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Research article

On Joint Type-II Generalized Progressive Hybrid Censoring Scheme

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Abstract: This paper discussed a new scheme is called joint Type-II generalized progressively hybrid censoring scheme (JGPHCS-II). It assumed that the lifetime distribution of the items from the two populations follow exponential distribution. Based on the JGPHCS-II , we first consider the maximum likelihood estimators of the unknown parameters along with thier asymptotic confidence intervals. Next, we provide the Bayesian inferences of the unknown parameters under the assumptions of independent gamma priors on the scale parameters using squared error (SE) , linear-exponential (LINEX) and general entropy (GE) loss functions. Using gamma conjugate priors, the Bayes estimators are developed relative to both symmetric and asymmetric loss functions. Two numerical application based on simulated and real data sets are analyzed to discuss how the applicability of the proposed methods in real phenomenon. Finally, to examine the performance of proposed methods, a Monte Carlo simulation study, simulated example and real-life data are carried out.

Keywords: Exponential distribution; Joint type-II generalized progressive hybrid censoring scheme; Maximum likelihood estimation; Bayesian estimation; Credible intervals; Loss functions.
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1. Introduction

Recently, a considerable body of literature has been devoted to the type-II progressively censoring scheme. It is a generalization of type-II right censoring scheme. One drawback of the type-II progressive censoring is that it can take a lot of time to get to the r^{th} failure time. In the last two decades, the progressive type-II hybrid censoring scheme (denoted as PHCS-II), introduced by Kundu and Joarder [20], has received considerable interest in the literature for a life-testing problems and analyzing highly reliable data. The PHCS-II is a mixture of progressive type-II and hybrid censoring schemes. However, under PHCS-II the time on experiment will be no more than T, namely, if the r^{th} failure does not occur before time point T, then all the remaining units are removed and the experiment terminates at the time point T. Some recent studies on progressive hybrid censoring schemes have been carried out, see, for example, Mokhtari et al. [24], Hemmati and khorram [18], and Lin et al. (2013).

One limitation of the progressive hybrid censoring scheme is that it cannot be applied when very few failures may occur before time T. Therefore MLE for a parameter of an underlying distribution of observations may not be computed or its accuracy will be extremely low. For this reason, Cho et al. [13] introduced a new censoring scheme called generalized progressive hybrid censoring scheme (GPHCS) which allows them to observe a pre-specified number of failures. So, a certain number of failures and their lifetimes are always provided under the GPHCS. The life-testing experiment based on the proposed censoring scheme can save both the total time on tests and the cost induced by failures of the units. For more details the readers may refer to Cho et al. [15], Gorny and Cramer [17], Lee et al. [21], Lee et al. [23], and Ashour and Elshahhat [8].

Under GPHCS the experimenter would ideally like to observe r^{th} failures, but is willing to accept a bare minimum of r^{th} failures. Lee et al. [23] refer to the GPHCS as a Type-I GPHCS and proposed the Type-II GPHCS to overcome the drawbacks in Type-II PHCS is that it might take a very long time to observe r^{th} failures and complete the life test. Type-II GPHCS is a modified for Type-II PHCS by guaranteeing that the test will be completed at time T_2 , therefore, T_2 represents the absolute longest time that the researcher is willing to allow the experiment to continue. They suggested this type of censoring scheme to a guarantee the experiment terminated at a pre-fixed time.

For more details on joint censoring schemes, Balakrishnan and Rasouli [9], and Rasouli and Balakrishnan [26] introduced the joint Type-II censoring and the joint progressive Type-II censoring schemes, respectively. The jointly censored samples are handled in various studies by Ashour and Abo-Kasem [4], [5], [6], [7], Balakrishnan and Su [10], Balakrishnan et al. [11], Doostparast et al. [16], Abo-Kasem et al. [3]. In these studies, mainly Type-II and related censoring schemes are handled. Recently, Su and Zhu [27], Abo-Kasem and Elshahhat [1] studied the joint generalized Type-I hybrid censoring scheme and the joint generalized Type-II hybrid censoring scheme for two exponential populations. Unlike similarities in mathematical structures of joint generalized Type-I hybrid censoring scheme and generalized progressive hybrid censoring scheme, there are significant differences between these censoring schemes.

Recently, Cetinkaya et al. [12] introduced a new joint generalized progressive hybrid censoring scheme (JGPHCS) for two independent samples from different populations. We introduce a new scheme called joint Type-II generalized progressive hybrid censoring scheme JGPHCS-II in which an experiment is implemented between two samples.We analyze estimation problems of two exponential populations. The Maximum likelihood estimators (MLEs) and Bayes estimators (BEs), with their associated confidence intervals, of the two exponential mean parameters are derived based on JGPHCS-II. Under the assumption of independent gamma priors, the BEs are developed against SE, LINEX and GE loss functions. To assess the performances of the proposed estimators, some Monte Carlo simulations are conducted. Comparison between different point estimates is made with respect to their average estimates (AEs), mean squared-error (MSE) and mean absolute bias (MAB). Also, the performance of 95% two-sided interval estimates is compared using average confidence lengths (ACLs) and coverage percentages (CPs). Finally, a practical example using simulated and real-life data sets representing the times of air conditioning system of a fleet is discussed for illustrative purposes.

The rest of the article is organized as follows: The next section 2 provides the description and necessary assumptions of the proposed model. The maximum likelihood and Bayesian inferential procedures for estimating the unknown exponential parameters are discussed in Sections. 3, and 4, respectively. A Monte Carlo simulation results are presented in Section 5. Section 6 deals with a simulatedand real-life data sets for illustration purposes. Finally, Section 7 offers some concluding remarks.

2. Model Description

Suppose we consider products from two different populations. we draw a random sample of size m from population (2.1) with distribution function $F(x)$ and density function$f(x)$, and a random sample of size n from population (3.1) with distribution function $G(y)$ and density function$g(y)$. The two independent samples are placed simultaneously in a life testing experiment. Further, $W_{(1)} \le W_{(2)} \le ... \le W_{(N)}$ denote the order statistics of the $N = m + n$ random variables$(X_1, ..., X_m; Y_1, ..., Y_n)$. The proposed JGPHCS-II can be described as follows. The integer $r < N$ is fixed at the beginning of the experiment, $R_1, ..., R_r$ are $r$pre-fixed integers satisfying $R_1 + ... + R_r + r = N$and the times $T_1$and T_2 is also fixed beforehand and $0 < T_1 < T_2 < \infty$. Let D_1 and D_2 denote the number of observed failures up to time $T_1$and T_2, respectively. At the time of first observed failure, R_1 of the remaining items are withdrawn from the test at random. Following the second observed failure, R_2 of the remaining items are withdrawn and so on. This process continues until the termination time $T^* = \max\{T_1, \min\{w_{(r)}, T_2\}\}$, at this time all of the remaining units are removed from the experiment. If $w_{(r)} < T_1$, then instead of terminating the test by withdrawing the remaining R_r items after the r^{th} failure, the experiment continue to observe failures but without any further withdrawals up to time T_1 , therefore, $R_j = 0$, for $j = r, r+1, ..., D_1$. If $T_1 < w_{(r)} < T_2$, terminate the test at $w_{(r)}$. If $w_{(r)} > T_2$, terminate the test at time T_2.

Based on the Type-II GPHCS, the observed data will be one of the following three forms:

$$\begin{array}{ll} Case-I : \{w_{(1)}, ..., w_{(r)}, w_{(r+1)}, ..., w_{(d_1)}\}, & \text{if } w_{(r)} < T_1 < T_2, \\ Case-II : \{w_{(1)}, ..., w_{(d_1)}, ..., w_{(r)}\}, & \text{if } T_1 < w_{(r)} < T_2, \\ Case-III : \{w_{(1)}, ..., w_{(d_2)}, ..., w_{(r)}\}, & \text{if } T_1 < T_2 < w_{(r)}. \end{array}$$

The data observed in this form will consist of (Z, W, S) , where $W = (w_{(1)}, ..., w_{(D)})$, $Z = (z_1, ..., z_D)$ with $z_j = 1$ or 0 according as whether $w_{(j)}$ is either X- or Y-failure, respectively, and $S = (s_1, ..., s_D)$. $R = (R_1, R_2, ..., R_D)$ has the decomposition$S + Q = (s_1, ..., s_D) + (q, ..., q_D)$.

