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Some Inferences on Three Parameters Birnbaum-Saunders Distribution: Statistical Properties, Characterizations and Applications

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Abstract: Three parameters Birnbaum-Saunders, BS (3P), distribution is an important statistical model which is useable in the fields of both pure and applied sciences. Since its inception, Birnbaum-Saunders (BS) distribution has received a considerable attention in view of its wide applications in many areas of research in applied sciences, such as engineering science, earth science, environmental science, medical science, material fatigue and reliability studies. The objective of this paper is a statistical analysis of the three parameter Birnbaum-Saunders, BS (3P), distribution and to draw some inferences on it. Several new distributional properties of this distribution have been discussed. Based on these distributional properties, we have established several new characterization results of the three-parameter Birnbaum-Saunders distribution. Finally, applications to some real-life data sets are analyzed to show the usefulness of this distribution. The results of this article will be useful for the researchers, scientists, and statisticians in fields of theoretical and applied sciences.

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1. Introduction

For modelling the fatigue life of a metal, subject to cyclic stress, Birnbaum and Saunders introduced a two-parameter lifetime distribution in 1969; see Birnbaum and Saunders [1]. It is named after Z. W. Birnbaum and S. C. Saunders. This distribution was initially introduced to model failures due to cracks in the buildings. The BS distribution is unimodal, asymmetric, and has two parameters that modify its shape and scale. The BS distribution is closed under scalar multiplication and under reciprocation; its median coincides with the BS scale parameter; it has different shapes for its probability density function (PDF), which cover high, medium, and low asymmetry levels [29].

After its inception in physics of materials and engineering, the BS distribution has been widely studied from theoretical and methodological perspectives, and has received special attention in view of its wide-range applications in many areas of research in applied sciences, such as engineering science, earth science, environmental science, medical science, material fatigue and reliability studies, among others. Bhattacharyya et. al. [28], established that a BS distribution can be obtained as an approximation of an inverse Gaussian (IG) distribution. Desmond [30] examined that a BS distribution can be viewed as a mixture of an IG distribution and its reciprocal. Further developments continued with the contributions of many authors and researchers. See, for example, Johnson et al. [2], Athayde et al. [3], Leiva [4], and Balakrishnan and Kundu [5].

It appears from the literature that despite extensive work already done on the two-parameter Birnbaum-Saunders distribution, not much attention has been paid to the three-parameter Birnbaum-Saunders distribution. Motivated by this fact, the objective of this paper is to study a three-parameter Birnbaum-Saunders distribution, its distributional properties, applications, characterizations and draw some inferences on it.

The paper divides the work into the following sections: In Section 2, we give a description of the Birnbaum-Saunders three-parameter distribution, BS (3P). Several new distributional properties are given in Section 3. The computations of percentage points are provided in Section 4. Some characterizations are given in Section 5. The estimation of the parameters and some real lifetime data to show the applications of Birnbaum-Saunders distribution are discussed in Section 6. The concluding remarks are enumerated in Section 7. An appendix defining and explaining the necessary terms from the statistical literature, and important abbreviation, used in the article, is given at the end of the article.

2. Birnbaum-Saunders Distribution with Three Parameters, BS (3P)

Using the two-parameter Birnbaum and Saunders distribution, BS (2P) [1], we shall derive a three-parameter Birnbaum and Saunders distribution, BS (3P). Let us consider that a positive continuous random variable X $\sim BS(\alpha,\beta)$, where α,β are parameters of BS(2P). Consider another positive continuous random variable Y, such that Y = X+λ, where 0$\leq\lambda<$ x. So, Y = G(X). We are interested in finding the distribution of Y. In this context, we use the method of direct transformation of random variables. In this method, we first find the CDF, and then PDF of Y. F_Y(y) = 0 if y < 0. If $Y\geq 0$, then P[Y$\leq$y] = P[X+$\lambda\leq$Y] = P[X$\leq$Y$-\lambda$], thus the CDF of Y is defined by (2.1).

$$
\begin{aligned}
F_Y(y) &= \int_{-\infty}^{y-\lambda} f_X(x)dx, \\
&= \int_{-\infty}^{y-\lambda} \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{x}\right)^{\frac{1}{2}} + \left(\frac{\beta}{x}\right)^{\frac{3}{2}}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right] dx, \\
&= \frac{1}{2\sqrt{2\pi}\alpha\beta}\int_{-\infty}^{y-\lambda} \left[\left(\frac{\beta}{x}\right)^{\frac{1}{2}} + \left(\frac{\beta}{x}\right)^{\frac{3}{2}}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right] dx.
\end{aligned} \tag{2.1}
$$

Since the PDF, $F_Y(y)$, of Y is given by $F_Y(y) = F'_Y(y)$, therefore by the first fundamental theorem of integral calculus (see APPENDIX), the PDF of Y is given by

$$
f_Y(y) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{y-\lambda}\right)^{\frac{1}{2}} + \left(\frac{\beta}{y-\lambda}\right)^{\frac{3}{2}}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{y-\lambda}{\beta}+\frac{\beta}{y-\lambda}-2\right)\right],
$$

where $y > 0, \lambda \geq 0, \alpha > 0, \ \beta > 0$. We can also express above PDF, $F_Y(y)$, in terms of standard normal PDF Φ (.)as follows:

$$
f_Y(y;\alpha,\beta,\lambda) = \frac{1}{2a(x-\lambda)}\left[\sqrt{\frac{y-\lambda}{\beta}} + \sqrt{\frac{\beta}{y-\lambda}}\right]\lambda\left[\frac{1}{\alpha}\left\{\sqrt{\frac{y-\lambda}{\beta}} - \sqrt{\frac{\beta}{y-\lambda}}\right\}\right], y > 0, \lambda > 0, \alpha > 0, \beta > 0. \tag{2.2}
$$

We see that the PDF of $BS(\alpha,\beta,\lambda)$ given by Equation (2.2) is the right-shift of the PDF of Y given by Equation (2.2); see Munir et al [8].

Now we first find the PDF of Y, then its CDF. The PDF of Y = X+λ is defined by

$$
f_Y(y) = f_X(x)\frac{dx}{dy} = f_X(y-\lambda),
$$

which again results in Equation (2.2).

By integrating Equation (2.2) from ∞ to y$-\lambda$, we get the CDF of the $BS(\alpha,\beta,\lambda)$ which again results in Equation (2.2).

We have concluded that Y has three-parameter Birnbaum-Saunders distribution i.e., $Y \sim BS(\alpha,\beta,\lambda)$. α is the shape, β is the scale and λ is the location/shift parameter.

PDF of $BS(\alpha,\beta,\lambda)$ tends to 0 when $Y \to 0$ as well as when $Y \to \infty$. the PDF is unimodal for all values of α, β and λ. The graphs of the pdf of $BS(\alpha,\beta,\lambda)$ are drawn for different values of α, β and λ in Figure 1. The BS(3P) is important as it is characterized by three parameter α, β and λ. The first two α, and β are the same as defined in Birnbaum-Saunders two parameter distribution, whereas the third one λ, called the location parameter, denots the right shift of the PDF which enlarges the scope of the application of the BS(α,β,λ).

The effects of parameter values on the PDF of $BS(\alpha,\beta,\lambda)$ distribution are easily seen from Figure 1. The $BS(\alpha,\beta,\lambda)$ distribution can have an upside-down bathtub probability density function depending on the values of its parameters.

We know that if X $\sim BS(\alpha,\beta)$, then Z $=\frac{1}{\alpha}\left(\sqrt{\frac{X}{\beta}} - \sqrt{\frac{\beta}{X}}\right) \sim$ N(0, 1). From this relation, we get

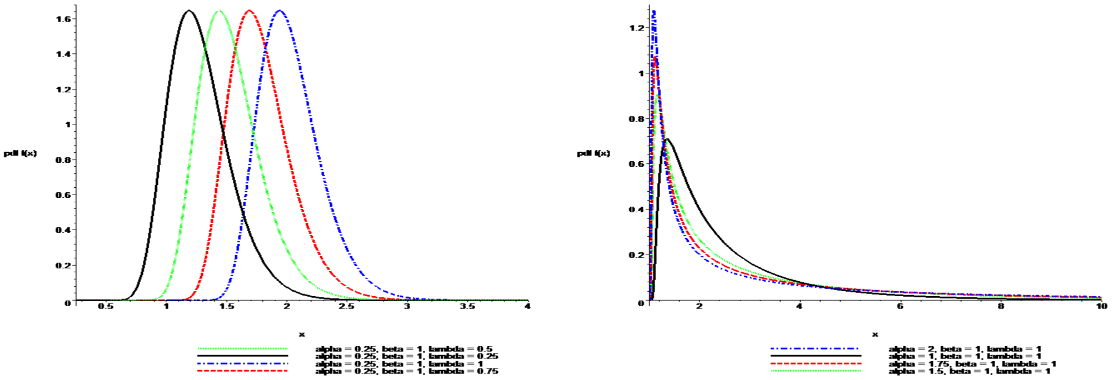

Figure 1. The PDF of $BS(\alpha,\beta,\lambda)$ for Different Values of Parameters

$$X = \beta\left(\frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1}\right)^2, \tag{2.3}$$

has $BS(\alpha,\beta)$. Now, if $0 < \lambda \leq X$, then

$$\begin{aligned} Y &= X + \lambda \\ &= \beta\left(\frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1}\right)^2 + \lambda. \end{aligned} \tag{2.4}$$

From this relation, we conclude that $Y \sim BS(\alpha,\beta,\lambda)$. The graph of the BS(3P) for different values of the location parameter λ is given in Figure 1. Now, since

$$F_Y(y;\alpha,\beta,\lambda) = P[Y \leq y] = P\left[Z \leq \frac{1}{\alpha}\left(\sqrt{\frac{y-\lambda}{\beta}} - \sqrt{\frac{\beta}{y-\lambda}}\right)\right], \tag{2.5}$$

which coincides with the CDF given in Equation (2.1). Thus

$$F_Y(y;\alpha,\beta,\lambda) = \Phi(\frac{1}{\alpha}\xi(\frac{x-\lambda}{\beta})), \tag{2.6}$$

where $\xi(y) = Y^{\frac{1}{2}} - Y^{-\frac{1}{2}} = \sinh(\log(y))$. Also $\Phi(z) = \int_{-\infty}^{z} \phi(u)du$, $z \in R$. BS(3P) is positively skewed (asymmetry to right), unimodal, and continuous as is depicted in Figure 1.

