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The Length-Biased Weighted Exponentiated Inverted Exponential Distribution: Properties and Estimation

Marwa M. Salama[1*], **Elsayed A. Elsherpieny**[1], **Abd-eltawab A. Abd-elaziz**[1]

[1] Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt; mrwzaghloul@gmail.com, ahmedc55@yahoo.com, abdeltawabgira@gmail.com.

* **Correspondence:** mrwzaghloul@gmail.com

Abstract: This paper introduces the length- biased weighted exponentiated inverted exponential distribution with sub models. The basic statistical properties are derived such as mean, mode, variance, reliability functions, moment generating function, characteristic function and order statistic distribution. Furthermore, maximum likelihood estimators of distribution parameters are obtained under censored Type I. Numerical study is applied. Variance-covariance matrix is obtained and Confidence intervals are calculated. Two real data sets are applied to compare performance of the proposed model with competitive distributions by using some criterions as measures of goodness fit.

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1. Introduction

Weighted distributions are used in many researches related to reliability analysis of family data, biomedicine and ecology. They provide us way to deal with model specification and data interpretation problem. This distribution was first introduced by Fisher [1]. He studied an effect of ascertainment methods on the form of the distribution for recorded observations and then Rao [2] introduced it in general form for modeling statistical data when the usual using of standard distributions were unsuitable. Patil and Rao [3] proposed statistical applications on weighted distribution and studied length biased (size biased) sampling related to human families and wildlife populations with applications. The weighted distribution is referred to length-biased distribution when the weight function is the length of the units, size biased distribution as generalization of length biased-sampling and doubly weighted distribution when using two weight functions. There are many researches have been discussed many researches on weighted distributions with application included important results on weighted distributions. Gupta and Kundu [4] introduced shape parameter to exponential distribution depends on Azzalini's method [5]. Saghir et al. [6] developed length–biased weighted exponentiated inverted Weibull distribution (LBWEIWD) and applied maximum likelihood estimation (MLE) method in estimation Alqallaf et al. [7] applied various methods of estimation for two parameters weighted exponential distribution (TPWED) which introduced by Gupta and Kundu [4]. Oguntunde [8] presented the exponentiated weighted exponential distribution (EWED) and discussed its statistical properties. Helal et al. [9] discussed statistical properties of weighted Shanker distribution. MLE and method of moments (MOM) were used for estimating the parameters of the model. Nasiru [10] presented another weighted Weibull distribution (WWD) based on Azzalini's [5] family with statistical properties and MLE was used in parameters estimation. Sobhi and Mashail [11] discussed moments of dual generalized order statistics and characterization for transmuted exponential mode. Rezzoky and Nadhel [12] proposed a weighted generalized exponential distribution (WGED).They estimated its scale parameter by using MOM and MLE.

Oguntunde et al. [13] presented a two parameter inverted weighted exponential distribution, discussed its statistical properties and estimated its unknown parameters Ocloo et al. [14] discussed extension of the Burr XII distribution with its applications and regression model. Almutiry [15] introduced inverted length-biased exponential distribution with statistical properties. The maximum likelihood strategy used to estimate the model parameters in the case of complete and Type II censored samples. Abubakari et al. [16] introduced Chen Burr-Hatke exponential distribution with its properties, regressions and biomedical applications. Haj Ahmad et al. [17] discussed statistical analysis of alpha power inverse Weibull distribution under hybrid censored scheme with applications to ball bearings technology and biomedical data. Almetwally [18] discussed Type I and Type II censoring under Marshall olkin alpha power extended Weibull distribution.

Ilori and Jolayemi [19] introduced weighted exponentiated inverted exponential distribution (WEIED) with statistical properties and estimation parameters by MLE. The probability density function (pdf) of WEIED is given by**:**

$$f_w(x) = \frac{(\alpha\lambda)^{(1+\beta)}}{\Gamma(1+\beta)} x^{-(2+\beta)} e^{\frac{-\alpha\lambda}{x}}; x > 0, \alpha, \beta, \lambda > 0, \tag{1.1}$$

where α,$\lambda$are scale parameters and β is the shape parameter.

The associated cumulative density function (cdf) is shown as below:

$$F_w(x) = \frac{\Gamma\left(1+\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(1+\beta)}; x > 0, \alpha, \beta, \lambda > 0, \tag{1.2}$$

where $\Gamma(\beta, t)$ is the upper incomplete gamma function and it is defined as$\Gamma(\beta, t)$= $\int_t^\infty t^{\beta-1} e^{-t}\, dt$.

The corresponding survival function is:

$$s(x) = 1 - \frac{\Gamma\left(1+\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(1+\beta)}; \quad x > 0. \tag{1.3}$$

The corresponding hazard function is given by:

$$h(x) = \frac{(\alpha\lambda)^{(1+\beta)} x^{-(2+\beta)} e^{\frac{-\alpha\lambda}{x}}}{\Gamma(1+\beta) - \Gamma\left(1+\beta, \frac{\alpha\lambda}{x}\right)}, \ x > 0, \alpha > 0, \beta > 0.. \tag{1.4}$$

Possible plots for the pdf and the hazard function at various parameters values are as shown in Figure 1.