The likelihood function (without the constant term) of (z, w, s) can be written as,

$$L \propto \begin{cases} \prod_{j=1}^{d_1} \left(f(w_{(j)})^{z_j} g(w_{(j)})^{1-z_j}\right) \left(\bar{F}(w_{(j)})\right)^{s_j} \left(\bar{G}(w_{(j)})\right)^{q_j} \left(1-\bar{F}(T_1)\right)^{R^*_{d_1+1}} & \text{for } Case\ I, \\ \prod_{j=1}^{r} \left(f(w_{(j)})^{z_j} g(w_{(j)})^{1-z_j}\right) \left(\bar{F}(w_{(j)})\right)^{s_j} \left(\bar{G}(w_{(j)})\right)^{q_j} & \text{for } Case\ II, \\ \prod_{j=1}^{d_2} \left(f(w_{(j)})^{z_j} g(w_{(j)})^{1-z_j}\right) \left(\bar{F}(w_{(j)})\right)^{s_j} \left(\bar{G}(w_{(j)})\right)^{q_j} \left(1-\bar{F}(T_2)\right)^{R^*_{d_2+1}} & \text{for } Case\ III, \end{cases} \tag{2.1}$$

where $R_j = s_j + q_j$, $R^*_{d_1+1} = N - d_1 - \sum_{j=1}^{d_1} R_j$, $R^*_{d_2+1} = N - d_2 - \sum_{j=1}^{d_2} R_j$ and $\bar{F} = 1 - F$, $\bar{G} = 1 - G$ are the survival function of the two populations.

3. Maximum Likelihood Estimation

Suppose the lifetimes of m units of population (2.1) , $X_1, ..., X_m$, are independent and identically distributed (i.i.d.) random variables from exponential ($Exp\ (\theta_1)$) population with density and distribution functions as,

$$f(x) = \theta_1\ e^{-\theta_1 x} \text{ and } F(x) = 1 - e^{-\theta_1 x} for\ x\ \geq 0\ ,\ \ \theta_1\ > 0 \tag{3.1}$$

respectively. Similarly, let the lifetimes of n units of population 3.1, $Y_1, ..., Y_n$ be i.i.d. random variables from $Exp\ (\theta_2)$ population with density and distribution functions as,

$$g(y) = \theta_2\ e^{-\theta_2 x} \text{ and } G(x) = 1 - e^{-\theta_2 x} fory\ \geq 0\ ,\ \ \theta_2\ > 0 \tag{3.2}$$

respectively. The log-likelihood function corresponding to Equations (2.1), (3.1), and (3.2) is given by.

$$\ln L = \begin{cases} m_{d_1}\ln\theta_1 + n_{d_1}\ln\theta_2 + \sum_{j=1}^{d_1} z_j \ln e^{-\theta_1 w_{(j)}} + \sum_{j=1}^{d_1}(1-z_j)\ln\ e^{-\theta_2 w_{(j)}} + \sum_{j=1}^{d_1} s_j \ln e^{-\theta_1 w_{(j)}} \\ + \sum_{j=1}^{d_1} q_j \ln e^{-\theta_2 w_{(j)}} + R^*_{d_1+1} \ln e^{-(\theta_1+\theta_2)T_1} \qquad \text{for } Case\ I \\ \\ m_r \ln\theta_1 + n_r \ln\theta_2 + \sum_{j=1}^{r} z_j \ln e^{-\theta_1 w_{(j)}} + \sum_{j=1}^{r}(1-z_j)\ln\ e^{-\theta_2 w_{(j)}} + \sum_{j=1}^{r} s_j \ln e^{-\theta_1 w_{(j)}} + \sum_{j=1}^{r} q_j \ln e^{-\theta_2 w_{(j)}} \\ \qquad \text{for } Case\ II \\ m_{d_2}\ln\theta_1 + n_{d_2}\ln\theta_2 + \sum_{j=1}^{d_2} z_j \ln e^{-\theta_1 w_{(j)}} + \sum_{j=1}^{d_2}(1-z_j)\ln\ e^{-\theta_2 w_{(j)}} + \sum_{j=1}^{d_2} s_j \ln e^{-\theta_1 w_{(j)}} \\ + \sum_{j=1}^{d_2} q_j \ln e^{-\theta_2 w_{(j)}} + R^*_{d_2+1} \ln e^{-(\theta_1+\theta_2)T_2} \qquad \text{for } Case\ III \end{cases} \tag{3.3}$$

Differentiating partially (3.3) with respect to θ_1 and θ_2 and equating them to zero, we get the following three equations.

$$Case\ I \quad \frac{\partial \ln L}{\partial\theta_1} = \frac{m_{d_1}}{\hat{\theta}_1} - U_1 = 0 \text{ and } \frac{\partial \ln L}{\partial\theta_2} = \frac{n_{d_1}}{\hat{\theta}_2} - U_2 = 0, \tag{3.4}$$

$$Case\ II \quad \frac{\partial \ln L}{\partial\theta_1} = \frac{m_r}{\hat{\theta}_1} - V_1 = 0 \text{ and } \frac{\partial \ln L}{\partial\theta_2} = \frac{n_r}{\hat{\theta}_2} - V_2 = 0, \tag{3.5}$$

$$Case\ III \quad \frac{\partial \ln L}{\partial\theta_1} = \frac{m_{d_2}}{\hat{\theta}_1} - P_1 = 0 \text{ and } \frac{\partial \ln L}{\partial\theta_2} = \frac{n_{d_2}}{\hat{\theta}_2} - P_2 = 0. \tag{3.6}$$

where $U_1 = \sum_{j=1}^{d_1} z_j w_{(j)} + \sum_{j=1}^{d_1} s_j w_{(j)} + R^*_{d_1+1}T_1$, $U_2 = \sum_{j=1}^{d_1}(1-z_j)w_{(j)} + \sum_{j=1}^{d_1} q_j w_{(j)} + R^*_{d_1+1}T_1$, $V_1 = \sum_{j=1}^{r} z_j w_{(j)} + \sum_{j=1}^{r} s_j w_{(j)}$, $V_2 = \sum_{j=1}^{r}(1-z_j)w_{(j)} + \sum_{j=1}^{r} q_j w_{(j)}$, $P_1 = \sum_{j=1}^{d_2} z_j w_{(j)} + \sum_{j=1}^{d_2} s_j w_{(j)} + R^*_{d_2+1}T_2$ and $P_2 = \sum_{j=1}^{d_2}(1-z_j)w_{(j)} + \sum_{j=1}^{d_2} q_j w_{(j)} + R^*_{d_2+1}T_2$

Upon solving Equations (3.4) , (3.5), and (3.6), we obtain the MLEs of θ_1 and θ_2 as

$$\begin{cases} \hat{\theta}_1 = \frac{m_{d_1}}{U_1} \text{ and } \hat{\theta}_2 = \frac{n_{d_1}}{U_2} & \text{for } Case\ I \\ \hat{\theta}_1 = \frac{m_r}{V_1} \text{ and } \hat{\theta}_2 = \frac{n_r}{V_2} & \text{for } Case\ II \\ \hat{\theta}_1 = \frac{m_{d_2}}{P_1} \text{ and } \hat{\theta}_2 = \frac{n_{d_2}}{P_2} & \text{for } Case\ III \end{cases} \tag{3.7}$$

The variance-covariance matrix for the maximum likelihood estimators of $\theta_1$and θ_2 can be obtained by inverting the information matrix with the elements that are negative of the expected values of the second order derivatives of logarithms of the likelihood functions. Cohen [14] concluded that the approximate variance covariance matrix may be obtained by replacing expected values by their MLEs. Now the Fisher information matrix associated with $\hat{\theta}_1$and $\hat{\theta}_2$ is defined as:

$$I(\hat{\theta}_1, \hat{\theta}_2) \cong \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial \theta_1{}^2} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \theta_2{}^2} \end{pmatrix}^{-1}_{(\theta_1,\theta_2)=(\hat{\theta}_1,\hat{\theta}_2)}$$

From *Case I*,

$$\frac{\partial^2 \ln L}{\partial \theta_1{}^2} = -\frac{m_{d_1}}{\hat{\theta}_1{}^2} \quad , \quad \frac{\partial^2 \ln L}{\partial \theta_2{}^2} = -\frac{n_{d_1}}{\hat{\theta}_2{}^2}$$

and by using *Case II*,

$$\frac{\partial^2 \ln L}{\partial \theta_1{}^2} = -\frac{m_r}{\hat{\theta}_1{}^2} \quad , \quad \frac{\partial^2 \ln L}{\partial \theta_2{}^2} = -\frac{n_r}{\hat{\theta}_2{}^2}$$

and by using *Case III*,

$$\frac{\partial^2 \ln L}{\partial \theta_1{}^2} = -\frac{m_{d_2}}{\hat{\theta}_1{}^2} \quad \text{and} \quad \frac{\partial^2 \ln L}{\partial \theta_2{}^2} = -\frac{n_{d_2}}{\hat{\theta}_2{}^2}$$

Then, the approximate $100\,(1-\alpha)\%$confidence intervals for θ_1 and θ_2 under *case I* are,

$$\hat{\theta}_1 \pm z_{(\alpha/2)} \sqrt{\frac{\hat{\theta}_1{}^2}{m_{d_1}}} \quad \text{and} \quad \hat{\theta}_2 \pm z_{(\alpha/2)} \sqrt{\frac{\hat{\theta}_2{}^2}{n_{d_1}}}$$

Also, approximate $100\,(1-\alpha)\%$confidence intervals for θ_1 and θ_2 under *case II* are,

$$\hat{\theta}_1 \pm z_{(\alpha/2)} \sqrt{\frac{\hat{\theta}_1{}^2}{m_r}} \quad \text{and} \quad \hat{\theta}_2 \pm z_{(\alpha/2)} \sqrt{\frac{\hat{\theta}_2{}^2}{n_r}}$$

and also approximate $100\,(1-\alpha)\%$confidence intervals for θ_1 and θ_2 under case III are,

$$\hat{\theta}_1 \pm z_{(\alpha/2)} \sqrt{\frac{\hat{\theta}_1{}^2}{m_{d_2}}} \quad \text{and} \quad \hat{\theta}_2 \pm z_{(\alpha/2)} \sqrt{\frac{\hat{\theta}_2{}^2}{n_{d_2}}}$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$ percentage point of the standard normal distribution.