Now we define the quantile function or $q \times 100$th quantile function of BS(3P).

Definition 3 We define the quantile function of $Y \sim$ BS(3P), which is an indicator in the statistical analysis, as follows [4]:

$$t_Y(q;\alpha,\beta,\lambda) = F_Y^{-1}(y;\alpha,\beta,\lambda), \tag{2.7}$$

where $F_Y(y;\alpha,\beta,\lambda)$ is the CDF of Y, respectively given by Equation (2.1).

Definition 4 The Hazard Function(HF) or the failure rate of the $Y \sim$ BS(3P), which is an important risk indicator, is defined by Balakrishnan and Kundu [5] as

$$h_Y(y;\alpha,\beta,\lambda) = \frac{f_Y(y;\alpha,\beta,\lambda)}{1-F_Y(y;\alpha,\beta,\lambda)}, \; with \;\; y>0 \; and \; 0 \leq F_Y(y;\alpha,\beta,\lambda) \leq 1, \tag{2.8}$$

where $f_Y(y;\alpha,\beta,\lambda)$ and $F_Y(y;\alpha,\beta,\lambda)$ are the PDF and CDF of Y, respectively given by Equation (2.2) and Equation (2.1).

The location, scale and shape parameters are important as regards the probability distribution is characterized by them. Location and scale parameters are used in modeling applications. In the context of PDF, the scale parameter β streches or compresses the PDF of the distribution. β= 1 keeps the distrbution in standard shape. If β> 1, then it streches the PDF. If β< 1, it compresses the PDF. On the other hand, the location parameter λ= 0 gives the standard shapes of the PDF. If λ> 0, then it shifts the PDF to the left and if λ< 0, then it shifts the PDF to the right. So without loss of any fear of generality, we can take β= 1 and λ= 0 in order to discuss the shape of the hazard function of BS(3).

Taking β= 1 and λ= 0, we consider a function defined by Kundu et al., [9],

$$g(t) = t^{\frac{1}{2}} - t^{-\frac{1}{2}}, \tag{2.9}$$

then

$$\begin{aligned} g'(t) &= \frac{d}{dt}(g(t)), \\ &= \frac{1}{2}(t^{-\frac{1}{2}} - t^{-\frac{3}{2}}). \end{aligned} \tag{2.10}$$

Similarly,

$$g''(t) = -\frac{1}{4t^2}(t^{\frac{1}{2}} + 3t^{-\frac{1}{2}}), \tag{2.11}$$

and

$$g^2(t) = t + \frac{1}{t} - 2. \tag{2.12}$$

The probability density function as given by Equation (2.2) of Y ~ BS(**??**) for β= 1 and λ= 0 becomes

$$f_Y(y;\alpha) = \frac{1}{\sqrt{2\pi}\alpha} g'(t) \exp\left[-\frac{1}{2\alpha^2} g^2(t)\right], \; 0 < x < \infty, \quad \alpha > 0. \tag{2.13}$$

Using Equation (2.13) in conjunction with the CDF given by Equation (2.1) in Equation (2.8), we get

$$\begin{aligned} h_Y(y;\alpha) &= \frac{f_Y(y;\alpha)}{1-F_Y(y;\alpha)}, \\ &= \frac{\frac{1}{\sqrt{2\pi}\alpha} g'(t) exp\left[-\frac{1}{2\alpha^2} g^2(t)\right]}{\Phi\left(-\frac{g(t)}{\alpha}\right)}. \end{aligned} \tag{2.14}$$

The graphs of the hazard function given in Equation (2.14) of $BS(\alpha,\beta,\lambda)$ are drawn for different values of α, β and λ in Figure . 2

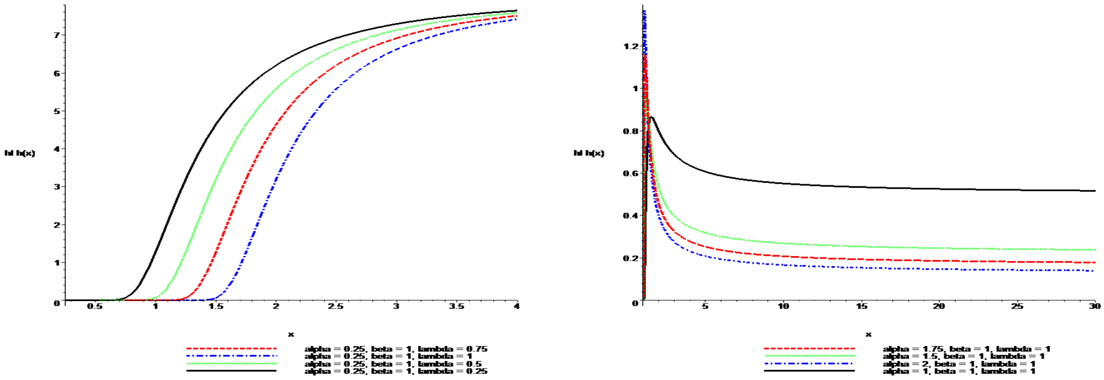

Figure 2. Hazard Function of $BS(\alpha, \beta, \lambda)$ for Different Values of Parameters

The effects of parameter values on the HF of $BS(\alpha, \beta, \lambda)$ distribution are easily inferred from Figure 2. It is to be noted that the hazard function of the $BS(\alpha, \beta, \lambda)$ distribution can have an upside-down bathtub shape depending on the values of its parameters.

3. Distributional Properties

In this section, we derive various moments of BS (3P) distribution. From now onward, we shall use j in stead of $\boldsymbol{Y}$. As such, assume that j is a positive continuous random variable possessing BS (3P) distribution, j ~BS(α, β, λ), with the PDF given by Equation (2.2), that is,

$$\boldsymbol{f}_j(j; \alpha, \beta, \lambda) = \frac{1}{2\alpha(j-\lambda)}\left[\sqrt{\frac{j-\lambda}{\beta}} + \sqrt{\frac{\beta}{j-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{j-\lambda}{\beta}} - \sqrt{\frac{\beta}{j-\lambda}}\right\}\right],$$

where $j > \lambda \geq 0,\ \alpha > 0, \beta > 0$.

3.1. j^{th}Moment of BS (3P) Distribution

The j^{th} moment, a_j, $j > 0$, of the BS (3P) distribution is defined by

$$\alpha_j = \boldsymbol{E}\left(j^j\right) = \int_\lambda^\infty j^j \frac{1}{2\alpha(j-\lambda)}\left[\sqrt{\frac{j-\lambda}{\beta}} + \sqrt{\frac{\beta}{j-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{j-\lambda}{\beta}} - \sqrt{\frac{\beta}{j-\lambda}}\right\}\right]dx. \quad (3.1)$$

Letting $x - \lambda = u$ in Equation (3.1), we have

$$\alpha_j = \boldsymbol{E}\left(j^j\right) = \int_0^\infty (u+\lambda)^j \frac{1}{2\alpha u}\left[\sqrt{\frac{u}{\beta}} + \sqrt{\frac{\beta}{u}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{u}{\beta}} - \sqrt{\frac{\beta}{u}}\right\}\right]du. \quad (3.2)$$

Now, using the binomial expansion for $(u + \lambda)^j$ in Equation (3.2) and simplifying, we obtain

$$\alpha_j = \boldsymbol{E}\left(j^j\right) = \sum_{\boldsymbol{m}=0}^{j}\binom{j}{\boldsymbol{m}}\lambda^{\boldsymbol{m}}\int_0^\infty u^{j-\boldsymbol{m}}\left\{\frac{1}{2\alpha u}\left[\sqrt{\frac{u}{\beta}} + \sqrt{\frac{\beta}{u}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{u}{\beta}} - \sqrt{\frac{\beta}{u}}\right\}\right]\right\}du. \quad (3.3)$$