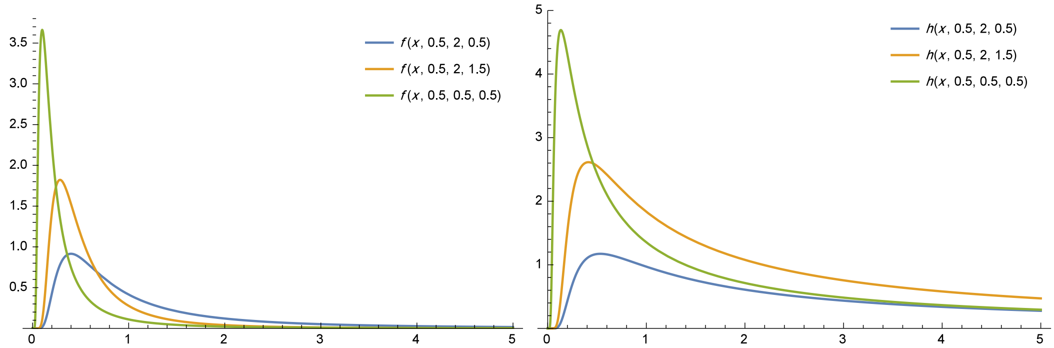

Figure 1. pdf plot (left) and Hazard plot (right) of the weighted exponentiated inverted exponential distribution

From Figure 1, the pdf and hazard rate function of WEIED is right skewed and decreasing at varied values of parameters.

The objective of this paper is to introduce the length-biased weighted exponentiated inverted exponential distribution (LBWEIED) to gain more flexibility for fitting life time data and estimate the unknown parameters by using MLE under complete and censored sample data. The asymptotic confidence intervals of unknown parameters are obtained. Simulation study is performed to examine the behavior of provided estimates. Finding variance-covariance matrix. Calculating confidence intervals of simulated estimates. Applying real data set to illustrate the applicability of the new distribution and compare LBWEIED with another competitive distributions.

The paper is organized as follows: in section 2 the LBWEIED is introduced with sub models. Some statistical properties in section 3. Section 4 presents methods of estimation. Section 5 provides a numerical illustration. An application of two real data sets in section 6. Section 7 introduces the conclusion.

2. The Length-Biased Weighted Exponentiated Inverted Exponential Distribution

In this section, we introduce the length biased version of WEIED based on the definition of the pdf of the weighted random variable X as follows:

$$f_w(x) = \frac{w(x)\, f(x)}{\mu(w)}, \quad \mu(w) < \infty, \tag{2.1}$$

where $w(x)$ is a non-negative weight function and $\mu(w) = E[W(X)] < \infty$.

The LBWEIED is obtained by using the pdf of WEIED, then $\mu(w)$ is derived as bellows:

$$\begin{aligned}\mu(w) &= \int_0^\infty w(x) f(x)\, dx \\ &= \frac{(\alpha\lambda)^{(1+\beta)}}{\Gamma(1+\beta)} \int_0^\infty x^{-(1+\beta)} e^{\frac{-\alpha\lambda}{x}}\, dx,\end{aligned}$$

let $t = \frac{\alpha\lambda}{x}, x = \frac{\alpha\lambda}{t}, dx = \frac{-\alpha\lambda}{t^2} dt$

$$\mu(w) = (\alpha\lambda)\int_0^\infty t^{\beta-1}e^{-t}\,dt$$
$$= \frac{(\alpha\lambda)\,\Gamma(\beta)}{\Gamma(1+\beta)}. \tag{2.2}$$

The pdf of LBWEIED is obtained by inserting (2.2) and (1.1) in (2.1) as follows:

$$f_w(x) = \frac{(\alpha\lambda)^\beta}{\Gamma(\beta)}x^{-(1+\beta)}e^{\frac{-\alpha\lambda}{x}};\, x > 0, \alpha, \beta, \lambda > 0, \tag{2.3}$$

where α,λ are scale parameters and β is the shape parameter.

The associated cdf is derived as:

$$F_w(x) = \frac{(\alpha\lambda)^\beta}{\Gamma(\beta)}\int_0^x x^{-(1+\beta)}e^{\frac{-\alpha\lambda}{x}}\,dx$$

let $t = \frac{\alpha\lambda}{x}, x = \frac{\alpha\lambda}{t}, dx = \frac{-\alpha\lambda}{t^2}dt$. Then, $F_w(x)$ can be written as

$$F_w(x) = \frac{(\alpha\lambda)^\beta}{\Gamma(\beta)}\int_{\frac{\alpha\lambda}{x}}^\infty \left(\frac{\alpha\lambda}{t}\right)^{-(1+\beta)} e^{-t}\left(\frac{\alpha\lambda}{t^2}\right)dt$$
$$= \frac{1}{\Gamma(\beta)}\int_{\frac{\alpha\lambda}{x}}^\infty t^{\beta-1}e^{-t}dt \tag{2.4}$$
$$= \frac{\Gamma\left(\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(\beta)},$$

where $\Gamma(., t)$ is the upper incomplete gamma function.

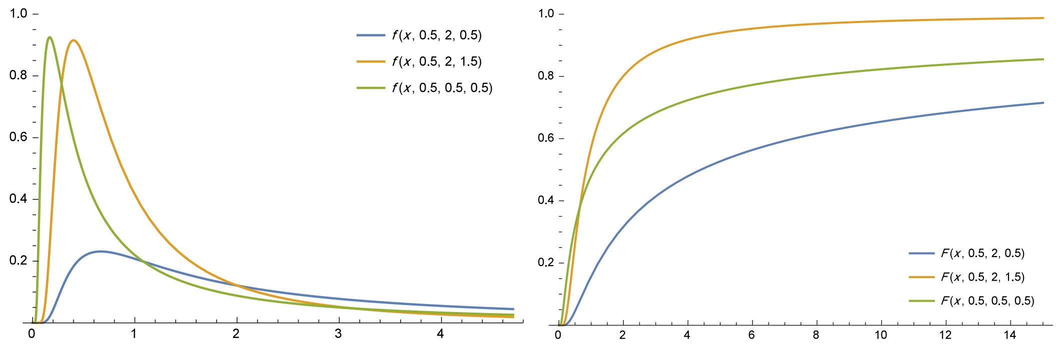

Figure 2. pdf (left) and cdf (right) plot of the length-biased weighted exponentiated inverted exponential distribution

Possible plots for the pdf and cdf of the LBWEIED at various parameter values are as shown in Figure 2. From Figure 2, the pdf of LBWEIED is right skewed and decreasing rate at the corresponding various values of parameters.