4. Bayes Estimation

In this section, we provide the Bayes estimators of the unknown parameters, and the corresponding credible intervals based on the JGPHCS-II as described before. Now, to compute the Bayes estimators of the unknown parameters, we need to assume some pecific form of the prior distributions of θ_1 and θ_2, and it is assumed that θ_1 has a $G(a_1, b_1)$ and θ_2 has a $G(a_2, b_2)$ distributions, respectively, with pdf given by,

$$\pi_1(\theta_1) = \frac{b_1{}^{a_1}}{\Gamma(a_1)} \theta_1{}^{a_1-1} e^{-\,\theta_1 b_1} \;, \quad \pi_2(\theta_2) = \frac{b_2{}^{a_2}}{\Gamma(a_2)} \theta_2{}^{a_2-1} e^{-\theta_2 b_2} \;, \quad \theta_1, \theta_2 > 0, \quad a_1, b_1 a_2, b_2 > 0 \tag{4.1}$$

and $\Gamma(.)$ denots the complet gamma function. There are three cases:

Case I: Using Bayes theorem, the joint posterior density of θ_1 and θ_2 is given by

$$\pi(\theta_1, \theta_2|data) = A_1\, \theta_1^{m_{d_1}+a_1-1}\; \theta_2^{n_{d_1}+a_2-1} \exp\left(-\left(\theta_1(U_1+b_1)+\theta_2(U_2+b_2)\right)\right) \tag{4.2}$$

where $A_1 = \frac{(U_1+b_1)^{m_{d_1}+a_1}(U_2+b_2)^{n_{d_1}+a_2}}{\Gamma(m_{d_1}+a_1)\Gamma(n_{d_1}+a_2)}$

From Equation (4.2) the joint posterior density function of θ_1 andθ_2 is a product of two independent density functions, and so the posterior density function of θ_1 andθ_2 , given he data, are $G\ (m_{d_1}+a_1, U_1+b_1)$ and $G\ (n_{d_1}+a_2, U_2+b_2)$, respectively. To obtain Bayesestimates, using three loss functions. The first loss function is SE loss function, and Bayes estimate in this case is the posterior mean. The second is LINEX loss function proposed by Varian (1975) in the form,

$$\ell\left(\hat{\theta}_{LIN}, \theta\right) \propto e^{\tau\left(\hat{\theta}_{LIN}-\theta\right)} - \tau\left(\hat{\theta}_{LIN}-\theta\right) - 1$$

where $\hat{\theta}_{LIN}$ is an estimator of the unknown parameter θ and τ is constants, when $\tau > 0$, overestimation is more serious than underestimation; however, when $\tau < 0$, the conclusion is opposite.. Bayes estimator of θ in this case, denoted by$\hat{\theta}_{LIN}$, is,

$$\hat{\theta}_{LIN} = -\frac{1}{\tau}\ln \mathrm{E}\left(e^{-\tau\theta}\right), \qquad \tau \neq 0 \tag{4.3}$$

The third loss function is GE loss function and is given by,

$$\ell(\hat{\theta}_{GE}, \theta) \propto \left(\frac{\hat{\theta}_{GE}}{\theta}\right)^{c} - c\ln\left(\frac{\hat{\theta}_{GE}}{\theta}\right) - 1$$

where c is a shape parameter for loss function whose minimum occurs at $(\hat{\theta}_{GE} = \theta)$.

Using GE loss function, the Bayes estimator of θ, denoted by$\hat{\theta}_{GE}$, can be obtained as follows.

$$\hat{\theta}_{GE} = \mathrm{E}\left((\theta^{-c})\right)^{-1/c} \tag{4.4}$$

We obtained Bayes estimators for *Case I* using the three loss functions. Based on the SE loss function, the Bayes estimators of θ_1 and θ_2 can be obtained as follows:

$$\hat{\theta}_{1SE} = \frac{m_{d_1}+a_1}{U_1+b_1} \quad \text{and} \quad \hat{\theta}_{2SE} = \frac{n_{d_1}+a_2}{U_2+b_2}$$

From (4.3), Bayes estimators of $\theta_1$and θ_2 under the LINEX loss function are,

$$\hat{\theta}_{1LIN} = -\frac{m_{d_1}+a_1}{\tau}\ln\left(\frac{U_1+b_1}{U_1+b_1+\tau}\right) \quad \text{and} \quad \hat{\theta}_{2LIN} = -\frac{n_{d_1}+a_2}{\tau}\ln\left(\frac{U_2+b_2}{U_2+b_2+\tau}\right)$$

Also, Bayes estimators of $\theta_1$and θ_2 using the GE loss function can be obtained from (4.4) as follows:

$$\hat{\theta}_{1GE} = \frac{1}{U_1+b_1}\left(\frac{\Gamma(m_{d_1}+a_1-c)}{\Gamma(m_{d_1}+a_1)}\right)^{-1/c} \quad \text{and} \quad \hat{\theta}_{2GE} = \frac{1}{U_2+b_2}\left(\frac{\Gamma(n_{d_1}+a_2-c)}{\Gamma(n_{d_1}+a_2)}\right)^{-1/c}$$

Case II: Similarly, the joint posterior density of θ_1 and θ_2 is given by,

$$\pi(\theta_1, \theta_2|data) = A_2\, \theta_1^{m_r+a_1-1}\; \theta_2^{n_r+a_2-1} \exp\left(-\left(\theta_1(V_1+b_1)+\theta_2(V_2+b_2)\right)\right) \tag{4.5}$$

where $A_2 = \frac{(V_1+b_1)^{m_r+a_1}(V_2+b_2)^{n_r+a_2}}{\Gamma(m_r+a_1)\Gamma(n_r+a_2)}$

From Equation (4.5) the joint posterior density function of θ_1 and θ_2 is a product of two independent density functions, and so the posterior density function of θ_1 andθ_2 , given he data, are $G\ (m_r + a_1, V_1 + b_1)$ and $G\ (n_r + a_2, V_2 + b_2)$, respectively. Likewise, the Bayes estimators under *Case II* can be obtained using the three loss functions as mentioned earlier. Bayes estimators of θ_1 and θ_2 under the SE loss function are,

$$\hat{\theta}_{1SE} = \frac{m_r + a_1}{V_1 + b_1} \quad \text{and} \quad \hat{\theta}_{2SE} = \frac{n_r + a_2}{V_2 + b_2}$$

and under the LINEX loss function are,

$$\hat{\theta}_{1LIN} = -\frac{m_r + a_1}{\tau} \ln\left(\frac{V_1 + b_1}{V_1 + b_1 + \tau}\right) \quad \text{and} \quad \hat{\theta}_{2LIN} = -\frac{n_r + a_2}{\tau} \ln\left(\frac{V_2 + b_2}{V_2 + b_2 + \tau}\right)$$

and under the GE loss function are,

$$\hat{\theta}_{1GE} = \frac{1}{V_1 + b_1}\left(\frac{\Gamma(m_r + a_1 - c)}{\Gamma(m_r + a_1)}\right)^{-1/c} \quad \text{and} \quad \hat{\theta}_{2GE} = \frac{1}{V_2 + b_2}\left(\frac{\Gamma(n_r + a_2 - c)}{\Gamma(n_r + a_2)}\right)^{-1/c}$$