Since the integral $\int_0^{\infty} u^{j-m}\left\{\frac{1}{2\alpha u}\left[\sqrt{\frac{u}{\beta}}+\sqrt{\frac{\beta}{u}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{u}{\beta}}-\sqrt{\frac{\beta}{u}}\right\}\right]\right\}du$ in above Equation (3.3) represents the $(j-\boldsymbol{m})\,\boldsymbol{th}$ moment of BS (2P) distribution for u ~BS(α, β), therefore, following Athayde et al. [3] and Leiva [4], we have

$$\boldsymbol{E}\left(u^{j-\boldsymbol{m}}\right)=\beta^{j-\boldsymbol{m}}\sum_{\boldsymbol{k}=0}^{j-\boldsymbol{m}}\binom{2(j-\boldsymbol{m})}{2\boldsymbol{k}}\sum_{\boldsymbol{i}=0}^{\boldsymbol{k}}\binom{\boldsymbol{i}}{\boldsymbol{k}}\frac{(2(j-\boldsymbol{m})-2\boldsymbol{k}+2\boldsymbol{i})!}{2^{j-\boldsymbol{m}-\boldsymbol{k}+\boldsymbol{i}}(j-\boldsymbol{m}-\boldsymbol{k}+\boldsymbol{i})!}\left(\frac{\alpha}{2}\right)^{2(j-\boldsymbol{m})-2\boldsymbol{k}+2\boldsymbol{i}}$$

Thus, using the above expression for $\boldsymbol{E}\left(u^{j-\boldsymbol{m}}\right)$ in Equation (3.3), we have

$$\alpha_j=\boldsymbol{E}\left(j^j\right)=\sum_{\boldsymbol{m}=0}^{j}\binom{j}{\boldsymbol{m}}\lambda^{\boldsymbol{m}}\boldsymbol{E}\left(u^{j-\boldsymbol{m}}\right),o \tag{3.4}$$

which is the required expression for the the j^{th} moment of j ~BS(α, β, λ). Furthermore, the mean of u ~BS(α, β)is given by

$$\boldsymbol{E}(u)=\frac{\beta}{2}\left(\alpha^2+2\right),$$

see Athayde et al. [3] or Leiva [4]. Thus, taking $j=1$ in Equation (3.4) and after some simplification, the first moment (or the mean), α_1, of of j ~BS(α, β, λ) is easily given by

$$\alpha_1=\boldsymbol{E}(j)=\frac{\beta}{2}\left(\alpha^2+2\right)+\lambda. \tag{3.5}$$

3.2. j^{th} (Central) Moment

:

The jth (central) moment of $X\sim$ BS(α,β,λ)can easily be derived as follows:

$$\beta_j=E[X-E(X)]^j=\int_{\lambda}^{\infty}[x-E(X)]^j f_X(x)dx=\sum_{m=0}^{j}(-1)^m\binom{j}{m}(E(X))^m E\left(X^{j-m}\right) \tag{3.6}$$

where $E\left(X^{j-m}\right)$ and $(E(X))^m$ can be obtained from the Equation (3.4) and Equation (3.5) respectively. Equation (3.6) specifies the second, the third, and the higher central moments for different values of j.

3.3. Variance, Coefficients of Skewness and Kurtosis

These concepts are defined as follow:

Variance: Taking $j=2$ in Equation (3.6), the variance (or the second central moment), β_2, after simplification, is given by

$$\beta_2=E[X-E(X)]^2=\int_{\lambda}^{\infty}[x-E(X)]^2 f_X(x)\,dx=E\left[X^2\right]-(E[X])^2=(a\beta)^2\left(1+\frac{5a^2}{4}\right) \tag{3.7}$$

Coefficients of Skewness and Kurtosis: By taking $j = 3$ and $j = 4$ in the Equation (3.6), the third and the fourth central moments are respectively given by

$$\beta_3 = E[X - E(X)]^3 = \sum_{m=0}^{3} (-1)^m \binom{3}{m} (E(X))^m E\left(X^{3-m}\right), \tag{3.8}$$

and

$$\beta_4 = E[X - E(X)]^4 = \sum_{m=0}^{4} (-1)^m \binom{4}{m} (E(X))^m E\left(X^{4-m}\right), \tag{3.9}$$

where $E\left(X^{j-m}\right)$ and $(E(X))^m$ can be obtained from the Equation (3.4) and Equation (3.5) respectively. Thus, using Equation (3.8) and Equation (3.9), the measure of skewness,γ_1, and kurtosis, γ_2, after simplification, are respectively given by

$$\gamma_1 = \frac{\sum_{m=0}^{3} (-1)^m \binom{3}{m} (E(X))^m E\left(X^{3-m}\right)}{\left(E[X^2] - E[X]^2\right)^{\frac{3}{2}}} = \frac{\beta_3}{(\beta_2)^{3/2}} = \frac{4\alpha\left(11\alpha^2 + 6\right)}{(5\alpha^2 + 4)^{3/2}} \tag{3.10}$$

and

$$\gamma_2 = \frac{\sum_{m=0}^{4} (-1)^m \binom{4}{m} (E(X))^m E\left(X^{4-m}\right)}{\left(E[X^2] - E[X]^2\right)^2} = \frac{\beta_4}{(\beta_2)^2} = 3 + \frac{6\alpha^2\left(93\alpha^2 + 40\right)}{(5\alpha^2 + 4)^2} \tag{3.11}$$

3.4. Moment Generating Function, Characteristic Function and r^{th} Cumulant

For $X \sim$ BS(α, β, λ), the moment generating respectively the characteristic functions of X aredefined by

$$M_X(t) = E\left(e^{tX}\right) = \sum_{j=0}^{\infty} \frac{(t)^j}{j!} E\left(X^j\right) \tag{3.12}$$

and

$$\phi_X(t) = M_X(it) = E\left(e^{itX}\right) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} E\left(X^j\right), \tag{3.13}$$

where $i = \sqrt{-1}$ is the imaginary number, $i^2 = -1$, and $E\left(X^j\right)$, the jth moment of $X \sim$ BS(α,β,λ) is given by Equation (3.3).

The r^{th} cumulant, κ_r, of $X \sim$ BS(α, β, λ) is given by taking the natural log of the characteristic function $(\phi_X(t)$ in Equation (3.13),

$$ln\left(\phi_X(t)\right) = \sum_{r=1}^{\infty} \kappa_r \frac{(it)^r}{r!},$$

Taking the Maclaurin series of the left-hand side of the above equation and equating the coefficients of various terms on both sides (see Equation (26.1.12), Page. 928, of Abramowitz and Stegun [10] or Stuart and Ord [11]), we get the following required *rth* cumulant κ_r:

$$\kappa_r = \frac{1}{i^r}\left[\frac{d^r\left(ln\left(\Phi_X(t)\right)\right)}{dt^r}\right]_{t=0}, r = 1, 2, \ldots, \tag{3.14}$$

From which, by successive differentiation, it can be easily seen that

$$\boldsymbol{\kappa_1 = E(j) = \alpha_1, \kappa_2 = Var(j) = \beta_2, \kappa_3 = E[j - E(j)]^3 = \beta_3,}$$

etc.,which can easily obtained by using the Equations (3.5), (3.7) and (3.8), respectively.

3.5. j^{th} Incomplete Moment of BS (3P) Distribution

For $j > 0$, the j^{th} incomplete moment of BS (3P) distribution is descibed by

$$\boldsymbol{I}_j(j) = \int_\lambda^j t^j \frac{1}{2\alpha(t-\lambda)}\left[\sqrt{\frac{t-\lambda}{\beta}} + \sqrt{\frac{\beta}{t-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{t-\lambda}{\beta}} - \sqrt{\frac{\beta}{t-\lambda}}\right\}\right]dt. \tag{3.15}$$

Letting $t - \lambda = u$ in above Equation (3.15), we have

$$I_j(x) = \int_0^{x-\lambda} (u+\lambda)^j \frac{1}{2\alpha u}\left[\sqrt{\frac{u}{\beta}} + \sqrt{\frac{\beta}{u}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{u}{\beta}} - \sqrt{\frac{\beta}{u}}\right\}\right]du. \tag{3.16}$$