2.1. Some special cases

Case 1: Putting β=1 in Equation (2.3), gives two parameter exponentiated inverted exponential distribution (TPEIED), see (Fatima & Ahmed [21]) with the following pdf:

$$f(x) = \frac{\alpha\lambda}{x^2}\left(e^{\frac{-\lambda}{x}}\right)^{\alpha}; x > 0, \alpha, \lambda > 0.$$

Case 2: Putting $\beta = 1, \alpha = 1, \lambda = \frac{1}{\lambda}$, the inverse exponential distribution (IED) is produced, (see Lin et al. [22]) with the following pdf:

$$f(x) = \lambda^{-1}x^{-2}\exp\left\{(\lambda x)^{-1}\right\}; x > 0, \alpha, \lambda > 0.$$

Case 3: Putting $\beta = 2, \alpha = 1$, the inverted length– biased exponential distribution (ILBED), (see Almutiry [15]) with the following pdf:

$$f(x) = \frac{\lambda^2}{x^3}\left(e^{\frac{-\lambda}{x}}\right); x > 0, \lambda > 0.$$

2.2. Reliability analysis

Survival function corresponding to (2.4) is:

$$\begin{aligned} S(x) &= 1 - F(x) \\ &= 1 - \frac{\Gamma\left(\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(\beta)}\ x > 0\ , \alpha > 0, \beta > 0. \end{aligned} \tag{2.5}$$

The corresponding hazard function is given by:

$$h(x) = \frac{(\alpha\lambda)^{\beta}\, x^{-(1+\beta)} e^{\frac{-\alpha\lambda}{x}}}{\Gamma(\beta) - \Gamma\left(\beta, \frac{\alpha\lambda}{x}\right)}\quad , x > 0, \alpha > 0, \beta > 0\ . \tag{2.6}$$

Plots for the pdf and hazard rate function of the LBWEIED at various parameter values are as shown in Figure 3.

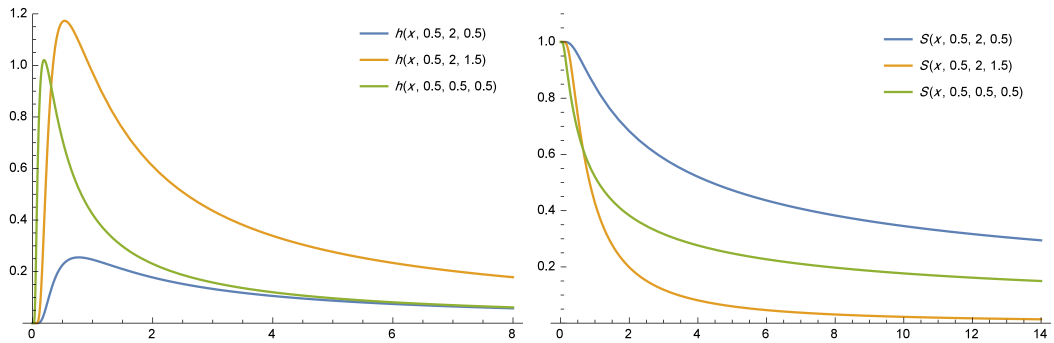

Figure 3. hazard plot (left) and survival plot (right) plot of the length-biased weighted exponentiated inverted exponential distribution

Figure 3 shows that the hazard rate function of LBWEIED is positively skewed and unimodal (inverted bathtub) and decreasing shapes. This implies that the LBWEIED can be used to describe or model real life phenomena with unimodal or decreasing failure rates.

3. Some Statistical properties

In this section, some basic statistical properties are derived and obtained such as moments, mean, variance, moment generating function, quantile function, median, mode, characteristic function and order statistic distribution.

3.1. Quantile Function and Median for LBWEIED

Let $Q(u)$ denoted to the quantile function for LBWEIED, where 0< u <1. Then $Q(u) = F^{-1}(u)$.

Since $F_w(x)$ for LBWEIED involves in complete gamma function in Equation (2.2), the quantile function can be obtained by numerical method only. For calculating median, let $x_o$is the median and substitute in Equation (2.2) then solve equation numerically by equating it to 0.5.

$$F_w(x_o) = \frac{\gamma\left(\beta, \frac{\alpha\lambda}{x_o}\right)}{\Gamma(\beta)} = 0.5.$$

3.2. Moments and Moment Generating Function

If X follows LBWEIED with pdf in Equation (2.3), the r^{th} non central moment is given by:

$$\mu_r^{'} = \frac{(\alpha\lambda)^r \Gamma(\beta - r)}{\Gamma(\beta)}, \beta > r. \tag{3.1}$$

To obtain (3.1), we employ the following:

$$\mu_r^{'} = \int_0^\infty x^r f(x)\,dx$$
$$= \frac{(\alpha\lambda)^\beta}{\Gamma(\beta)} \int_0^\infty x^{-(1-r+\beta)} e^{\frac{-\alpha\lambda}{x}}\,dx.$$