Case III: Similarly, the joint posterior density of θ_1 and θ_2 is given by,

$$\pi(\theta_1, \theta_2|data) = A_3\, \theta_1^{m_{d_2}+a_1-1}\ \theta_2^{n_{d_2}+a_2-1} \exp\left(-(\theta_1(P_1 + b_1) + \theta_2(P_2 + b_2))\right) \tag{4.6}$$

where $A_3 = \frac{(P_1+b_1)^{m_{d_2}+a_1}(P_2+b_2)^{n_{d_2}+a_2}}{\Gamma(m_{d_2}+a_1)\Gamma(n_{d_2}+a_2)}$

From Equation (4.6) the joint posterior density function of θ_1 andθ_2 is a product of two independent density functions, and so the posterior density function of θ_1 andθ_2 , given he data, are $G\ (m_{d_2} + a_1, P_1 + b_1)$ and $G\ (n_{d_2} + a_2, P_2 + b_2)$, respectively. Likewise, the Bayes estimators under *Case II* can be obtained using the three loss functions as mentioned earlier. Bayes estimators of θ_1 and θ_2 under the SE loss function are,

$$\hat{\theta}_{1SE} = \frac{m_{d_2} + a_1}{P_1 + b_1} \quad \text{and} \quad \hat{\theta}_{2SE} = \frac{n_{d_2} + a_2}{P_2 + b_2}$$

and under the LINEX loss function are,

$$\hat{\theta}_{1LIN} = -\frac{m_{d_2} + a_1}{\tau} \ln\left(\frac{P_1 + b_1}{P_1 + b_1 + \tau}\right) \quad \text{and} \quad \hat{\theta}_{2LIN} = -\frac{n_{d_2} + a_2}{\tau} \ln\left(\frac{P_2 + b_2}{P_2 + b_2 + \tau}\right)$$

and under the GE loss function are,

$$\hat{\theta}_{1GE} = \frac{1}{P_1 + b_1}\left(\frac{\Gamma(m_{d_2} + a_1 - c)}{\Gamma(m_{d_2} + a_1)}\right)^{-1/c} \quad \text{and} \quad \hat{\theta}_{2GE} = \frac{1}{P_2 + b_2}\left(\frac{\Gamma(n_{d_2} + a_2 - c)}{\Gamma(n_{d_2} + a_2)}\right)^{-1/c}$$

The posterior distributions in Equations (4.2) , (4.5), and (4.6) can be used to obtain the credible intervals of θ_1 and θ_2. Let $K_1 = 2\theta_1(U_1 + b_1)$, and $K_2 = 2\theta_2(U_2 + b_2)$, where K_1 and K_2 are greater than zero, then the two pivots K_1 and K_2 follow χ^2 distributions with $2(m_{d_1} + a_1)$ and $2(n_{d_1} + a_2)$ degrees of freedom, provided that $2(m_{d_1} + a_1)$ and $2(n_{d_1} + a_2)$ are positive integers (see Kundu & Joarder [20]). Thus, the 100 $(1 - \alpha)\%$ Bayes credible intervals for θ_1 and θ_2 under *Case I* are,

$$\left(\frac{\chi^2_{2(m_{d_1}+a_1),1-\frac{\alpha}{2}}}{2(U_1+b_1)}, \frac{\chi^2_{2(m_{d_1}+a_1),\frac{\alpha}{2}}}{2(U_1+b_1)}\right) \quad \text{and} \quad \left(\frac{\chi^2_{2(n_{d_1}+a_2),1-\frac{\alpha}{2}}}{2(U_2+b_2)}, \frac{\chi^2_{2(n_{d_1}+a_2),\frac{\alpha}{2}}}{2(U_2+b_2)}\right),$$

respectively, where $\chi^2_{v,\alpha}$ is the α percentage point of the χ^2_v distribution.

Similarly, the 100 $(1-\alpha)$%Bayes credible intervals for θ_1 and θ_2 under *Case II* can be obtained as,

$$\left(\frac{\chi^2_{2(m_r+a_1),1-\frac{\alpha}{2}}}{2(V_1+b_1)}, \frac{\chi^2_{2(m_r+a_1),\frac{\alpha}{2}}}{2(V_1+b_1)}\right) \quad \text{and} \quad \left(\frac{\chi^2_{2(n_r+a_2),1-\frac{\alpha}{2}}}{2(V_2+b_2)}, \frac{\chi^2_{2(n_r+a_2),\frac{\alpha}{2}}}{2(V_2+b_2)}\right)$$

Similarly, the 100 $(1-\alpha)$%Bayes credible intervals for θ_1 and θ_2 under *Case III* can be obtained as,

$$\left(\frac{\chi^2_{2(m_{d_2}+a_1),1-\frac{\alpha}{2}}}{2(P_1+b_1)}, \frac{\chi^2_{2(m_{d_2}+a_1),\frac{\alpha}{2}}}{2(P_1+b_1)}\right) \quad \text{and} \quad \left(\frac{\chi^2_{2(n_{d_2}+a_2),1-\frac{\alpha}{2}}}{2(P_2+b_2)}, \frac{\chi^2_{2(n_{d_2}+a_2),\frac{\alpha}{2}}}{2(P_2+b_2)}\right)$$

respectively. Notice that if $2(U_1+b_1)$ and $2(U_2+b_2)$for *Case I* or $2(V_1+b_1)$ and $2(V_2+b_2)$ for *Case II* or $2(P_1+b_1)$ and $2(P_2+b_2)$for *Case III* are not integers, then gamma distribution can be used instead of χ^2 distribution to construct the credible intervals.

5. Monte Carlo Simulation

To evaluate the performance of the proposed methodologies including the classical and Bayesian frameworks discussed in the preceding sections, an extensive Monte Carlo simulation study is performed. To run the experiment according to a JGPHCS-II sampling from two exponential populations, we propose the following generation process:

Step-1: Set the parameter values of θ_1 and θ_2.

Step-2: Generate independent observations X and $Y$of sizes m and n from $Exp(\theta_1)$ and $Exp(\theta_2)$, respectively.

Step-3: Combine the two generated samples and rearrange them in ascending order.

Step-4: For a specific values of m, n, r, R and T_i, $i = 1, 2$, generate an ordinary JPCS-II sample using the algorithm proposed by Doostparast et al. [16] as:

1. At the first failure $W_{(1)}$ occurs, R_1 of survival units are randomly withdrawn from the test and divided R_1 by two groups belonging to lines A and B as R'_1 and R''_1, respectively, thus $R_1 = R'_1 + R''_1$.
2. At the second failure $W_{(2)}$ occurs, R_2 of survival units are randomly withdrawn from the test and set $R_2 = R'_2 + R''_2$ and so on.
3. Determine d_1 and d_2 at given the threshold points T_1 and T_2, respectively.

1. Finally, under JGPHCS-II, the simulated sample data will consist of one of the following observations:
2. If $W_{(r)} < T_1$, the experiment stops at T_1 with failure times $W_{(j)}$, $j = 1, 2, ..., d_1$ and censoring $(R_1, R_2, ..., R_{r-1}, 0, ..., 0, R^*_{d_1+1})$, that is *Case-I*.
3. If $T_1 < W_{(r)} < T_2$, the experiment stops at $r-th$ failure with failure times $W_{(j)}$, $j = 1, 2, ..., r$ and progressive censoring $(R_1, R_2, ..., R_r)$, that is *Case-II*.

Table 1. Censoring information in simulation

$(m,n) = (20,20)$			$(m,n) = (50,50)$		
FI	CS	CS^{N}_{FI}	FI	CS	CS^{N}_{FI}
50%	(1*20)	$U^{40}_{50\%}$	50%	(1*50)	$U^{100}_{50\%}$
	(4*5,0*15)	$L^{40}_{50\%}$		(5*10,0*40)	$L^{100}_{50\%}$
	(0*7,4*5,0*8)	$M^{40}_{50\%}$		(0*20,5*10,0*20)	$M^{100}_{50\%}$
	(0*15,4*5)	$R^{40}_{50\%}$		(0*40,5*10)	$R^{100}_{50\%}$
70%	(1*12,0*16)	$U^{40}_{70\%}$	70%	(1*30,0*40)	$U^{100}_{70\%}$
	(4*3,0*25)	$L^{40}_{70\%}$		(3*10,0*60)	$L^{100}_{70\%}$
	(0*12,4*3,0*13)	$M^{40}_{70\%}$		(0*30,3*10,0*30)	$M^{100}_{70\%}$
	(0*25,4*3)	$R^{40}_{70\%}$		(0*60,3*10)	$R^{100}_{70\%}$
90%	(1*4,0*32)	$U^{40}_{90\%}$	90%	(1*10,0*80)	$U^{100}_{90\%}$
	(4*1,0*35)	$L^{40}_{90\%}$		(10*1,0*89)	$L^{100}_{90\%}$
	(0*17,4*1,0*18)	$M^{40}_{90\%}$		(0*44,10*1,0*45)	$M^{100}_{90\%}$
	(0*35,4*1)	$R^{40}_{90\%}$		(0*89,1*10)	$R^{100}_{90\%}$

4. If $T_2 < W_{(r)}$, the experiment stops at T_2 with failure times $W_{(j)},\ j = 1,2,...,d_2$ and censoring $(R_1, R_2, ..., R_{d_1}, ..., R^*_{d_2+1})$, that is *Case-III*.