Now, using the binomial expansion for $(u + \lambda)^j$ in Equation (3.16) and simplifying, we obtain

$$I_j(x) = \sum_{m=0}^{j}\binom{j}{m}\lambda^m \int_0^{x-\lambda} u^{j-m}\left\{\frac{1}{2\alpha u}\left[\sqrt{\frac{u}{\beta}} + \sqrt{\frac{\beta}{u}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{u}{\beta}} - \sqrt{\frac{\beta}{u}}\right\}\right]\right\}du = P_j(x) \tag{3.17}$$

where the integral $\int_0^{x-\lambda} u^{j-m}\left\{\frac{1}{2\alpha u}\left[\sqrt{\frac{u}{\beta}} + \sqrt{\frac{\beta}{u}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{u}{\beta}} - \sqrt{\frac{\beta}{u}}\right\}\right]\right\}du$ in Equation (3.17) represents the incomplete $(j - m)$ *th* moment of BS (2P) distribution for $U \sim$ BS(α, β), and cannot be evaluated analytically in closed form and so requires some quadrature formulas for computations. Taking $j = 1$ in Equation (3.17), then a little simplification gives the first incomplete moment of BS(3P) distribution is given by

$$\begin{aligned} P_1(j) &= \int_\lambda^j t\frac{1}{2\alpha(t-\lambda)}\left[\sqrt{\frac{t-\lambda}{\beta}} + \sqrt{\frac{\beta}{t-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{t-\lambda}{\beta}} - \sqrt{\frac{\beta}{t-\lambda}}\right\}\right]dt \\ &= \int_0^{j-\lambda}(u+\lambda)\frac{1}{2\alpha u}\left[\sqrt{\frac{u}{\beta}} + \sqrt{\frac{\beta}{u}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{u}{\beta}} - \sqrt{\frac{\beta}{u}}\right\}\right]du,\ Letting(t-\lambda = u) \qquad (3.18) \\ &= (j)\,\Phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{j-\lambda}{\beta}} - \sqrt{\frac{\beta}{j-\lambda}}\right\}\right] - \int_0^{j-\lambda}\Phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{u-\lambda}{\beta}} - \sqrt{\frac{\beta}{u-\lambda}}\right\}\right]du \end{aligned}$$

where $\Phi(.)$ denotes the CDF of $X \sim$ BS(α, β, λ), and the integral $\int_0^{x-\lambda} F\left[\frac{1}{\alpha}\left\{\sqrt{\frac{u-\lambda}{\beta}} - \sqrt{\frac{\beta}{u-\lambda}}\right\}\right]du$ in Equation (3.18) cannot be evaluated analytically in closed form and so requires some quadrature formulas for computations.

3.6. *Shannon Entropy:*

Referring to [12], the Shannon entropy measure of a continuous real random variable X is given by

$$H_X[f_X(X)] = E[-ln(f_X(X)] = -\int_{-\infty}^{\infty} f_X(x)\ ln\ [f_X(x)]\ dx,$$

Therefore, the Shannon entropy of BS (3P) distribution is defined by

$$\begin{aligned} H_X[f_X(X)] = &-\int_{\lambda}^{\infty}\left[\frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]\right] \\ &\times\left[ln\frac{1}{2a(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\Phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]\right]dx \\ &= ln(2a) + E(ln(x-\lambda)) - E\left(ln\left(\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\Phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]\right)\right) \end{aligned} \tag{3.19}$$

where the two expected values in Equation (3.19) cannot be evaluated analytically in closed forms and so requires some quadrature formulas for computations.

4. Percentile Points

The percentage points, x_p, of the BS (3P) distribution computed numerically by solving the equation $F(x_p) = \int_{\lambda}^{x_p} f_X(u)du = p$ (say), for any $0 < p < 1$, for given different sets of values of the parameters α, β, λ, are given in Table 1.

Table 1. Percentile Points of BS (3P) Distribution

Parameters	Percentiles p	0.75	0.80	0.85	0.90	0.95	0.99
$\alpha = 1, \beta = 1, \lambda = 1$	x_p	2.93928	3.26727	3.70443	4.34327	5.48245	8.27443
$\alpha = 1.5, \beta = 1, \lambda = 1$	x_p	3.64562	4.28976	5.17756	6.51398	8.96187	15.10587
$\alpha = 1.75, \beta = 1, \lambda = 1$	x_p	4.06721	4.91374	6.09339	7.88452	11.18757	19.51993
$\alpha = 2, \beta = 1, \lambda = 1$	x_p	4.53702	5.61670	7.13374	9.45117	13.74370	24.60521
$\alpha = 0.25, \beta = 1, \lambda = 0.25$	x_p	1.43344	1.48370	1.54484	1.62580	1.75436	2.02480
$\alpha = 0.25, \beta = 1, \lambda = 0.5$	x_p	1.68344	1.73370	1.79484	1.87580	2.00436	2.27480
$\alpha = 0.25, \beta = 1, \lambda = 0.75$	x_p	1.93344	1.98370	2.04484	2.12580	2.25436	2.52480
$\alpha = 0.25, \beta = 1, \lambda = 1$	x_p	2.18343	2.23370	2.29484	2.37580	2.50436	2.77480

5. Characterizations

In this section, we give some important characterization results of the BS (3P) which are important as regards that it is the only distribution that satisfies these specified conditions. These methods have been described in Ahsanullah [13], and references therein.

5.1. *Characterization by Truncated Moment:*

The following theorems characterize the Birbaum-Sanders (3P) distribution.

Theorem 5.1. If the random variable X satisfies the Assumption 5.1 with $\omega = \lambda$ and $\delta = \infty$, then $E(X|X = x) = g(x)\frac{f(x)}{F(x)}$, where

$$(x) = \frac{P_1(x)}{\frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]}, \tag{5.1}$$

where $P_1(x)$ is given by Equation (3.18), if and only if X has the pdf

$$f(x) = \frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]. \tag{5.2}$$

Proof. Suppose that$E(X|X \leq x) = g(x)\frac{f(x)}{F(x)}$. Then, since$E(X|X \leq x) = \frac{\int_\lambda^x u f(u)\,du}{F(x)}$, we have$g(x) = \frac{\int_\gamma^x u f(u)\,du}{f(x)}$. Now, if the random variable X satisfies the Assumption 10 and has the distribution with the PDF given in Equation (2.1), then we have

$$\begin{aligned} g(x) &= \frac{\int_\lambda^x u f(u)\,du}{f(x)} \\ &= \frac{P_1(x)}{f(x)} \\ &= \frac{P_1(x)}{\frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]}, \end{aligned}$$

where $P_1(x)$ is given by Equation (3.18). Consequently, the proof of "if" part of Theorem 5.1 follows from Lemma A.2. Conversely, suppose that

$$g(x) = \frac{P_1(x)}{\frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]}$$

where $P_1(x)$ is given by (3.18). Now, from Lemma (A.2), we have
$g(x) = \frac{\int_\lambda^x u f(u)\,du}{f(x)}$, or
$\int_\lambda^x u f(u)\,du = f(x)g(x)$. After differentiating the above equation with respect to respect to x, we obtain

$$xf(x) = f'(x)g(x) + f(x)g'(x),$$

from which, using the definition of PDF (2.2) and $f^{/}(x)$ being given by Equation (5.2), we easily obtain

$$g'(x) = x - g(x)\frac{\left[\sqrt{\frac{\beta}{x-\lambda}} - \sqrt{\frac{x-\lambda}{\beta}}\right]\left[\sqrt{\frac{\beta}{x-\lambda}} + \sqrt{\frac{x-\lambda}{\beta}}\right]^2 + \left[3\sqrt{\frac{\beta}{x-\lambda}} + \sqrt{\frac{x-\lambda}{\beta}}\right]}{2\alpha^2 (x-\lambda)\frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{\beta}{x-\lambda}} + \sqrt{\frac{x-\lambda}{\beta}}\right]}$$

or,

$$\frac{x - g^{/}(x)}{g(x)} = \frac{\left[\sqrt{\frac{\beta}{x-\lambda}} - \sqrt{\frac{x-\lambda}{\beta}}\right]\left[\sqrt{\frac{\beta}{x-\lambda}} + \sqrt{\frac{x-\lambda}{\beta}}\right]^2 + \left[3\sqrt{\frac{\beta}{x-\lambda}} + \sqrt{\frac{x-\lambda}{\beta}}\right]}{2\alpha^2 (x-\lambda)\frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{\beta}{x-\lambda}} + \sqrt{\frac{x-\lambda}{\beta}}\right]}. \tag{5.3}$$

Since, by Lemma (A.2), we have

$$\frac{x - g^{/}(x)}{g(x)} = \frac{f^{/}(x)}{f(x)}, \tag{5.4}$$

see Shakil, et al. [14]. Therefore, from Equation (5.3) and Equation (5.4), it follows that

$$\frac{f^{/}(x)}{f(x)} = \frac{\left[\sqrt{\frac{\beta}{x-\lambda}} - \sqrt{\frac{x-\lambda}{\beta}}\right]\left[\sqrt{\frac{\beta}{x-\lambda}} + \sqrt{\frac{x-\lambda}{\beta}}\right]^2 + \left[3\sqrt{\frac{\beta}{x-\lambda}} + \sqrt{\frac{x-\lambda}{\beta}}\right]}{2\alpha^2 (x-\lambda)\frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{\beta}{x-\lambda}} + \sqrt{\frac{x-\lambda}{\beta}}\right]} \tag{5.5}$$

Now, integrating Equation (5.5) with respect to x and simplifying, we easily have

$$ln(f(x)) = ln\left(c\frac{1}{(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]\right)$$

or

$$f(x) = c\left(\frac{1}{(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]\right), \tag{5.6}$$

where c is the normalizing constant to be determined. Thus, on integrating the above Equation (5.6) with respect to x from $x = \lambda$ to $x = \infty$, and using the condition $\int_{\gamma}^{\infty} f(x)\,dx = 1$, we obtain

$$\frac{1}{c} = \int_{\lambda}^{\infty}\left(\frac{1}{(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]\right)dx. \tag{5.7}$$

Now, letting $\frac{x-\lambda}{\beta} = u$ in Equation (5.7), we have

$$\frac{1}{c} = \int_{0}^{\infty}\left(\frac{1}{u}\left[\sqrt{u} + \sqrt{\frac{1}{u}}\right]\Phi\left[\frac{1}{\alpha}\left\{\sqrt{u} - \sqrt{\frac{1}{u}}\right\}\right]\right)du$$

from which, on multiplying and dividing by 2α, $(\alpha > 0)$, we obtain

$$\frac{1}{c} = (2a) \int_0^\infty \left(\frac{1}{(2a)\, u} \left[\sqrt{u} + \sqrt{\frac{1}{u}} \right] \Phi \left[\frac{1}{\alpha} \left\{ \sqrt{u} - \sqrt{\frac{1}{u}} \right\} \right] \right) du \tag{5.8}$$

Since the expression within the integral on the right side of the Equation (5.8) defines the pdf of the standard Birnbaum-Saunders (or fatigue life), BS(α), distribution, we easily obtain$c = \frac{1}{2\alpha}$. This completes the proof of Theorem 5.1.