Let $t = \frac{\alpha\lambda}{x}, x = \frac{\alpha\lambda}{t}, dx = \frac{-\alpha\lambda}{t^2}dt$. Then by substituting it in previous equation

$$\mu_r^{'} = \frac{(\alpha\lambda)^r}{\Gamma(\beta)} \int_0^\infty t^{-(1+r-\beta)} e^{-t}\,dt$$
$$= \frac{(\alpha\lambda)^r \Gamma(\beta - r)}{\Gamma(\beta)}, \quad \beta > r.$$

From the r^{th} non central moment, then the mean and variance, respectively are $\mu_1^{'}, \mu_2^{'}$ as follows:
Putting $r = 1, \mu_1^{'} = \frac{(\alpha\lambda)\Gamma(\beta-1)}{\Gamma(\beta)}, \beta > 1$.

$$\sigma^2 = \mu_2^{'} - \left(\mu_1^{'}\right)^2 = \frac{(\alpha\lambda)^2 \Gamma(\beta - 2)}{\Gamma(\beta)} - \frac{(\alpha\lambda)^2 \Gamma^2(\beta - 1)}{\Gamma^2(\beta)}$$
$$= \frac{(\alpha\lambda)^2 \left(\Gamma(\beta)\Gamma(\beta - 2) - \Gamma^2(\beta - 1)\right)}{\Gamma^2(\beta)}.$$

Furthermore, the moment generating function $M_x(t)$ for LBWEIED is given as following:

$$M_x(t) = \sum_{r=0}^{\infty}\left[\frac{t^r}{r!}\frac{(\alpha\lambda)^r\,\Gamma(\beta-r)}{\Gamma(\beta)}\right], \tag{3.2}$$

where $M_x(t) = E(e^{xt}) = \int_0^\infty e^{xt} f(x)\ dx$.

To get (3.2), using Taylor series as follows

$$M_x(t) = \int_0^\infty \left(1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots + \frac{t^rx^r}{r!}\right) f(x)\ \ dx.$$
$$= \sum_{r=0}^{\infty}\left[\frac{t^rE(X^r)}{r!}\right] = \sum_{r=0}^{\infty}\left[\frac{t^r}{r!}E(X^r)\right].$$

Insert Equation (3.1), then the resultant is given by: $M_x(t) = \sum_{r=0}^{\infty}\left[\frac{t^r}{r!}\frac{(\alpha\lambda)^r\Gamma(\beta-r)}{\Gamma(\beta)}\right]$.

3.3. The Mode

Based on the pdf for LBWEIED in Equation (2.3), the mode can be calculated by taking the logarithm of $f_w(x)$ and obtain the first derivative with respect to zero, then solving to obtain x_o.

The mode can be derived as follows:

$$\log[f_w(x)] = \beta\log(\alpha) + \beta\log(\lambda) - \log\Gamma(\beta) - \log(x) - \beta\log(x) - \frac{\alpha\lambda}{x}.$$

Then the first derivative respect to x is given as

$$\frac{\partial}{\partial x}\log[f_w(x)] = \frac{1}{x}\left(-1 - \beta + \frac{\alpha\lambda}{x}\right) = 0\ .$$

Therefore $\left(-1 - \beta + \frac{\alpha\lambda}{x}\right) = 0$. Then the mode of LBEIED is as follows

$$x_o = \frac{\alpha\lambda}{1+\beta}. \tag{3.3}$$

3.4. Charachteristic function of LBWEIED

Let $X \sim \text{LBWEIED}(\alpha, \beta, \lambda)$, the characteristic function $\varphi_x(t)$ can be derived as follows:

$$\varphi_x(t) = E\left(e^{itx}\right) = \int_0^\infty e^{itx} f(x)\ dx$$

Using Taylor series

$$\varphi_x(t) = \sum_{r=0}^{\infty}\left[\frac{(it)^r\,E(X^r)}{r!}\right] = \sum_{r=0}^{\infty}\left[\frac{(it)^r}{r!}E(X^r)\right]. \tag{3.4}$$

Insert Equation (3.1), the resultant is given by:

$$\varphi_x(t) = \sum_{r=0}^{\infty}\left[\frac{(it)^r}{r!}\frac{(\alpha\lambda)^r\,\Gamma(\beta-r)}{\Gamma(\beta)}\right]. \tag{3.5}$$

3.5. Order Statistic

The pdf of kth order statistics for a random sample of size n from LBWEIED is shown as:

$$f_{k:n}(x) = \frac{n!}{(k-1)!\,(n-k)!}\frac{(\alpha\lambda)^{(\beta)}}{\Gamma(\beta)}x^{-(1+\beta)}e^{\frac{-\alpha\lambda}{x}}\left[\frac{\Gamma\left(\beta,\frac{\alpha\lambda}{x}\right)}{\Gamma(\beta)}\right]^{k-1}\left[1-\frac{\Gamma\left(\beta,\frac{\alpha\lambda}{x}\right)}{\Gamma(\beta)}\right]^{n-k}.$$

In particular, the distribution of minimum and maximum order statistics is given by:

$$f_{1:n}(x) = n\frac{(\alpha\lambda)^{(\beta)}}{\Gamma(\beta)}x^{-(1+\beta)}e^{\frac{-\alpha\lambda}{x}}\left[1-\frac{\Gamma\left(\beta,\frac{\alpha\lambda}{x}\right)}{\Gamma(\beta)}\right]^{n-1}, \tag{3.6}$$

and

$$f_{n:n}(x) = n\frac{(\alpha\lambda)^{(\beta)}}{\Gamma(\beta)}x^{-(1+\beta)}e^{\frac{-\alpha\lambda}{x}}\left[\frac{\Gamma\left(\beta,\frac{\alpha\lambda}{x}\right)}{\Gamma(\beta)}\right]^{n-1}. \tag{3.7}$$

4. Methods of Estimations

This section gives maximum likelihood and moment estimators for the parameters α, β and λ of the LBWEIED under complete sample. Furthermore, the parameters estimators for the LBWEIED are discussed under Type I censored sample.