Using two sets of the true parameter values (θ_1, θ_2) namely Set-1:(0.4,0.6) and Set-2:(1.5,1.2), a large 5,000 JGPHCS-II samples for different combinations of total sample size $N = m+n$, two threshold time points (T_1 and T_2), effective sample size r and progressive censoring R are simulated. For each parameter set, for fixed (m,n) values, two different choices of (T_1, T_2) are used such as (2,4) and (3,5) for Set-1 as well as (0.5,1.5) and (1,2) for Set-2, respectively. Further, for given m and n, the number of failed items r is specified using various percentages of failure information (FI), $(r/N)100\%$, such as 50, 70 and 90%.

Furthermore, to assess the performance of removal patterns R_j, j $= 1,2,...,r$, four different censoring schemes namely: uniform (U), left (L), middle (M) and right (R) censoring plans are also used. For short, $R = (1,0,0,0,1)$ is denoted by $R = (1,0*3,1)$. All proposed censoring schemes (CSs) are provided in Table 1.

One of the main issues in Bayesian analysis is to determine the value of hyper-parameters when an informative prior of the density parameter is taken into account. It is known, when the gamma improper information is available, that the joint posterior density of $\theta_i,\ i = 1,2$ reduced in proportion to the corresponding joint likelihood function. Following Kundu [19], the hyper-parameter values are chosen in such a way that the prior mean became the expected value of the corresponding population parameter. Thus, to see the effects of the priors on the Bayesian estimates, two different informative sets of the hyper-parameters $a_i, b_i,\ i = 1,2$ for each parameter set are used, namely:

1. For Set-1; Prior-I: $(a_1, a_2, b_1, b_2) = (0.8, 1.2, 2, 2)$ and Prior-II: $(a_1, a_2, b_1, b_2) = (2, 3, 5, 5)$.
2. For Set-2; Prior-I: $(a_1, a_2, b_1, b_2) = (3.0, 2.4, 2, 2)$ and Prior-II: $(a_1, a_2, b_1, b_2) = (7.5, 6, 5, 5)$.

In this study, the Bayes estimates are developed based on SE, LINEX (when $\tau(= -2, +2)$) and GE (when $c(= -3, +3)$) loss functions. The average estimates (AEs) of θ_1 (as an example) is given by $\overline{\hat{\theta}_1} = \frac{1}{5000}\sum_{j=1}^{5000}\hat{\theta}_1^{(j)}$, where $\hat{\theta}_1^{(j)}$ is the computed (maximum likelihood or Bayesian) estimate obtained at j^{th} sample of the unknown parameter θ_1.

Comparison between different point estimates of θ_1 and θ_2 is made using two criteria MSE and MAB values using the following formulae as

$$\text{MSE}\left(\hat{\theta}_1\right) = \frac{1}{5000}\sum_{j=1}^{5000}\left(\hat{\theta}_1^{(j)} - \theta_1\right)^2 \quad \text{and} \quad \text{MAB}\left(\hat{\theta}_1\right) = \frac{1}{5000}\sum_{j=1}^{5000}\left|\hat{\theta}_1^{(j)} - \theta_1\right|$$

, respectively.

Moreover, the behavior of asymptotic/credible intervals estimates are evaluated using their ACLs and CPs using the following formulae, respectively, as

$$\text{ACL}\left(\theta_1\right) = \frac{1}{5000}\sum_{j=1}^{5000}\left(U_{\hat{\theta}_1^{(j)}} - L_{\hat{\theta}_1^{(j)}}\right) \quad \text{and} \quad \text{CP}\left(\theta_1\right) = \frac{1}{5000}\sum_{j=1}^{5000}\text{I}\left(L_{\hat{\theta}_1^{(j)}}; U_{\hat{\theta}_1^{(j)}}\right)$$

,
where I(·) is the indicator function, $L(\cdot)$ and $U(\cdot)$ denote the lower and upper bounds, respectively, of $(1-\alpha)100\%$ asymptotic (or credible) interval. In a similar pattern, the AE, MSE, MAB, ACL and CP of θ_2 can be easily computed.

All computational algorithms were coded in R statistical software version 4.1.2. The average maximum likelihood and Bayes estimates of θ_1 and θ_2 with their MSEs and MABs are calculated and reported in Tables 2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17. Further, the ACL and CP values of 95% asymptotic/credible intervals of θ_1 and θ_2 are provided in Tables 18-19.

From Tables 2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17, using the simulated MSE, MAB, ACL and CP values of θ_1 and θ_2, we can make the following remarks:

1. Generally, the calculated point/interval estimates of θ_1 and θ_2 using both classical and Bayes approaches have shown good behavior in terms of the lowest MSEs, MABs and ACLs as well as highest CPs.
2. As N (or FI) increases, for all given tests, the MSEs, MABs and ACLs of all estimates of θ_1 and θ_2 decrease while their CPs increase as expected. A similar behavior is also observed when the total number of removal items during the test decreases. Therefore, to get good results, one may tend to increase the total (or effective) sample size.
3. Using gamma informative prior, the Bayes estimates of θ_1 and θ_2 are better as they include additional prior information than those obtained by the ML approach in terms of the smallest MSEs and MABs. Thus, the credible intervals provide satisfactory estimates compared to the asymptotic intervals in terms of smallest ACLs and highest CPs.
4. Since the variance of Prior-II is lower than the variance Prior-I, for each set of parameter values, it can be seen that the Bayes estimates using symmetric (or asymmetric) loss based on Prior-II performed quite satisfactory compared to other prior in terms of smallest MSEs, MABs and ACLs as well as highest CPs. Similar behavior is observed in the case of credible interval estimation.
5. As T_i, $i = 1, 2$ increase, in most cases, the MSEs and MABs for all estimates of θ_1 and θ_2 based on Set-1 decrease while that based on Set-2 increase.
6. As T_i, $i = 1, 2$ increase, based on both sets 1 and 2, the ACLs for all estimates of θ_1 and θ_2 decrease while their CPs increase.

Table 2. The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_1 when $N = 40$ using Set-1