Theorem 5.2. If the random variable X satisfies the Assumption (A.1) with $\omega = \lambda$ and$\delta = \infty$, then $(X/X \geq x) = \tilde{h}(x)\frac{f(x)}{1-F(x)}$, where $\tilde{h}\,(x) = \frac{(E(X)-g(x)f(x))}{f(x)} g(x)$, $g(x)$ being given by Equation (5.1) and $E(X)$ being given by Equation (3.5), if and only if X has the pdf

$$f\,(x) = \frac{1}{2\alpha\,(x-\lambda)} \left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}} \right] \phi \left[\frac{1}{\alpha} \left\{ \sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}} \right\} \right]$$

Proof. Suppose that $E\,(X/X \geq x) = \tilde{h}(x)\frac{f(x)}{1-F(x)}$. Then, since $E\,(X/X \geq x) = \frac{\int_x^\infty uf(u)}{1-F(x)} du$, we have $\tilde{h}\,(x) = \frac{\int_x^\infty uf(u)}{1-F(x)} du$ Now, if the random variable X satisfies the Assumptions (A.1) and has the distribution with the PDF (2.2), then, using the Theorem 5.1, we have

$$\begin{aligned} \tilde{h}\,(x) &= \frac{\int_x^\infty uf(u)}{f(x)} du = \frac{\int_\gamma^\infty uf(u)du - \int_\gamma^x uf(u)du}{f(x)} \\ &= \frac{(E\,(X) - g(x)f(x))}{f(x)}, \end{aligned}$$

where $f\,(x)$ denotes the PDF of the BS (3P) distribution given by Equation (2.2), $g\,(x)$ being given by Equation (5.1) and $E\,(X)$ being given by Equation (3.5). Consequently, the proof of "if" part of the Theorem 5.2 follows from Lemma (A.3). Conversely, suppose that $\tilde{h}\,(x) = \frac{(E(X)-g(x)f(x))}{f(x)}$. Now, from Lemma (A.3)., we have

$$\tilde{h}\,(x) = \frac{\int_x^\infty uf(u)}{f(x)} du,$$

or

$$\int_x^\infty uf(u) = f(x)\tilde{h}\,(x)\,.$$

Differentiating the above equation with respect to respect to x, we obtain

$$xf(x) = f'\,(x)\,\tilde{h}\,(x) + f\,(x) \left[\tilde{h}\,(x)\right]'.$$

Thus, proceeding in the same way as in Theorem 5.1 and following the similar arguments, we easily obtain

$$f\,(x) = c \left(\frac{1}{(x-\lambda)} \left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}} \right] \lambda \left[\frac{1}{\alpha} \left\{ \sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}} \right\} \right] \right),$$

where $c = \frac{1}{2\alpha}$. This completes the proof of Theorem 5.2.

5.2. Characterizations by Order Statistics:

If $X_1, X_2, \ldots, X_n$ be the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and PDF $f(x)$, and if $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ be the corresponding order statistics that is known from Ahsanullah et al. see Shakil, et al. [15], Chapter 5, or Arnold et al. see Shakil, et al. [16], Chapter 2, that $X_{j,n}|X_{k,n} = x$, for $1 \leq k < j \leq n$, is distributed as the $(j-k)$ *th* order statistics from $(n-k)$ independent observations from the random variable V having the PDF $f_V(v|x)$ where $f_V(v|x) = \frac{f(v)}{1-F(x)}$, $0 \leq v < x$, and $X_{i..n}|X_{k,n} = x$, $1 \leq i < k \leq n$, is distributed as *ith* order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where $f_W(w|x) = \frac{f(w)}{F(x)}$, $w < x$. Let$S_{k-1} = \frac{1}{k-1}(X_{1,n} + X_{2,n} + \ldots + X_{k-1,n})$, and $T_{k,n} = \frac{1}{n-k}(X_{k+1,n} + X_{k+2,n} + \ldots + X_{n.n})$.
Theorem 5.3: Suppose the random variable X satisfies the Assumption (A.1) with $\omega = \lambda$ and $\delta = \infty$, then $E(S_{k-1}|X_{k,n} = x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ being given by Equation (5.1), if and only if X has the PDF

$$f(x) = \frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right].$$

Proof: It is known that$E(S_{k-1}|X_{k,n} = x) = E(X|X \leq x)$; see Ahsanullah et al. see Shakil, et al. [15], and David and Nagaraja see Shakil, et al. [17]. Hence, by Theorem 5.1, the result follows.

Theorem 5.4: Suppose the random variable X satisfies Assumption (A.1) with $\omega = \lambda$ and $\delta = \infty$, then $E(T_{k,n}/X_{k,n} = x) = \tilde{h}(x)\frac{f(x)}{1-F(x)}$, where

$$\tilde{h}(x) = \frac{(E(X) - g(x)f(x))}{f(x)},$$

$g(x)$ being given by Equation (5.1) and $E(X)$ being given by Equation (3.5), if and only if X has the pdf

$$f(x) = \frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right].$$

Proof: Since $E(T_{k,n}|X_{k,n} = x) = E(X|X \geq x)$, see Ahsanullah et al. see Shakil, et al. [15], and David and Nagaraja see Shakil, et al. [17], the result follows from Theorem 5.2.**5.3. Characterization by Upper Record Values:** For details on record values, see Ahsanullah see Shakil, et al. [18]. Let$X_1, X_2, \ldots$ be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and PDF$f(x)$. If $Y_n = max(X_1, X_2, \ldots, X_n)$ for $n \geq 1$ and$Y_j > Y_{j-1}$, $j > 1$, then X_j is called an upper record value of$\{X_n, n \geq 1\}$. The indices at which the upper records occur are given by the record times $\left\{U(n) > min(j|j > U(n+1), X_j > X_{U(n-1)}, n > 1)\right\}$ and $U(1) = 1$. Let the *nth* upper record value be denoted by$X(n) = X_{U(n)}$. **Theorem 5.5:** Suppose the random variable X satisfies the Assumption A.11 with $\omega = \lambda$ and $d = 8$, then

$$E(X(n+1)/X(n) = x) = \tilde{h}(x)\frac{f(x)}{1-F(x)},$$

where $\tilde{h}(x) = \frac{(E(X)-g(x)f(x))}{f(x)}$, $g(x)$ being given by Equation (5.1) and $E(X)$ being given by Equation (3.5), if and only if X has the PDF

$$f(x) = \frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right]\phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right].$$

Proof: It is known from Ahsanullah et al. see Shakil, et al. [15], and Nevzorov see Shakil, et al. [19] that$E(X(n+1)|X(n)=x) = E(X|X=x)$. Then, the result follows from Theorem 5.2.

6. Estimation of Parameters and Applications

In this section, we provide the estimation of the parameters of BS (3P) distribution by

1. The method of moment(MOM)- Sub-Section 6.1,
2. The method of maximum likelihood(MLE) - Sub-Section 6.2.

The parameters of BS (3P) distribution are estimated using MLEs by considering two real-world data set examples in Sub-Section 6.3,. The results are presented in Table 2 by comparing BS (3P) distribution with some well-known skew distributions for testing the goodness of fit of BS (3P) distribution.

6.1. *The Method of Moments:*

If $\{X_i\}_{i=1}^{n}$ is an *iid* sample from a distribution with an m-dimensional parameter vector ϕ, then according to the method of moment (MOM), the estimator $\Phi^{\sim}$ is the solution of the following system of equations:

$$E_{\phi^{\sim}}\left(X^j\right) = \frac{\sum_{i=1}^{n} X_i^{\,j}}{n}, \qquad j = 1, 2, 3, \ldots, m \tag{6.1}$$

Thus, using the above-mentioned Definition (6.1) of the MOM, we can obtain the respective moments from the Equation (6.1) of the jthmoment, $E\left(X^j\right)$ of BS (3P) distribution by taking the respective values of$j, j = 1, 2, 3$ and evaluating the respective expressions of the respective moments numerically. Then, the moment estimations of the respective parameters of BS (3P) distribution can be determined by solving the system of respective equations thus obtained by Newton-Raphson's iteration method, and using some computer packages like Maple, or Mathematica, or R, or MathCad, or other software.