4.1. Method of Moments

To obtain the moment estimators of the unknown parameters. We equal the three theoretical moments with the sample moments, where the sampling moments are given by:

$$\frac{1}{n}\sum_{i=1}^{n}x_i,\ \frac{1}{n}\sum_{i=1}^{n}x_i^2,\ \frac{1}{n}\sum_{i=1}^{n}x_i^3.$$

Hence by equating the theoretical with sampling. We have the following:

$$\frac{1}{n}\sum_{i=1}^{n}x_i = \frac{(\alpha\lambda)\,\Gamma(\beta-1)}{\Gamma(\beta)}, \tag{4.1}$$

$$\frac{1}{n}\sum_{i=1}^{n}x_i^2 = \frac{(\alpha\lambda)^2\,\Gamma(\beta-2)}{\Gamma(\beta)}, \tag{4.2}$$

$$\frac{1}{n}\sum_{i=1}^{n}x_i^3 = \frac{(\alpha\lambda)^3\,\Gamma(\beta-3)}{\Gamma(\beta)}. \tag{4.3}$$

By solving three equations with the system (4.1), (4.2), and (4.3), the moment's estimates of α, β and λ can be obtained.

4.2. Parameters Estimation of LBWEIED using MLE

Let $\underline{X} = (x_1, x_2, ..., x_n)$ is a random sample of size n from LBWEIED. From Equation (2.1). The likelihood function is given by:

$$L(\underline{x}, \alpha, \beta, \lambda) = \left(\frac{(\alpha\lambda)^{\beta}}{\Gamma(\beta)}\right)^{n} \prod_{i=1}^{n} x_i^{-(1+\beta)} e^{\sum_{i=1}^{n} \frac{-\alpha\lambda}{x_i}}. \tag{4.4}$$

Let log-likelihood function be denoted by L. So, its log likelihood function is obtained as:

$$l = n\beta \ln(\alpha) + n\beta \ln(\lambda) - n \ln \Gamma(\beta) - (1+\beta)\sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} \frac{\alpha\lambda}{x_i}. \tag{4.5}$$

Taking the partial derivatives of Equation (4.5) with respect to α,$\beta$and λ then equating it by zero.

$$\frac{\partial L}{\partial \alpha} = \frac{n(\beta)}{\alpha} - \sum_{i=1}^{n} \frac{\lambda}{x_i} = 0, \tag{4.6}$$

$$\frac{\partial L}{\partial \beta} = n \ln\alpha + n \ln\lambda - n\psi(\beta) - \sum_{i=1}^{n} \ln x_i = 0, \tag{4.7}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n(\beta)}{\lambda} - \sum_{i=1}^{n} \frac{\alpha}{x_i} = 0. \tag{4.8}$$

As seen Equations (4.6)-(4.8), have no explicit numerical solution, so numerical technique must be applied to get the solution after setting them with zero. To get the approximate confidence intervals, we must obtain the second partial derivative of all parameter as below

$$\frac{\partial^2 L}{\partial \alpha^2}_{\beta=\hat{\beta},\alpha=\hat{\alpha}} = -\frac{n(\beta)}{\alpha^2},$$

$$\frac{\partial^2 L}{\partial \alpha \partial \beta}_{\alpha=\hat{\alpha}} = \frac{\partial^2 L}{\partial \beta \partial \alpha} = \frac{n}{\alpha},$$

$$\frac{\partial^2 L}{\partial \alpha \partial \lambda} = \frac{\partial^2 L}{\partial \lambda \partial \alpha} = -\sum_{i}^{n} \frac{1}{x_i},$$

$$\frac{\partial^2 L}{\partial \beta^2}_{\beta=\hat{\beta}} = -n\psi(1,\beta),$$

$$\frac{\partial^2 L}{\partial \beta \partial \lambda}_{\lambda=\hat{\lambda}} = \frac{\partial^2 L}{\partial \lambda \partial \beta} = \frac{n}{\lambda},$$

$$\frac{\partial^2 L}{\partial \lambda^2}_{\beta=\hat{\beta},\lambda=\hat{\lambda}} = -\frac{n(\beta)}{\lambda^2}.$$

It has to be solved numerically, the above approach is used to derive the approximate $100(1-\tau)\%$ confidence intervals of the parameters α,β and λ as in the following forms: $\hat{\alpha} \pm Z_{\frac{\tau}{2}} \sqrt{\mathrm{var}(\hat{\alpha})}$, $\hat{\beta} \pm Z_{\frac{\tau}{2}} \sqrt{\mathrm{var}(\hat{\beta})}$ and $\hat{\lambda} \pm Z_{\frac{\tau}{2}} \sqrt{\mathrm{var}(\hat{\lambda})}$, where $\frac{\tau}{2}$ is the upper the percentile of the standard normal distribution.