(T_1, T_2)	CS_{FI}^{N}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(2,4)		0.2319	0.2386	0.2478	0.2444	0.2533	0.2333	0.2426	0.2599	0.2677	0.194	0.206
	$U_{50\%}^{40}$	0.0341	0.0315	0.0282	0.0302	0.0269	0.0329	0.0295	0.0258	0.0231	0.0468	0.0417
		0.1698	0.1629	0.1536	0.1579	0.1488	0.1677	0.1583	0.1436	0.1355	0.2062	0.1941
		0.5282	0.5004	0.4744	0.5312	0.4973	0.475	0.4545	0.541	0.5095	0.4156	0.4014
	$L_{50\%}^{40}$	0.1596	0.0977	0.0857	0.1326	0.1012	0.0932	0.0729	0.1348	0.1046	0.0696	0.0535
		0.3079	0.2825	0.2518	0.3096	0.2746	0.2583	0.2312	0.3172	0.283	0.2134	0.1898
		0.4388	0.3957	0.3644	0.4296	0.3849	0.3707	0.3474	0.4285	0.392	0.3272	0.3069
	$M_{50\%}^{40}$	0.3648	0.209	0.1393	0.2796	0.1705	0.1673	0.1158	0.2506	0.1672	0.1464	0.0953
		0.4014	0.3515	0.3	0.3908	0.3285	0.3193	0.2756	0.3874	0.3304	0.3014	0.2647
		0.6178	0.5693	0.5246	0.6166	0.5575	0.5317	0.497	0.6236	0.5697	0.4549	0.4302
	$R_{50\%}^{40}$	0.2226	0.1468	0.092	0.2004	0.1179	0.1122	0.0737	0.1882	0.1168	0.0846	0.0562
		0.3522	0.2972	0.2439	0.3378	0.2705	0.2661	0.2224	0.3296	0.2688	0.2407	0.2019
		0.2565	0.2609	0.2671	0.2656	0.2716	0.2565	0.2628	0.2773	0.2827	0.227	0.2349
	$U_{70\%}^{40}$	0.0243	0.0229	0.021	0.0219	0.02	0.0239	0.0219	0.0189	0.0174	0.0328	0.03
		0.1454	0.1409	0.1346	0.1368	0.1306	0.1449	0.1385	0.126	0.1203	0.1734	0.1655
		0.7074	0.6821	0.6514	0.7093	0.6743	0.6576	0.6306	0.717	0.6828	0.6104	0.587
	$L_{70\%}^{40}$	0.1348	0.1108	0.065	0.1325	0.0802	0.0771	0.0542	0.1214	0.08	0.0625	0.0439
		0.2682	0.2358	0.2036	0.262	0.2221	0.2151	0.1883	0.2609	0.2242	0.193	0.1694
		0.7155	0.6675	0.6186	0.7089	0.649	0.6331	0.5921	0.7101	0.6552	0.5795	0.543
	$M_{70\%}^{40}$	0.3085	0.1979	0.1188	0.2711	0.1454	0.1544	0.101	0.2295	0.1342	0.1378	0.0923
		0.3693	0.3356	0.2958	0.3682	0.3225	0.3068	0.2719	0.3735	0.3296	0.2819	0.2416
		0.6034	0.5778	0.5496	0.608	0.5737	0.5516	0.5282	0.619	0.5857	0.4922	0.4748
	$R_{70\%}^{40}$	0.1173	0.0896	0.0638	0.1131	0.0786	0.0721	0.0525	0.1133	0.0808	0.0526	0.038
		0.2546	0.2266	0.1952	0.2522	0.215	0.2053	0.1784	0.255	0.2194	0.1794	0.1557
		0.3044	0.3061	0.3087	0.3118	0.314	0.3007	0.3036	0.3218	0.3236	0.2739	0.2781
	$U_{90\%}^{40}$	0.0232	0.0215	0.0194	0.0217	0.0194	0.0215	0.0195	0.0201	0.0179	0.0261	0.0239
		0.139	0.1344	0.1282	0.1332	0.1268	0.1357	0.1296	0.1262	0.1201	0.1526	0.1459
		0.6055	0.5847	0.5607	0.6104	0.5819	0.562	0.5415	0.621	0.5931	0.5098	0.494
	$L_{90\%}^{40}$	0.0956	0.0759	0.0565	0.0937	0.0685	0.0621	0.047	0.0955	0.0713	0.0446	0.0333
		0.2382	0.2161	0.1902	0.2387	0.2086	0.1965	0.174	0.2443	0.215	0.165	0.1457
		0.7669	0.7332	0.6935	0.7659	0.7203	0.7043	0.6694	0.7713	0.7274	0.6551	0.624
	$M_{90\%}^{40}$	0.195	0.1559	0.1172	0.188	0.1388	0.1305	0.0995	0.1873	0.1412	0.1012	0.0756
		0.3689	0.3163	0.272	0.3447	0.2872	0.2967	0.2602	0.3256	0.2773	0.2591	0.2279
		0.544	0.5341	0.5215	0.5513	0.5368	0.5182	0.5074	0.5632	0.5482	0.4741	0.4667
	$R_{90\%}^{40}$	0.0364	0.031	0.0251	0.0378	0.0303	0.0256	0.0208	0.041	0.0332	0.0161	0.0129
		0.1476	0.1374	0.1247	0.1536	0.1389	0.123	0.1119	0.1644	0.1493	0.092	0.0834

Table 3. Continue: The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_1 when $N = 40$ using Set-1

(T_1, T_2)	CS_{FI}^{N}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(3,5)	$U_{50\%}^{40}$	0.1979	0.205	0.2148	0.2092	0.2189	0.201	0.2109	0.2233	0.2321	0.1666	0.1786
		0.0451	0.0421	0.0381	0.0407	0.0369	0.0434	0.0394	0.0358	0.0327	0.0577	0.0521
		0.2024	0.1952	0.1873	0.1916	0.1845	0.1991	0.1892	0.1901	0.2334	0.2212	0.2214
	$L_{50\%}^{40}$	0.5641	0.5526	0.5382	0.5702	0.5537	0.5365	0.5239	0.5809	0.5642	0.4945	0.485
		0.0497	0.0423	0.0341	0.0506	0.0404	0.0355	0.0301	0.0536	0.0433	0.0351	0.0311
		0.1723	0.1615	0.1485	0.1761	0.1593	0.1538	0.1424	0.1849	0.168	0.1535	0.143
	$M_{50\%}^{40}$	0.2802	0.2689	0.2605	0.2838	0.2713	0.2568	0.2512	0.2913	0.2803	0.2222	0.2193
		0.1435	0.1084	0.0855	0.1278	0.0937	0.0967	0.0799	0.1174	0.0899	0.0963	0.0811
		0.306	0.2857	0.2647	0.2971	0.272	0.2772	0.2589	0.2893	0.2665	0.2813	0.2636
	$R_{50\%}^{40}$	0.5999	0.5577	0.5176	0.6017	0.549	0.5223	0.491	0.6111	0.5622	0.4452	0.4243
		0.1828	0.1249	0.0811	0.1684	0.1032	0.0963	0.0654	0.1617	0.1038	0.0709	0.0491
		0.3338	0.2854	0.2372	0.3238	0.2633	0.2557	0.216	0.3198	0.2642	0.2251	0.1912
	$U_{70\%}^{40}$	0.2005	0.2055	0.2127	0.2084	0.2155	0.2028	0.2099	0.2185	0.2251	0.1788	0.187
		0.0418	0.0398	0.037	0.0388	0.036	0.0408	0.0379	0.0351	0.0324	0.0505	0.0469
		0.1995	0.1945	0.1854	0.1912	0.1814	0.1972	0.1774	0.1685	0.1816	0.1749	0.213
	$L_{70\%}^{40}$	0.3758	0.3709	0.3661	0.3863	0.3792	0.3573	0.3543	0.4011	0.3932	0.3079	0.3097
		0.0444	0.0384	0.0323	0.0433	0.0354	0.0349	0.0288	0.0435	0.0356	0.0244	0.0196
		0.1719	0.1601	0.1453	0.1711	0.1561	0.1457	0.1326	0.172	0.1572	0.1168	0.106
	$M_{70\%}^{40}$	0.5172	0.5006	0.4818	0.5224	0.4998	0.4814	0.4656	0.5326	0.5104	0.4345	0.4228
		0.0944	0.0762	0.0589	0.0916	0.0689	0.0646	0.0511	0.0919	0.0704	0.0519	0.0415
		0.239	0.2198	0.1973	0.2384	0.2122	0.2038	0.1842	0.2415	0.2162	0.1812	0.1641
	$R_{70\%}^{40}$	0.6103	0.5845	0.5558	0.6151	0.5803	0.5579	0.534	0.6263	0.5923	0.4978	0.48
		0.1137	0.0879	0.0635	0.1107	0.078	0.0708	0.0522	0.112	0.0808	0.0504	0.0369
		0.2639	0.2357	0.2038	0.2624	0.2247	0.2133	0.1859	0.2662	0.2299	0.1819	0.1585
	$U_{90\%}^{40}$	0.2475	0.2506	0.2551	0.2544	0.2587	0.247	0.2516	0.2634	0.2674	0.2242	0.2298
		0.0319	0.0304	0.0283	0.0299	0.0278	0.0309	0.0289	0.0275	0.0256	0.0374	0.0349
		0.17	0.1659	0.1601	0.1645	0.1586	0.1674	0.1616	0.1575	0.1519	0.1848	0.1784
	$L_{90\%}^{40}$	0.4733	0.4659	0.457	0.4818	0.4709	0.4514	0.4442	0.4948	0.4834	0.4062	0.4026
		0.0341	0.0288	0.0231	0.0345	0.0272	0.0244	0.0198	0.0363	0.0289	0.0192	0.0158
		0.1459	0.1359	0.1234	0.1474	0.1332	0.1259	0.1149	0.1519	0.1377	0.1116	0.1022
	$M_{90\%}^{40}$	0.5995	0.5843	0.5655	0.6046	0.5831	0.5658	0.5494	0.6146	0.5931	0.522	0.5088
		0.0718	0.0601	0.0475	0.072	0.0563	0.0505	0.0402	0.0749	0.0593	0.036	0.0283
		0.208	0.1925	0.1733	0.2116	0.1898	0.1754	0.1585	0.22	0.1983	0.142	0.1278
	$R_{90\%}^{40}$	0.5177	0.5093	0.4986	0.5252	0.5128	0.4946	0.4855	0.5371	0.5242	0.452	0.4461
		0.0326	0.0279	0.0225	0.0337	0.027	0.0232	0.0189	0.0364	0.0295	0.0155	0.0125
		0.1363	0.127	0.1154	0.1399	0.1266	0.1158	0.1056	0.1474	0.134	0.0952	0.0868

Table 4. The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_1 when $N = 100$ using Set-1.

