right-skewed shapes. The hazard rate function show different kinds of increasing-constant-decreasing, reversed-J, bathtub, and upside down bathtub shapes. The statistical properties including quantile function, generating functions, inequality measures, order statistics, mean and median deviations, moments and incomplete moments are studied. Using numerical integration, the first four moments of the CFrL distribution were obtained. These moments were then used in estimating the SD, CV, CS, and CK. The skewness is always positive and the kurtosis is increasing. This is evident in the skewness and kurtosis plots. Actuarial measures including VaR, TVaR, and TV are derived and studied. The numerical values of the actuarial measures show that increasing confidence levels are associated with increasing VaR, TVaR, and TV. This is evident in the VaR, TVaR, and TV plots. This shows that the CFrL is a heavy tailed distribution. The maximum likelihood estimators are studied and simulations carried out to ascertain the behavior of the estimators. It is observed that the estimators are consistent since increasing sample size was associated with decreasing AB and RMSE estimates. The usefulness of the

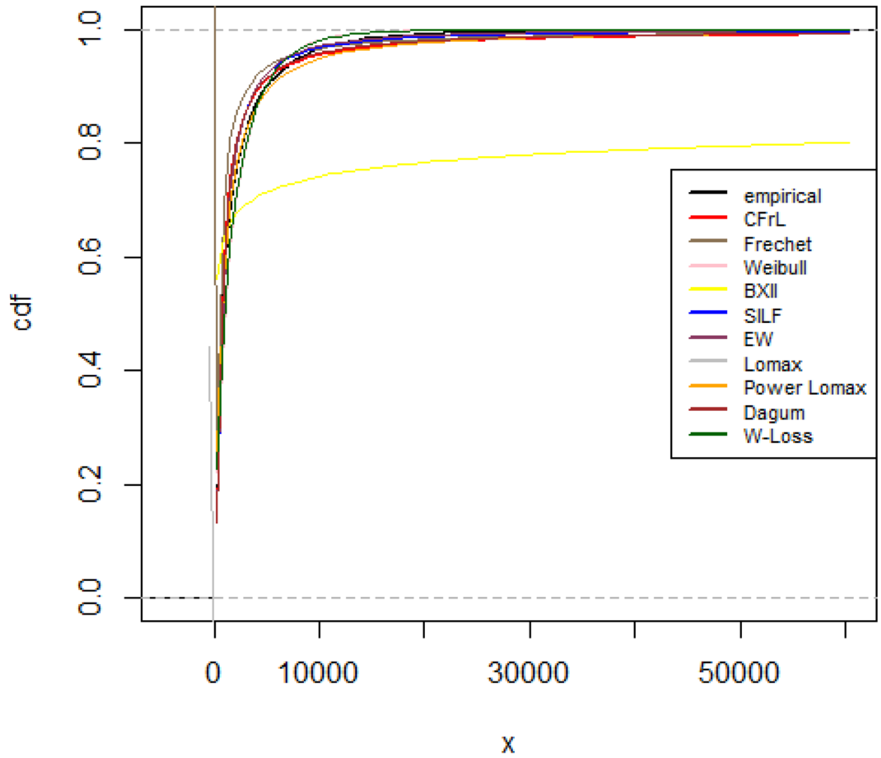

Figure 12. Empirical and CDF plots of vehicle insurance loss

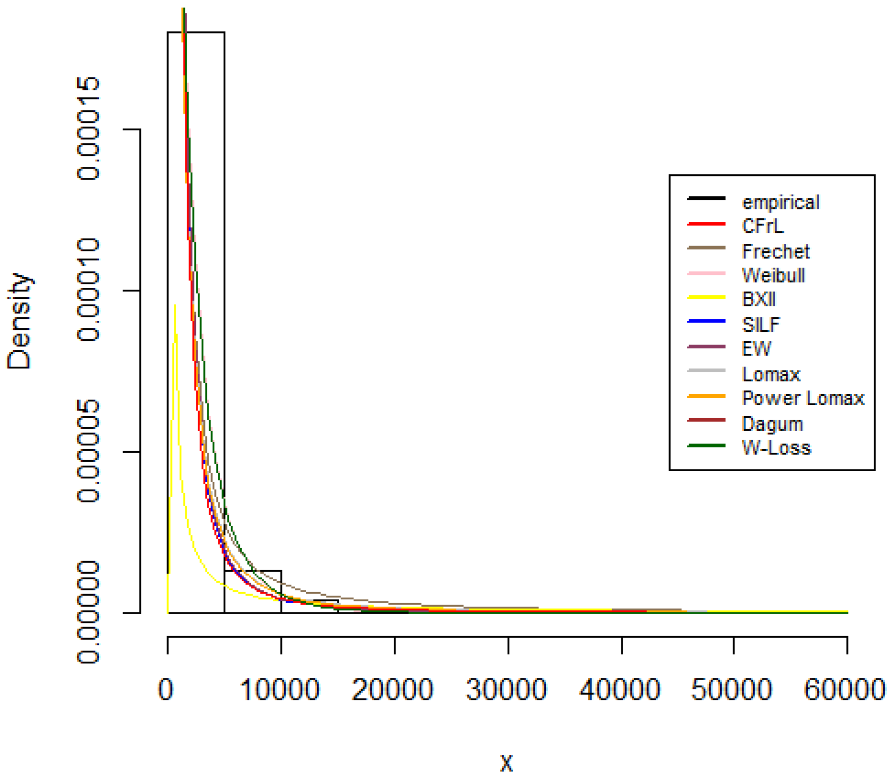

Figure 13. Empirical and PDF plots of vehicle insurance loss

Table 8. Maximum likelihood estimates of the parameters and standard errors for vehicle insurance loss dataset

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{c}$	$\hat{k}$	$\hat{\lambda}$
CFrL	292.247	1.002	270.504				
	(4.095)	(0.012)	(0.013)				
BXII					0.074	1.972	
					(0.028)	(0.745)	
Dagum	1.074	29.039					28.973
	(0.016)	(11.067)					(13.088)
EW	0.156			2.321			4.990
	(0.002)			(0.025)			(0.004)
Fréchet	5.276	0.650					