4.3. Estimation Parameters of LBWEIED under Censored Type I

Let $\underline{X} = (x_1 < x_2 < ... < x_n)$ is a random sample of size n from LBWEIED, then $x_{(i)}$is the failure items and k is the number of failures at predetermined time (T), where n and T are fixed. The likelihood function under censored Type I is defined by:

$$L(\underline{x}, \alpha, \beta, \lambda) = C \prod_{i=1}^{k} f\left(x_{i,\theta}\right) \left[1 - F\left(T, \theta\right)\right]^{n-k}, \tag{4.9}$$

where C is $\frac{n!}{(n-k)!}$

Insert Equation (2.3) and (2.5) in Equation (4.9) gives

$$L(\underline{x}, \alpha, \beta, \lambda) = C \left(\frac{(\alpha\lambda)^{\beta}}{\Gamma(\beta)} \right)^{k} \prod_{i=1}^{k} x_i^{-(1+\beta)} e^{\sum_{i=1}^{k} \frac{-\alpha\lambda}{x_i}} \left[1 - \frac{\Gamma\left(\beta, \frac{\alpha\lambda}{T}\right)}{\Gamma(\beta)} \right]^{n-k}. \tag{4.10}$$

Let log-likelihood function be denoted by L and put $A = \left[1 - \frac{\Gamma\left(\beta, \frac{\alpha\lambda}{T}\right)}{\Gamma(\beta)}\right]$, then it can be write as follows:

$$l(\underline{x}, \alpha, \beta, \lambda) = k\beta \ln(\alpha) + k\beta \ln(\lambda) - k \ln \Gamma(\beta) - (1+\beta) \sum_{i=1}^{k} \ln(x_i) - \sum_{i=1}^{k} \frac{\alpha\lambda}{x_i} + (n-k) \ln(A). \tag{4.11}$$

Therefore differentiate (4.11) with respect to the unknown parameters α,β, and λ respectively and equating to zero. As the maximum –likelihood equations in many cases have no analytical solution and cannot be solved in closed form. It has to be solved by using numerical optimization method.

5. A Numerical Illustration

In this section, a numerical example is applied to illustrate different maximum likelihood estimators and their variance covariance matrix by using different sample sizes like n = 30, 60 and 100 from LBWEIED with following sets of parameters (α, β, λ) ={(0.8,1.5,1), (0.2,1.5,1), (0.5,0.5,1)} . For generating random numbers form LBWEIED generate n random numbers from uniform distribution (0, 1) and equating it with the cumulative of the LBWEIED $\frac{\Gamma\left(\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(\beta)} = u_i$. It cannot be solved in closed form it has to be solved by using numerical optimization method as using programming software like Mathematica. For estimating the unknown parameters, it is iterated 10000 times, mainly the bias (Bias) of the gained values, mean square errors (MSE) and standard deviation (SD) are calculated. Also, confidence interval for the estimators are obtained. This simulation study results are tabulated in Table 1.

The simulation study is applied as the following process:

1. Generating random sample by equating the cumulative of the LBWEIED $\frac{\Gamma\left(\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(\beta)} = u_i$; with n random numbers from uniform distribution (0, 1).

2. Estimating unknown values for α, λ and β by using the MLE.

3. Repeating steps 1–2 10,000 times, calculating Bias of the gained values, MSE and SD by the following formula:

$$Bias(\hat{\theta}) = E\left(\hat{\theta}\right) - \theta.$$

$$SD(\hat{\theta}) = \sqrt{\text{variance}(\hat{\theta})}.$$

$$MSE(\hat{\theta}) = \text{variance}(\hat{\theta}) - (\text{bias}(\hat{\theta}))^2,$$

where $\theta = (\alpha, \beta, \lambda)$

4. Confidence intervals of the parameters α, β and λ as following forms: $\hat{\alpha} \pm Z_{\frac{\tau}{2}} \sqrt{\text{var}(\alpha)}$, $\hat{\beta} \pm Z_{\frac{\tau}{2}} \sqrt{\text{var}(\hat{\beta})}$ and $\hat{\lambda} \pm Z_{\frac{\tau}{2}} \sqrt{\text{var}(\hat{\lambda})}, \frac{\tau}{2}$ is the upper percentile of the standard normal distribution.

From Table 1, the following points can be noticed:

1. The bias of all estimators of the parameters decrease with increasing in sample size.
2. MSE of simulated estimates is decreasing when sample size is increasing.
3. SD decreases with the increase of sample size.

6. Application to Real Data

The analysis is provided to check the goodness fit of the LBWEIED, exponentiated inverted exponential distribution (EIED) and Pareto distribution by using two real data sets. The corresponding measures of fit statistic using log likelihood function, Kolmogorov-Smirnov (K-S) statistic with corresponding P-value and some criterions such as Akaike information criterion (AIC), correct Akaike information criterion (CAIC), Basyian information criterion (BIC) and Hannan-Quinn information criterion (HQIC), where sample size is denoted by n, m is the number of parameters for statistical model and $lnL(\underline{x};\theta)$ is the value of the highest log likelihood function for considered model which can be denoted by LL. The better distribution has less value of AIC, CAIC, BIC and HQIC than the other competitive distributions and highest log-likelihood value.

i. The first data set

The first data set was used by Ghitany et al. [20]. It represents waiting time (minutes) before service of 100 bank customer. Table 3 shows maximum likelihood estimators of some selected distributions parameters for complete data in Table 2. From this Figure 4, it is observed that LBWEIED is fitted to the data. Also, it is better than the EIED and Pareto distribution.

ii. The second data set

Data set Table 5 represents the marks in Mathematics for 48 students in the slow pace programme in the year 2003. See Gupta and Kundu [4]. Table 6 presents maximum likelihood estimators of some selected distributions parameters for complete data.