6.2. *The Method of Maximum Likelihood:*

The parameters of BS (3P) distribution are estimated by the use of the method of maximum likelihood (MLE). Given a sample $\{x_i\}$, $i = 1, 2, 3, \ldots, n$, the likelihood function of BS (3P) distribution PDF (**??**) is given by $L = \prod_{i=1}^{n} f(x_i)$, that is,

$$L = \prod_{i=1}^{n} \frac{1}{2\alpha(x_i-\lambda)}\left[\sqrt{\frac{x_i-\lambda}{\beta}} + \sqrt{\frac{\beta}{x_i-\lambda}}\right] \times \phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x_i-\lambda}{\beta}} - \sqrt{\frac{\beta}{x_i-\lambda}}\right\}\right].$$

The log-likelihood function of L is given by

$$R = ln(L) = \sum_{i=1}^{n} ln\left[\sqrt{\frac{x_i-\lambda}{\beta}} + \sqrt{\frac{\beta}{x_i-\lambda}}\right] + \sum_{i=1}^{n} ln\lambda\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x_i-\lambda}{\beta}} - \sqrt{\frac{\beta}{x_i-\lambda}}\right\}\right] - \sum_{i=1}^{n} ln(x_i-\lambda) - ln[2a] \tag{6.2}$$

After differentiating Equation (6.2) partially with respect to the respective parameters, the system of maximum likelihood of equations becomes

$$\begin{aligned}
\frac{\partial R}{\partial \alpha} &= \sum_{i=1}^{n} \frac{(x_i-\lambda-\beta)^2}{a^3\beta(x_i-\lambda)} - \frac{1}{\alpha} = 0,\\
\frac{\partial R}{\partial \beta} &= \sum_{i=1}^{n}\left[\frac{x_i}{\beta} - \frac{x_i}{(x_i-\lambda)} + \frac{\lambda}{(x_i-\lambda)}\right] + \frac{n}{2a^2\beta} - \frac{n\lambda}{2a^2\beta^2} + \frac{n}{2\beta} = 0,\\
\frac{\partial R}{\partial \lambda} &= \frac{1}{2}\sum_{i=1}^{n} \frac{(\beta+\lambda-x_i)}{(x_i-\lambda)(\beta-\lambda+x_i)} - \sum_{i=1}^{n}\frac{\beta}{2a^2(x_i-\lambda)^2} - \sum_{i=1}^{n}\frac{1}{(x_i-\lambda)} - \frac{n}{2a^2\beta} = 0.
\end{aligned} \tag{6.3}$$

Solving Equation (6.3) numerically by Newton-Raphson's iteration method using some computer software like Maple, or Mathematica, or R, the maximum likelihood estimates (MLE) of the parameters α, β, and λ are obtained.

6.3. Applications:

In this paper, we consider data from two real-world problems in the context of parameter estimation. Some more examples on parameter estimation as given in [31, 32, 33, 34, 35] can be considered by this method. In this sub-section, by considering two real-world data set examples, the goodness of fit tests of BS (3P) distribution is provided by comparing it with some well-known skew distributions, namely, BS (2P), log-logistic, lognormal, generalized extreme value, inverse Gaussian, and Weibull distributions. Probability density functions of these distributions and their parameters are listed in Table 9.

6.3.1. Example I:

We have a random sample of the "data for weights (pounds) of discarded glass garbage for one week" as reported in Triola see Shakil, et al. [20], as given in Table 2. We tested the chi-squared goodness-of-fit of BS (3P) distribution to this data and compared it with some well-known skew distributions, namely, BS (2P), log-logistic, lognormal and Weibull distributions, using the Kolmogorov-Smirnov, Anderson-Darling, and Chi-Squared goodness-of-fit (GOF) tests. Details on GOF tests can be seen in Massey [21] , and Stephens [22]. Moreover, for details on log-logistic, lognormal and Weibull distributions, the interested readers are referred to Patel et al. [23], Johnson et al. [2], Balakrishnan and Nevzorov [24], and Forbes et al.[25].

From Figure 3, it is obvious that the shape of the data is skewed to the right. This is also confirmed from the skewness (2.2477) and kurtosis (6.7303) of the data. Since fitting of any probability

Table 2. Weights (Pounds) of Discarded Glass Garbage for One Week(Sample Size: n = 62)

0.86, 3.46, 4.52, 4.92, 6.31, 2.49, 0.51, 5.81, 1.96, 17.67, 3.21, 4.94, 3.10, 1.39, 5.21, 2.03, 1.74, 3.99, 6.26, 3.52, 2.01, 2.21, 0.25, 0.09, 6.85, 2.33, 5.45, 2.04, 4.98, 3.54, 1.06, 2.70, 1.14, 12.24, 5.67, 2.43, 4.02, 6.45, 1.89, 1.78, 2.93, 1.82, 2.89, 0.99, 1.93, 3.61, 2.53, 3.76, 1.32, 2.64, 12.33, 1.79, 3.99, 4.44, 9.25, 4.02, 1.38, 1.59, 0.85, 8.87, 3.64, 3.03

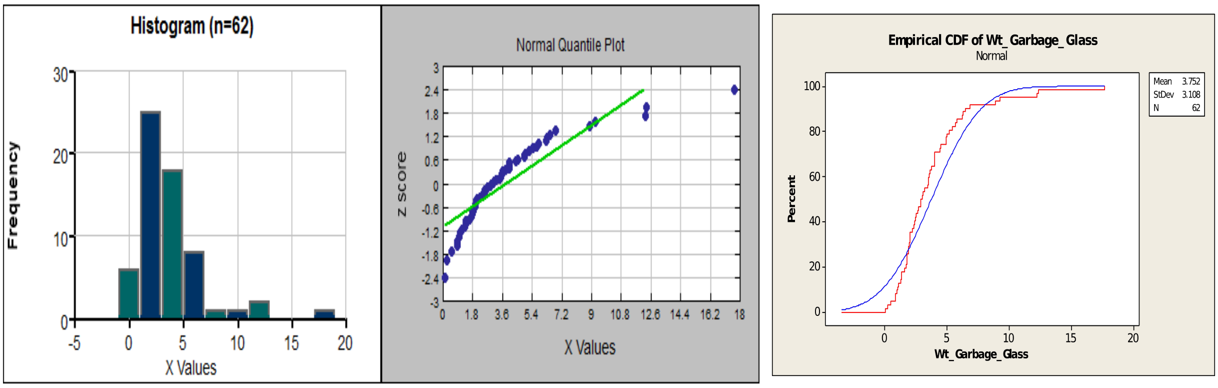

Figure 3. Histogram (left), Normal Quantile Plot (center) and Empirical CDF (right)

distribution to the data for a specific period is generally useful in predicting the probability of data or forecasting the frequency of occurrence of that data set, it is, therefore, imperative that the weights (pounds), **y**, of the discarded glass garbage for a one week, could possibly be modeled by some skewed distributions. Moreover, our data are skewed in nature, we fit the BS (3P) distribution to this data and make its comparison with some well-known skew distributions, viz., BS (2P), log-logistic, lognormal and Weibull distributions, based on Kolmogorov-Smirnov, Anderson-Darling, and Chi-Squared goodness-of-fit (GOF) tests.

Table 3. Fitting Results for Glass Garbage data

#	Distributions	Parameter Estimates
1	**BS (3P)**	α= 0.61813, β= 3.8231, λ = - 0.79923
2	**Weibull**	α = 1.362, β = 3.9985
3	**Lognormal**	σ= 0.87317, μ= 1.0109
4	**Log-Logistic**	α = 1.8814, β = 2.6654
5	**BS (2P)**	α = 1.0616, β = 2.3278

From Table 4, we observed that the BS (3P), BS (2P), log-logistic, lognormal and Weibull distributions fitted reasonably well to the weights (pounds) of discarded glass garbage for one week data. However, based on these three tests, the BS (3P) distribution model produces the highest P-Value and the smallest test statistic value, and therefore fitted better than the BS (2P), log-logistic, lognormal and Weibull distributions. For the parameters estimated in Table 3, the probability density functions of the BS (3P), BS (2P), log-logistic, lognormal and Weibull distributions have been superimposed on the histogram of the weights (pounds) of discarded glass garbage for one week data, as given in Figure 4,

Table 4. Comparison Criteria and Ranking of Fitted Distributions (Test Statistics, Critical Value and P-Value at Level of Significance = 0.05) for Glass Garbage data

#	Distribution	KolmogorovSmirnov	AndersonDarling	Chi-Squared
		Level of Significance = 0.05	**Level of Significance = 0.05**	**Level of Significance = 0.05**
1	**BS (3P)**	Test Statistic: 0.06335, Critical Value: 0.16956, P-Value: 0.95123	Test Statistic: 0.2518, Critical Value: 2.5018, P-Value: 0.98993	Test Statistic: 1.613, Critical Value: 11.07, P-Value: 0.89967
2	**Weibull**	Test Statistic: 0.08175, Critical Value: 0.16956, P-Value: 0.77097	Test Statistic: 0.57526, Critical Value: 2.5018, P-Value: 0.93117	Test Statistic: 2.6225, Critical Value: 11.07, P-Value: 0.75795
3	**Lognormal**	Test Statistic: 0.10682, Critical Value: 0.16956, P-Value: 0.44811	Test Statistic: 0.77256 , Critical Value: 2.5018, P-Value: 0.92692	Test Statistic: 2.806, Critical Value: 11.07, P-Value: 0.72987
4	**Log-Logistic**	Test Statistic: 0.11598, Critical Value: 0.16956, P-Value: 0.34774	Test Statistic: 0.88457, Critical Value: 2.5018, P-Value: 0.88847	Test Statistic: 3.294, Critical Value: 11.07, P-Value: 0.65475
5	**BS (2P)**	Test Statistic: 0.19806, Critical Value: 0.16956, P-Value: 0.99843	Test Statistic: 2.9197 , Critical Value: 2.5018, P-Value: 0.41532	Test Statistic: 9.7515 , Critical Value: 11.07, P-Value: 0.07109

from which we have also observed that BS (3P) distribution models the weights (pounds) of discarded glass garbage for one week data reasonably well by BS (3P).