	(9.626)	(0.010)					
Lomax	1.200						987.405
	(0.032)						(39.069)
Power-Lomax	1.264	0.984					9.108
	(0.061)	(0.010)					(0.001)
SILF	44.170	4.859		0.723			
	(0.039)	(0.258)		(0.008)			
Weibull	0.765			0.003			
	(0.008)			(0.002)			
W-Loss	7.519		4.798	0.004			
	(0.009)		(0.002)	(0.003)			

proposed distribution was investigated using two insurance loss datasets namely the Belgium Fire Loss and vehicle insurance loss and the performance compared with other known classical heavy-tailed distributions. The results showed that the proposed model provide a better parametric fit compared to the other heavy-tailed distributions.
The limitations of this study has to do with choosing initial values to include in the simulations so as achieve convergence. In the future a pricing model could be developed using the proposed model. Also, a regression model could be developed for the CFrL distribution.

Data Availability: The data that support the findings of this study are available from the corresponding author.

Conflict of Interest: There is no conflict of interest.

Table 9. Information Criteria and Goodness-of-fit of vehicle insurance loss dataset

Model	$-2l$	AIC	BIC	HQIC	K-S(p-value)
CFrL	76179.360	76183.360	76196.240	76187.310	0.084(0.852)
BXII	89930.440	89934.440	89947.320	89938.970	0.541(0.429)
Dagum	77184.080	77190.080	77209.400	77196.970	0.088(0.803)
EW	77307.260	77313.260	77332.580	77320.060	0.094(0.710)
Fréchet	77177.030	77183.560	77202.870	77190.350	0.094(0.710)
Lomax	78514.300	78518.300	78531.180	78522.830	0.910(0.226)
Power-Lomax	78549.520	78555.520	78574.830	78562.310	0.207(0.364)
SILF	77311.480	77317.480	77336.790	77324.270	0.095(0.694)
Weibull	78955.570	78967.570	78972.950	78731.760	0.087(0.813)
W-Loss	78966.050	78972.050	78991.370	78978.850	0.187(0.311)

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