Table 7 shows the K-S statistics, p-value, LL, AIC, CAIC, BIC and HQIC for the data set. From Table 7, the p-value of K-S for the LBWEIED is 0.503224. So, it is fitted to the data set compared with Pareto distribution and EIED. Also, it can be noted that proposed model fits data set more than Pareto distribution and EIED. It is evident that the LBWEIED is a strong competitor to other distributions. Figure 5 shows the empirical cdf of the real data set 5 compared with the estimated cdf. From Figure 5, it is observed that LBWEIED is fitted to data set 5 and more fitted than EIED and Pareto distribution.

iii. Estimation parameters under censored Type I based on first data set

From likelihood function of Type I censored for LBWEIED in Equation (4.6). It have to add information about predetermined failure time (T), where k is number of failures out of n. Analysis of estimation parameters α, β and λ for LBWEIED based on censored Type I is applied under first data sets is illustrated in Table 8. Let T=11, it was 64 censored items and 36 uncensored items while T=10,

Table 1. MLE of the parameters and 95% confidence intervals with lower (L_i) and upper bounds (U_i) (complete data)

$\lambda = 1$			MLE			Interval	
n	(α,β)		SD	Bias	MSE	Li	Ui
30	(0.8,1.5)	$\hat{\alpha}$	0.12285	0.03681	0.01646	0.05512	1.61695
		$\hat{\beta}$	0.42356	0.14341	0.19997	0.85149	2.42832
		$\hat{\lambda}$	0.15357	0.04601	0.0257	0.0689	2.02119
	(0.2,1)	$\hat{\alpha}$	0.03191	0.0101	0.00112	0.01408	0.552266
		$\hat{\beta}$	0.27452	0.09308	0.08403	0.04824	5.06222
		$\hat{\lambda}$	0.03191	0.7899	0.62496	0.131986	0.921887
	(0.5,0.5)	$\hat{\alpha}$	0.09184	0.03059	0.00937	0.246013	0.795698
		$\hat{\beta}$	0.12595	0.04228	0.01765	0.033277	3.41207
		$\hat{\lambda}$	0.18369	0.06118	0.03748	0.492025	1.59139
60	(0.8,1.5)	$\hat{\alpha}$	0.08232	0.01861	0.00712	0.44296	1.1923
		$\hat{\beta}$	0.27037	0.06523	0.07736	1.09785	0.04094
		$\hat{\lambda}$	0.08232	0.1814	0.03968	0.68953	1.35709
	(0.2,1)	$\hat{\alpha}$	0.02166	0.00504	0.00049	0.098275	0.311949
		$\hat{\beta}$	0.17381	0.04318	0.03207	0.686344	2.59511
		$\hat{\lambda}$	0.10828	0.02519	0.01236	0.491373	1.55975
	(0.5,0.5)	$\hat{\alpha}$	0.06123	0.0128	0.00391	0.269127	0.75166
		$\hat{\beta}$	0.08098	0.01817	0.00689	0.642982	2.56348
		$\hat{\lambda}$	0.12246	0.02561	0.01565	0.538254	1.50332
100	(0.8,1.5)	$\hat{\alpha}$	0.06273	0.01029	0.00404	0.33338	1.28496
		$\hat{\beta}$	0.20257	0.03969	0.04261	0.875643	2.25203
		$\hat{\lambda}$	0.06273	0.18971	0.03993	0.208362	0.803102
	(0.2,1)	$\hat{\alpha}$	0.0164	0.00504	0.00028	0.114563	0.291439
		$\hat{\beta}$	0.13187	0.02579	0.01805	0.865192	2.28839
		$\hat{\lambda}$	0.08199	0.01567	0.00697	0.572815	1.45719
	(0.5,0.5)	$\hat{\alpha}$	0.03911	0.00677	0.00158	0.328209	0.686332
		$\hat{\beta}$	0.20471	1.04123	1.12607	0.83323	2.32052
		$\hat{\lambda}$	0.07821	0.01353	0.0063	0.656418	1.37366

Table 2. waiting time (minutes) of 100 bank customer

0.8,0.8,1.3,1.5,1.8,1.9,1.9,2.1,2.6,2.7,2.9,3.1,3.2,3.3,3.5,3.6,4.,
4.1,4.2,4.2,4.3,4.3,4.4,4.4,4.6,4.7,4.7,4.8,4.9,4.9,5.,5.3,5.5,5.7,
5.7,6.1,6.2,6.2,6.2,6.3,6.7,6.9,7.1,7.1,7.1,7.1,7.4,7.6,7.7,8.,8.2,
8.6,8.6,8.6,8.8,8.8,8.9,8.9,9.5,9.6,9.7,9.8,10.7,10.9,11.,11.,11.1
,11.2,11.2,11.5,11.9,12.4,12.5,12.9,13,13.1,13.3,13.6,13.7,13.9,
14.1,15.4,15.4,17.3,17.3,18.1,18.2,18.4,18.9,19.,19.9,20.6,21.3,
21.4,21.9,23.,27.,31.6,33.1,38.5

Table 3. The maximum likelihood estimates of the parameters among competing distributions.

Distribution	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$
LBWEIED	1.4729	1.5978	5.80108
EIED	2.11935		2.52323
Pareto	0.44558		0.8

Table 4. Analysis of the performance of the competing distributions

Distribution	K-S	P-value	LL	AIC	CAIC	BIC	HQIC
LBWEIED	0.10805	0.1799	-330.77	667.534	667.784	667.534	663.342
EIED	0.16745	0.00647	-336.56	677.16	677.24	677.16	674.012
Pareto	0.35185	1.9×10-11	-382.95	769.88	769.898	765.88	767.102