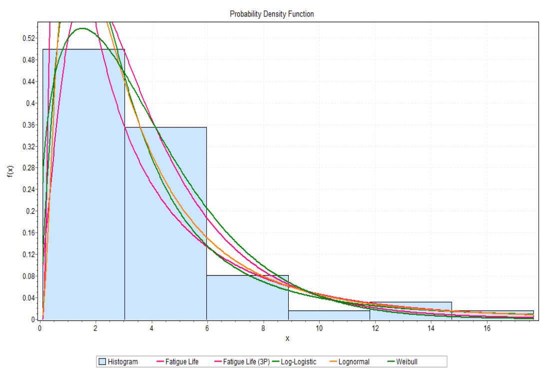

Figure 4. Fitting of the pdfs of the BS (3P), BS (2P), Log-Logistic, Lognormal and Weibull Distributions

6.3.2. Example II:

In this example, we considers a random sample of the data for the white blood cell counts (1000 cells / μ L) for females" as reported in Triola see Shakil, et al. [20], and depicted in Table 5. We have tested the goodness-of-fit of BS (3P) distribution to this data and compared it with some well-known skew distributions, namely, generalized extreme value, inverse Gaussian, lognormal and Weibull distributions, based on the Kolmogorov-Smirnov, Anderson-Darling, and Chi-Squared goodness-of-fit

(GOF) tests.

Table 5. White Blood Cell Counts (1000 cells /μ L) for Females (Sample Size: n = 40)

9.6, 7.1, 7.5, 6.8, 5.6, 5.4, 6.7, 8.6, 10.2, 4.1, 13.0,
9.2, 5.9, 8.0, 7.0, 9.1, 5.7, 4.6, 6.0, 5.7, 8.9, 6.4, 8.1,
7.9, 4.4, 4.9, 5.3, 5.3, 4.7, 9.8, 5.3, 4.9, 6.3, 5.4, 7.0,
13.5, 10.0, 10.3, 5.1, 6.6

The histogram and the normal quantile plot and empirical CDF (cumulative distributive function) of data for the white blood cell counts (1000 cells / µ L) for females are provided in Figure 6.

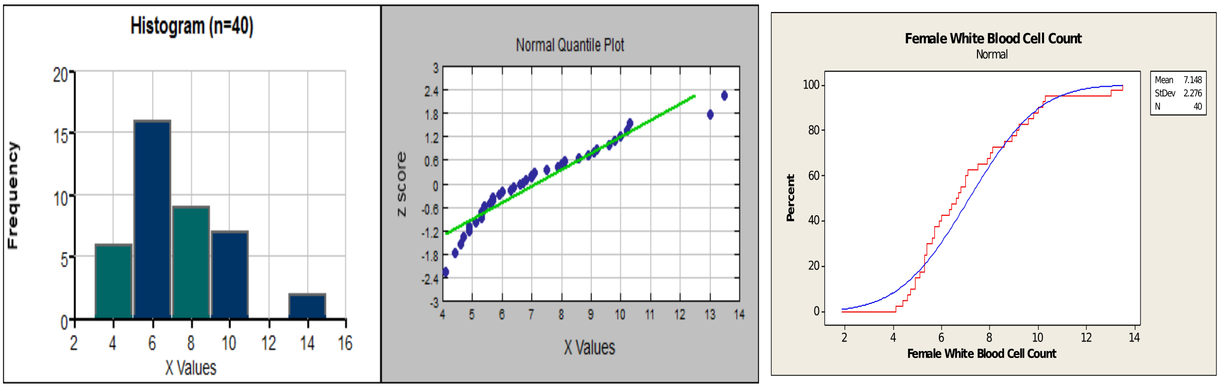

Figure 5. Histogram (left), Normal Quantile Plot (center) and Empirical CDF (right) for White Blood Cell

From Figure 5, it is obvious that the shape of the data for the white blood cell counts (1000 cells / μ L) for females is skewed to the right. This is also confirmed from the skewness (1.0171) and kurtosis (0.71481) of the data. Since fitting of a probability distribution to the data for the white blood cell counts (1000 cells / μ L) for females may be helpful in predicting the probability or forecasting the frequency of occurrence of the data for the white blood cell counts (1000 cells / µ L) for females, this suggests that **y**, the white blood cell counts (1000 cells / μ L) for females, can be modeled by some skewed distributions. Moreover, our data for the white blood cell counts (1000 cells / μ L) for females are skewed in nature, we fit BS (3P) distribution to this data and made its comparison with some well-known skew distributions, namely, generalized extreme value, inverse Gaussian, lognormal and Weibull distributions, based on the Kolmogorov-Smirnov, Anderson-Darling, and Chi-Squared goodness-of-fit (GOF) tests. The parameters estimated for the BS (3P), generalized extreme value, inverse Gaussian, lognormal and Weibull distributions are provided in Table6.

Goodness of Fit (Chi-Squared Test): The goodness-of-fit test results, based on the chi-squared test, using the P-Values and test statistics analysis, are provided in Table 7.

Based on the chi-squared test for goodness-of-fit in Table 7, BS (3P) distribution was found to be the best fit (Rank 1) for the white blood cell counts (1000 cells / µ L) for females' data, followed by Gen. Extreme Value (Rank 2), Lognormal (Rank 3), Inv. Gaussian (Rank 4) and Weibull (Rank 5).

Goodness of Fit (Kolmogorov-Smirnov and Anderson-Darling Tests): The goodness-of-fit test results, based on the Kolmogorov-Smirnov and Anderson-Darling Tests, are provided in Table8.

Based on the Kolmogorov-Smirnov and Anderson-Darling tests for goodness-of-fit as given in Table

Table 6. Fitting Results fir White Blood Cell data

#	Distributions	Parameter Estimates
1	**BS (3P)**	α= 0.60371, β= 3.3158, λ = 3.2258
2	**Gen. Extreme Value**	k = 0.08116, σ= 1.6735, μ= 6.0363
3	**Inv. Gaussian**	λ= 70.519, μ= 7.1475
4	**Lognormal**	σ= 0.2956, μ= 1.9216
5	**Weibull**	α = 3.8932, β = 7.7181

Table 7. Comparison Criteria and Ranking of Fitted Distributions (Based on the Chi-Squared Test for Goodness-of-Fit at the Level of Significance = 0.05)(P-Value and Test Statistic Analysis)

	BS (3P)	**Gen. Extreme Value**	**Log-normal**	**Inv. Gaussian**	**Weibull**
Rank	1	2	3	4	5
Test Statistic	2.2572	2.4862	2.6512	3.4236	5.3854
Critical Value	5.9886	5.9886	5.9886	5.9886	5.9886
P-Value	0.68857	0.64711	0.61777	0.4896	0.24999

Table 8. Comparison Criteria and Ranking of Fitted Distributions: Based on Kolmogorov-Smirnov and Anderson-Darling Tests of Goodness-of-Fit at the Level of Significance = 0.05

#	Distribution	KolmogorovSmirnov		AndersonDarling	
		Statistic	**Rank**	**Statistic**	**Rank**
1	**BS (3P)**	0.06618	1	0.19933	1
2	**Gen. Extreme Value**	0.08112	2	0.28121	2
3	**Inv. Gaussian**	0.08905	3	0.38007	3
4	**Lognormal**	0.10494	4	0.45884	4
5	**Weibull**	0.11217	5	1.2661	5

8, BS (3P) distribution was found to be the best fit (Rank 1) for the white blood cell counts (1000 cells / µ L) for females data, followed by Gen. Extreme Value (Rank 2), Inv. Gaussian (Rank 3), Lognormal (Rank 4) and Weibull (Rank 5).For the parameters estimated in Table 6, the probability density functions (PDF's) of the BS (3P), generalized extreme value, inverse Gaussian, lognormal and Weibull distributions respectively have been superimposed on the histogram of the white blood cell counts (1000 cells / µ L) for females data, which is provided in Figure 6 below.