(T_1, T_2)	CS_{FI}^{N}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(2,4)	$U_{50\%}^{100}$	0.2334	0.2363	0.2404	0.2385	0.2426	0.2341	0.2383	0.2453	0.2492	0.2179	0.2226
		0.0307	0.0297	0.0283	0.0291	0.0276	0.0303	0.0289	0.027	0.0257	0.0358	0.034
		0.1667	0.1638	0.1597	0.1616	0.1575	0.166	0.1618	0.1549	0.151	0.1821	0.1775
	$L_{50\%}^{100}$	0.6868	0.6778	0.6653	0.6882	0.675	0.6678	0.656	0.6922	0.6791	0.6487	0.6375
		0.0999	0.0933	0.0846	0.1002	0.0907	0.0869	0.0789	0.1022	0.0926	0.0767	0.0694
		0.2868	0.2778	0.2653	0.2882	0.275	0.2678	0.256	0.2922	0.2791	0.2487	0.2375
	$M_{50\%}^{100}$	0.9903	0.93	0.8611	0.9734	0.8949	0.8922	0.8309	0.9636	0.891	0.8615	0.8003
		0.687	0.5343	0.3908	0.6292	0.4495	0.4603	0.3429	0.5886	0.4309	0.4326	0.3158
		0.6247	0.5637	0.494	0.6061	0.5268	0.527	0.4649	0.5943	0.5209	0.503	0.4407
	$R_{50\%}^{40}$	0.5963	0.5779	0.5558	0.597	0.5719	0.5607	0.5409	0.602	0.5778	0.5288	0.5107
		0.1669	0.1398	0.1105	0.1601	0.125	0.1229	0.0982	0.157	0.1239	0.1091	0.0866
		0.2883	0.2677	0.2424	0.2835	0.2554	0.2538	0.2308	0.2815	0.2546	0.243	0.2208
	$U_{70\%}^{40}$	0.2536	0.2555	0.2583	0.2573	0.2601	0.2538	0.2566	0.2623	0.2649	0.2419	0.245
		0.0229	0.0223	0.0214	0.0218	0.021	0.0227	0.0219	0.0204	0.0197	0.0263	0.0253
		0.1465	0.1446	0.1418	0.1428	0.14	0.1463	0.1435	0.1379	0.1352	0.1581	0.1551
	$L_{70\%}^{100}$	0.1413	0.1427	0.1447	0.1435	0.1455	0.1419	0.1439	0.1464	0.1484	0.1351	0.1373
		0.0781	0.0772	0.0759	0.0771	0.0758	0.0773	0.076	0.0759	0.0746	0.08	0.0787
		0.2675	0.2659	0.2636	0.2659	0.2635	0.266	0.2637	0.2639	0.2616	0.2706	0.2682
	$M_{70\%}^{100}$	0.8246	0.8033	0.7752	0.8228	0.7926	0.7851	0.7589	0.825	0.7955	0.7594	0.7342
		0.2401	0.2144	0.1832	0.2355	0.2002	0.1957	0.1679	0.2349	0.2008	0.1758	0.1501
		0.4264	0.405	0.3769	0.4243	0.3941	0.3869	0.3607	0.4263	0.3967	0.3622	0.3369
	$R_{70\%}^{100}$	0.5715	0.5638	0.5536	0.5745	0.5635	0.5537	0.5443	0.5806	0.5696	0.5297	0.5213
		0.0666	0.0604	0.0526	0.0665	0.0578	0.0549	0.048	0.0679	0.0592	0.047	0.0409
		0.1895	0.1814	0.1707	0.1907	0.1791	0.1728	0.1628	0.1945	0.183	0.1575	0.1482
	$U_{90\%}^{100}$	0.2634	0.2649	0.2671	0.2665	0.2687	0.2634	0.2656	0.2706	0.2727	0.2536	0.2559
		0.0198	0.0194	0.0188	0.019	0.0184	0.0198	0.0192	0.018	0.0174	0.0225	0.0218
		0.1377	0.1362	0.1339	0.1347	0.1325	0.1376	0.1354	0.1306	0.1285	0.1474	0.1449
	$L_{90\%}^{100}$	0.2782	0.2773	0.2762	0.2817	0.2803	0.2731	0.2722	0.2869	0.2853	0.2577	0.2575
		0.0646	0.0628	0.0606	0.0641	0.0616	0.0618	0.0597	0.0638	0.0613	0.0617	0.0598
		0.2188	0.2156	0.2111	0.2181	0.2133	0.2134	0.2091	0.2178	0.2131	0.2132	0.209
	$M_{90\%}^{100}$	0.7487	0.7365	0.7197	0.7491	0.7314	0.7245	0.7086	0.7523	0.7348	0.7044	0.6892
		0.1486	0.1375	0.1232	0.1479	0.1321	0.128	0.1149	0.1495	0.1339	0.1149	0.1029
		0.3487	0.3365	0.3197	0.3491	0.3314	0.3245	0.3086	0.3523	0.3348	0.3044	0.2892
	$R_{90\%}^{100}$	0.5097	0.507	0.5031	0.5131	0.5089	0.5011	0.4975	0.5185	0.5142	0.4837	0.4807
		0.0172	0.0163	0.015	0.0178	0.0165	0.0148	0.0137	0.0191	0.0176	0.0115	0.0106
		0.1102	0.1074	0.1036	0.1134	0.1093	0.1017	0.0981	0.1187	0.1144	0.0856	0.0825

Table 5. Continue: The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_1 when $N = 100$ using Set-1

(T_1, T_2)	CS^{N}_{FI}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(3,5)	$U^{100}_{50\%}$	0.2001	0.2032	0.2075	0.2048	0.2092	0.2015	0.2059	0.2109	0.2151	0.1874	0.1921
		0.042	0.0408	0.039	0.0402	0.0386	0.0413	0.0396	0.0379	0.0371	0.047	0.045
		0.1999	0.1968	0.1956	0.1976	0.1945	0.1998	0.1967	0.1934	0.1904	0.2126	0.2079
	$L^{100}_{50\%}$	0.1132	0.1145	0.1166	0.1151	0.1171	0.114	0.116	0.1175	0.1195	0.1085	0.1106
		0.0892	0.0883	0.0872	0.0882	0.087	0.0885	0.0873	0.087	0.0858	0.0911	0.0899
		0.2879	0.2865	0.2844	0.2861	0.284	0.2869	0.2848	0.2838	0.2818	0.2921	0.29
	$M^{100}_{50\%}$	0.5904	0.5772	0.5605	0.5924	0.5739	0.5632	0.548	0.5982	0.58	0.5346	0.5208
		0.1163	0.101	0.0839	0.1137	0.0935	0.0903	0.0756	0.1136	0.0943	0.0785	0.0653
		0.2652	0.2508	0.2322	0.264	0.2437	0.2386	0.2216	0.2656	0.2458	0.2222	0.2062
	$R^{40}_{50\%}$	0.6105	0.5914	0.5681	0.6111	0.5848	0.5735	0.5527	0.6161	0.5906	0.541	0.522
		0.1676	0.1419	0.1136	0.162	0.1282	0.1252	0.1012	0.16	0.1279	0.1098	0.0884
		0.3113	0.2899	0.2633	0.3066	0.2772	0.2749	0.2508	0.3048	0.2766	0.2614	0.2382
	$U^{40}_{70\%}$	0.1992	0.2013	0.2044	0.2024	0.2055	0.2002	0.2033	0.2066	0.2096	0.1906	0.1938
		0.0411	0.0403	0.039	0.0398	0.0384	0.0407	0.0395	0.0382	0.0362	0.0446	0.0432
		0.2008	0.1987	0.1925	0.1952	0.1908	0.1985	0.1941	0.1891	0.1849	0.2094	0.2062
	$L^{100}_{70\%}$	0.2081	0.2092	0.2107	0.2116	0.2131	0.2068	0.2084	0.2164	0.2177	0.1944	0.1965
		0.0632	0.0624	0.0614	0.0624	0.0613	0.0625	0.0614	0.0615	0.0604	0.0648	0.0637
		0.2152	0.2134	0.2108	0.2133	0.2106	0.2136	0.211	0.2111	0.2085	0.2197	0.2169
	$M^{100}_{70\%}$	0.5998	0.5921	0.5817	0.6025	0.5913	0.5822	0.5725	0.6081	0.5969	0.5597	0.5509
		0.0663	0.061	0.0542	0.0668	0.0592	0.0559	0.0497	0.0686	0.061	0.0473	0.042
		0.2132	0.2053	0.1945	0.2149	0.2033	0.1962	0.1861	0.2191	0.2075	0.1785	0.1691
	$R^{100}_{70\%}$	0.5777	0.5696	0.5587	0.5806	0.5689	0.5592	0.5491	0.5866	0.5748	0.5352	0.5261
		0.0725	0.0659	0.0576	0.0725	0.0631	0.0601	0.0526	0.0739	0.0646	0.0516	0.0451
		0.2052	0.1965	0.1849	0.2059	0.1934	0.1878	0.1769	0.209	0.1966	0.1735	0.1632
	$U^{100}_{90\%}$	0.2155	0.2172	0.2197	0.2182	0.2207	0.2162	0.2186	0.2218	0.2242	0.2078	0.2104
		0.0348	0.0342	0.0333	0.0339	0.0329	0.0346	0.0337	0.0326	0.0317	0.0376	0.0366
		0.185	0.1833	0.1809	0.1823	0.1799	0.1843	0.1818	0.1788	0.1764	0.1926	0.1899
	$L^{100}_{90\%}$	0.5495	0.5456	0.5401	0.5523	0.5465	0.5391	0.5339	0.5572	0.5513	0.5222	0.5175
		0.033	0.0312	0.0287	0.0337	0.031	0.0289	0.0266	0.0351	0.0324	0.0241	0.0222
		0.1508	0.1468	0.1413	0.1533	0.1475	0.1406	0.1354	0.1579	0.152	0.1252	0.1205
	$M^{100}_{90\%}$	0.5869	0.5815	0.5741	0.5893	0.5815	0.574	0.567	0.5941	0.5861	0.5562	0.5497
		0.0501	0.0471	0.043	0.0508	0.0463	0.0437	0.04	0.0524	0.0479	0.0374	0.0341
		0.1878	0.1825	0.175	0.1901	0.1822	0.1752	0.1681	0.1946	0.1867	0.1585	0.152
	$R^{100}_{90\%}$	0.4874	0.4851	0.4818	0.4907	0.4872	0.4796	0.4766	0.4961	0.4925	0.4628	0.4603
		0.0173	0.0163	0.015	0.0177	0.0163	0.015	0.0139	0.0187	0.0172	0.0123	0.0113
		0.0989	0.0964	0.0928	0.1008	0.097	0.0923	0.0889	0.1042	0.1004	0.0829	0.0799