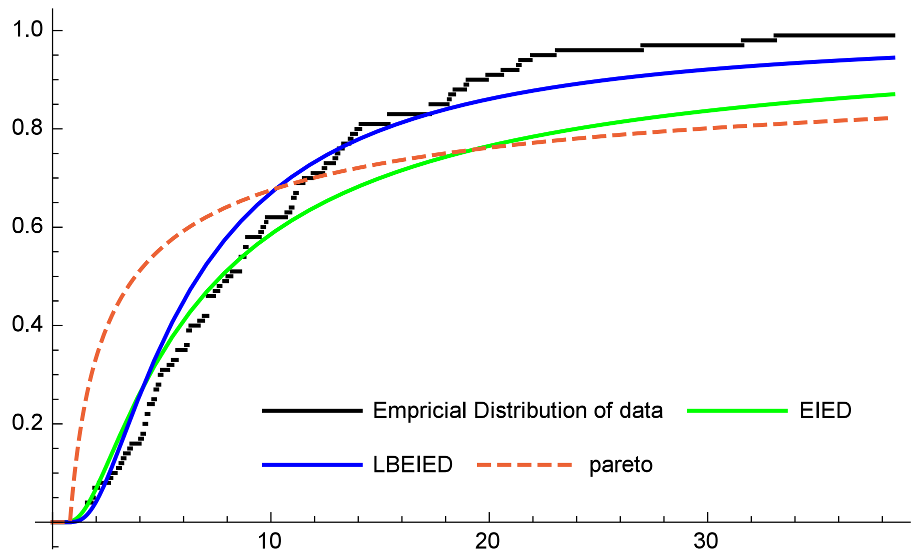

Figure 4. Estimated and empirical cdf for all models (data set 1)

Table 5. The marks in Mathematics for 48 students in the slow pace programme

29,25,50,15,13,27,15,18,7,7,8,19,12,18,5,21,15,86,21,15,14,39,15,14,
70,44,6,23,58,19,50,23,11,6,34,18,28,34,12,37,4,60,20,23,40,65,19,31

Table 6. The maximum likelihood estimates of the parameters for competing distribution (Data set 2)

Distribution	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$
LBWEIED	2.12941	2.02789	15.3198
EIED	3.14365	-	4.99164
Pareto	0.61492	-	4

Table 7. Analysis of the performance of the competing distributions (Data set 2)

Distribution	K-S	P-value	LL	AIC	CAIC	BIC	HQIC
LBWEIED	0.11585	0.50322	-198.94	403.88	404.42	402.924	399.234
EIED	0.17998	0.0784	-205.05	414.104	414.364	413.465	411.007
Pareto	0.32446	5.3E-05	-215.94	435.886	435.886	435.249	432.789

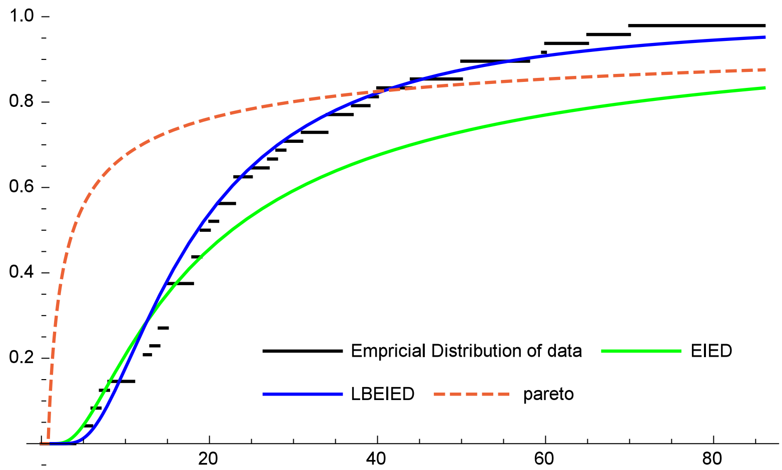

Figure 5. Estimated and empirical cdf for all models (data set 2)

censored item are 62 and 38 uncensored items. Maximum likelihood estimates under censored Type I and some goodness fit measures as AIC, BIC and HQIC are showed in Table 8.

Table 8. Maximum likelihood estimates of parameter for LBWEIED under censored (data set 1)

T	k	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	LL	AIC	BIC	HQIC
10	62	5.05602	1.0711	1.17714	-209.19	424.378	420.184	424.378
15	81	3.43912	1.31023	2.06478	-268.79	543.942	539.748	543.942

Table 8 presents the statistics of LL, AIC, BIC and HQIC for the life time data set when applying different predetermined times in the experiment. From Table 8, it is clear that the LBWEIED has the highest log likelihood, lowest AIC, BIC and HQIC at predetermined failure time T=10 with 62 censored items and 38 uncensored items.

iv. Estimation parameters under censored Type I based on second data set

Let T=27, it was 31 censored items and 17 uncensored items while T=22, censored item are 27and 21 uncensored items. Analysis of maximum likelihood estimates under censored Type I and some measures of goodness fit are shown in Table 9.

Table 9 presents the statistics of LL, AIC, BIC and HQIC for the life time data set when applying different predetermined times in the experiment. From Table 9, the model has the highest log likelihood, lowest AIC, BIC and HQIC at T= 22 with 27 censored items and 21 uncensored items.

Table 9. Maximum likelihood estimates of parameter for LBWEIED under censored (data set 2)

T	k	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	LL	AIC	BIC	HQIC
27	31	7.26932	1.59441	3.51977	-129.66	265.314	264.358	260.668
22	27	4.65805	1.57954	5.45676	-113.23	232.458	231.502	227.812

7. Conclusion

The LBWEIED has been introduced. Sub models from LBWEIED was obtained such as IED, ILBED and TPEIED. Also, the basic statistical properties have been successfully derived. The model has decreasing failure rates depending on the value of the parameters. MLE for complete and censored sample data has been applied to estimate the unknown parameters. Numerical study is applied for LBWEIED. Variance–covariance matrix is obtained with the confidence intervals. The real life application is provided by using two real data sets, then we note that performance of LBWEIED better than the competitive distributions. Finally, we can conclude that proposed model is a good competitive model.

Conflict of interest

The authors state that they have no financial or other conflicts of interest to disclose with connection to this research.

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