7. Conclusion

The main points of the research article are enumerated as follow:

1. We have considered the BS (3P) distribution. We have reviewed the BS (3P) distribution first, and then established its several new statistical properties, including the estimation of the parameters, computations of percentage points and characterizations.
2. We have shown the applications of the BS (3P) distribution by considering two real-world data

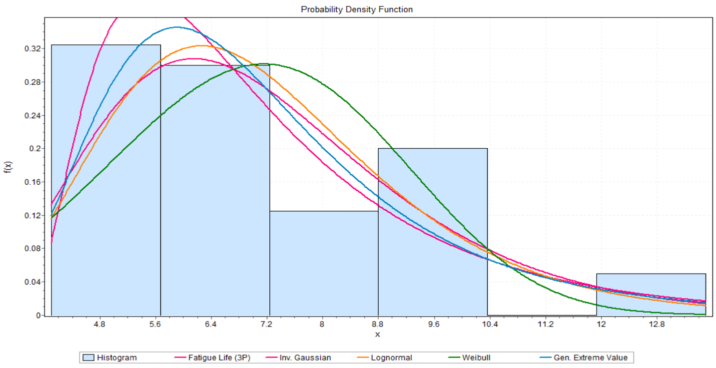

Figure 6. Fitting of PDF's to the Females' White Blood Cell Data

set examples, namely the "data for weights (pounds) of discarded glass garbage for one week" and the "data for the white blood cell counts (1000 cells / μ L) for females".

3. The goodness of fit tests of BS (3P) distribution is provided by comparing it with some well-known skew distributions, namely, BS (2P), log-logistic, lognormal, generalized extreme value, inverse Gaussian, and Weibull distributions. Based on the Kolmogorov-Smirnov, Anderson-Darling, and Chi-Squared goodness-of-fit (GOF) tests. SB (3P) distribution was found to be the best fit (Rank 1) for these data sets.
4. The findings of this paper will be quite helpful to the researchers and practitioners in various fields.
5. We can develop bootstrap control charts for the percentiles of the SB (3P) distribution, which is an important area of studies in quality and reliability engineering.

Conflict of interest

The authors declare no conflict of interests.

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Definition 1 If X has a Birbaum-Saunders distribution with two parameters denoted by X~BS(α,β), then its CDF is defined by $F_X(x;\alpha,\beta) = \Phi\left(\frac{1}{\alpha}\left[\left(\frac{x}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{x}\right)^{\frac{1}{2}}\right]\right), \quad 0 < y < \infty, \quad \alpha,\beta > 0.$ (A.1)
Here α is the shape parameter and β is the scale parameter.
Definition 2 The Probability Density Function(PDF) of X~BS(α,β) is given by

$$f_X(x;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{x}\right)^{\frac{1}{2}} + \left(\frac{\beta}{x}\right)^{\frac{3}{2}}\right] \exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta} + \frac{\beta}{x} - 2\right)\right], \tag{A.2}$$

$0 < x < \infty, \quad \alpha,\beta > 0$. For details on the BS(2P) distribution, the interested readers are referred to Birnbaum and Saunders [1, 7], Johnson et al. [2], and Balakrishnan and Kundu [5].
Let X and Y be two random variables taking values respectively from the sets U and V, and $g{:}U \to V$ be a function defined by $g(X) = Y$. Moreover, the distribution of X is known, we are interested in finding the distribution of Y.
Definition 2 Let $B \subseteq V$, then the inverse image of B under g, denoted by $g^{-1}[B]$, is defined by

$g^{-1}[B]$ =$\{x \in U : g(x) \in B\}$. (A.3)

Let X have a known distribution function F, and probability density function f, and those of Y are the G and g, then

Definition 3 Letting $x = r^{-1}(y)$, then by change of variables formula

$$g(y) = f(x) \left| \frac{dx}{dy} \right| \tag{A.4}$$

In this formula, x is to be written explicitly in terms of y.

Definition 4 Letting Y = a + bX, a≠0, b∈R, then the probabilty distribution function, g(y), is given by

$$g(y) = \left|\frac{1}{b}\right| f\left(\frac{y-a}{b}\right). \tag{A.5}$$

This transformation is called affine transformation. The multivariate analogue of this result can be obtained similaly.

Assumption A.1. Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function$f(x)$. We assume that$\omega = inf\{x|F(x) > 0\}$, and$d = sup\{x|F(x) < 1\}$.

We also assume that $f(x)$ is a differentiable for all$x$, and$E(X)$ exists.

Lemma A.2. If the random variable X satisfies the Assumption 5.1 with $\omega = \lambda$ and $\delta = \infty$, where $0 < \lambda < +\infty$, and if $E(X|X \leq x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_\lambda^x \frac{u - g^{/}(u)}{g(u)} du$ is finite for $\lambda \leq x$, $0 < \lambda < +\infty$, then $f(x) = ce^{\int_\lambda^x \frac{u - g^{/}(u)}{g(u)} du}$, where c is a constant determined by the condition $\int_\lambda^\infty f(x)dx = 1$.

Proof. See Shakil et al. [14].

Lemma A.3. If the random variable X satisfies the Assumption 5.1 with $\omega = \lambda$ and $\delta = \infty$, and if $E(X/X \geq x) = \tilde{h}(x)r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and $\tilde{h}(x)$ is a continuous differentiable function of x with the condition that $\int_x^\infty \frac{u + [\tilde{h}(x)]'}{\tilde{h}(x)} du$ is finite for $\lambda \leq x$, $0 < \lambda < +\infty$, then $f(x) = ce^{-\int_x^\infty \frac{u + [\tilde{h}(x)]'}{\tilde{h}(x)} du}$, where c is a constant such that $\int_\lambda^\infty f(x)dx = 1$.

Proof. See Shakil et al. [14].

Table 9. Distributions Used in Our Data Analysis

S. No.	Distributions	$f(x)$	Parameters
1	**Birnbaum-Saunders (or Fatigue-Life) (3P)or BS (3P)**	$f(x) = \frac{1}{2\alpha(x-\lambda)}\left[\sqrt{\frac{x-\lambda}{\beta}} + \sqrt{\frac{\beta}{x-\lambda}}\right] \times \phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x-\lambda}{\beta}} - \sqrt{\frac{\beta}{x-\lambda}}\right\}\right]$, where $\Phi()$ denotes the standard normal pdf.	$\alpha\,(> 0)$: shape parameter$\beta\,(> 0)$: scale parameter γ (real): location parameter, where $\lambda = x < \infty$
2	**Birnbaum-Saunders (2P)or BS (2P)**	$f(x) = \frac{1}{2\alpha x}\left[\sqrt{\frac{x}{\beta}} + \sqrt{\frac{\beta}{x}}\right] \times \phi\left[\frac{1}{\alpha}\left\{\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}}\right\}\right]$, where $\Phi()$ denotes the standard normal pdf.	$\alpha\,(> 0)$: shape parameter$\beta\,(> 0)$: scale parameterand $0 \leq x < +\infty$
3	**Log-Logistic** (2P)	$f(x) = \frac{a}{\beta}\left(\frac{x}{\beta}\right)^{a-1}\left(1 + \left(\frac{x}{\beta}\right)^{a}\right)^{-2}$	$a\,(> 0)$: shape parameter $\beta\,(> 0)$: scale parameterand $0 \leq x < +\infty$
4	**GeneralizedExtreme Value**	$f(x) = \begin{cases} \frac{1}{\sigma}exp\left(-(1+kz)^{\frac{-1}{k}}\right) \times (1+kz)^{-1-\frac{1}{k}}, & k \neq 0 \\ \frac{1}{\sigma}\,exp(-z - exp(-z)), & k = 0 \end{cases}$ where $z = \frac{(x-\mu)}{s}$	k (real): Shape Parameter$\sigma\,(> 0)$: Scale Parameter μ (real): Location ParameterDomain:$1 + k\frac{(x-\mu)}{\sigma} > 0$, for $k \neq 0$$-\infty < x < +\infty$, for $k = 0$
5	**Lognormal (2P)**	$f(x) = \frac{1}{(x)s\sqrt{2p}} \times exp\left(-\frac{1}{2}\left(\frac{ln(x) - \mu}{s}\right)^{2}\right)$	$\sigma\,(> 0)$: scale parameterμ (real): location parameterand $0 < x < +\infty$
6	**Inverse Gaussian(2P)**	$f(x) = \sqrt{\frac{\lambda}{2p(x)^{3}}}\,exp\left(-\frac{\lambda((x)-\mu)^{2}}{2\mu^{2}(x)}\right)$	$\lambda\,(> 0)$: scale parameter $\mu\,(> 0)$: location parameterand $0 < x < +\infty$
7	**Weibull**	$f(x) = \frac{a}{\beta}\left(\frac{x}{\beta}\right)^{a-1} exp\left(-\left(\frac{x}{\beta}\right)^{a}\right)$	$a\,(> 0)$: shape parameter $\beta\,(> 0)$: scale parameter, and $0 \leq x < +\infty$

Table 10. Abbreviations and their Meanings

S.No.	Abbreviations	Meanings
1.	BS (3P)	Three parameters Birnbaum-Saunders distribution
2.	BS (2P)	Two parameters Birnbaum-Saunders distribution
3.	BS(α,β)	Birnbaum-Saunders distribution with parameters α, and β
4.	BS(α,β,λ)	Birnbaum-Saunders distribution with parameters α, β, and λ
6.	CDF	Cumulative Distribution Function
7.	PDF	Probabilty Density Function
8.	HF	Hazard Function
9	MOM	Method ogf Moments
10	MLE	**The Method of Maximum Likelihood**
11	GOF	goodness-of-fit