Table 6. The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_2 when $N = 40$ using Set-1.

(T_1, T_2)	CS^{N}_{FI}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(2,4)	$U^{40}_{50\%}$	0.2351	0.2402	0.2475	0.2419	0.2492	0.2385	0.2458	0.2471	0.2542	0.2261	0.2337
		0.1342	0.1306	0.1254	0.1294	0.1242	0.1318	0.1265	0.1257	0.1207	0.1408	0.1352
		0.3649	0.3598	0.3525	0.3581	0.3508	0.3462	0.3434	0.3529	0.3458	0.3821	0.3774
	$L^{40}_{50\%}$	0.1015	0.1053	0.1108	0.1062	0.1117	0.1044	0.1099	0.1099	0.1153	0.096	0.1017
		0.2577	0.2543	0.2501	0.254	0.2489	0.2549	0.2504	0.2515	0.2473	0.262	0.2568
		0.5063	0.5034	0.4992	0.5031	0.4989	0.4583	0.4562	0.5006	0.4964	0.465	0.4628
	$M^{40}_{50\%}$	0.7902	0.6962	0.6292	0.7806	0.6791	0.6361	0.5893	0.7563	0.6763	0.5703	0.5312
		0.6047	0.3612	0.2602	0.4962	0.3113	0.299	0.2198	0.4074	0.2936	0.2778	0.2005
		0.5785	0.473	0.4115	0.5464	0.4262	0.4077	0.3729	0.5071	0.4099	0.4216	0.3869
	$R^{40}_{50\%}$	0.932	0.8583	0.7921	0.9383	0.8472	0.7957	0.7463	0.9249	0.8462	0.7195	0.68
		0.3677	0.2319	0.14	0.3381	0.232	0.2195	0.174	0.2856	0.2237	0.2013	0.1583
		0.475	0.3925	0.3151	0.4631	0.3924	0.2734	0.253	0.4312	0.3861	0.2766	0.2568
	$U^{40}_{70\%}$	0.2552	0.2591	0.2649	0.2607	0.2664	0.2577	0.2634	0.2648	0.2704	0.2477	0.2536
		0.1196	0.1169	0.1131	0.1159	0.1121	0.1179	0.1141	0.1131	0.1094	0.1283	0.1212
		0.3348	0.3409	0.3351	0.3393	0.3336	0.3367	0.3345	0.3352	0.3296	0.3581	0.355
	$L^{40}_{70\%}$	0.0937	0.0966	0.1008	0.0969	0.1011	0.0963	0.1005	0.0994	0.1036	0.0908	0.0951
		0.2571	0.2543	0.2494	0.2537	0.2498	0.2545	0.25	0.2506	0.2459	0.26	0.2557
		0.4985	0.4947	0.4892	0.4938	0.4883	0.3421	0.3398	0.4901	0.4847	0.3529	0.3508
	$M^{40}_{70\%}$	0.8521	0.8054	0.7608	0.8625	0.8031	0.7586	0.7246	0.8581	0.8054	0.6961	0.6689
		0.2521	0.1637	0.1034	0.2316	0.1367	0.1211	0.0801	0.204	0.1286	0.1012	0.0656
		0.357	0.3047	0.253	0.3538	0.2877	0.2595	0.2211	0.3413	0.2827	0.2491	0.2153
	$R^{40}_{70\%}$	0.8367	0.8047	0.7701	0.8512	0.8072	0.7648	0.7374	0.8522	0.8112	0.7067	0.6855
		0.1534	0.1154	0.081	0.1534	0.1046	0.0885	0.0636	0.145	0.1017	0.0699	0.0499
		0.3093	0.2733	0.2334	0.3125	0.2637	0.1951	0.1743	0.3067	0.2616	0.2026	0.1859
	$U^{40}_{90\%}$	0.2589	0.2656	0.2751	0.2683	0.2779	0.2629	0.2724	0.2755	0.2847	0.2454	0.2556
		0.1192	0.1147	0.1083	0.1129	0.1066	0.1164	0.11	0.1083	0.1023	0.1248	0.1207
		0.3411	0.3344	0.3249	0.3317	0.3221	0.3659	0.3617	0.3245	0.3153	0.3465	0.3441
	$L^{40}_{90\%}$	0.5862	0.5817	0.5777	0.6098	0.6015	0.557	0.5563	0.6215	0.6129	0.4991	0.505
		0.0469	0.039	0.031	0.0462	0.0354	0.0348	0.0285	0.0443	0.0342	0.0295	0.0226
		0.1734	0.159	0.1421	0.1734	0.1531	0.1288	0.11	0.1709	0.1517	0.1221	0.1085
	$M^{40}_{90\%}$	0.8065	0.7843	0.7588	0.8211	0.7896	0.7518	0.7312	0.8246	0.7946	0.7014	0.6856
		0.1019	0.0795	0.0581	0.1042	0.0746	0.0614	0.0455	0.1007	0.0738	0.047	0.0344
		0.2379	0.2139	0.1862	0.245	0.2116	0.1604	0.1324	0.2444	0.2128	0.1445	0.1222
	$R^{40}_{90\%}$	0.7397	0.7272	0.7123	0.7561	0.7374	0.7012	0.6894	0.7628	0.7443	0.6542	0.6467
		0.049	0.0402	0.0309	0.0525	0.0399	0.0311	0.0242	0.0527	0.0407	0.0229	0.0177
		0.1637	0.15	0.1333	0.1731	0.1531	0.1168	0.1165	0.1763	0.1568	0.1445	0.145

Table 7. Continue: The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_2 when $N = 40$ using Set-1

(T_1, T_2)	CS_{FI}^{N}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(3,5)	$U_{50\%}^{40}$	0.1908	0.194	0.2007	0.1951	0.2018	0.1929	0.1996	0.1996	0.2062	0.1738	0.1907
		0.1692	0.1655	0.1602	0.1647	0.1593	0.1664	0.161	0.1611	0.1558	0.1831	0.1691
		0.4092	0.406	0.3993	0.4049	0.3982	0.3998	0.3967	0.4004	0.3938	0.4334	0.4214
	$L_{50\%}^{40}$	0.0794	0.0826	0.0874	0.0832	0.0879	0.082	0.0868	0.0862	0.0909	0.0753	0.0802
		0.2769	0.2739	0.2695	0.2735	0.2691	0.2743	0.2699	0.2707	0.2664	0.2805	0.2759
		0.5236	0.5212	0.5176	0.521	0.5174	0.486	0.484	0.5189	0.5153	0.4915	0.4894
	$M_{50\%}^{40}$	0.5727	0.5325	0.5036	0.5824	0.5365	0.4949	0.4764	0.5787	0.5414	0.4357	0.4249
		0.3091	0.2051	0.1481	0.2736	0.1736	0.1694	0.1323	0.2362	0.163	0.1635	0.1312
		0.4375	0.3829	0.3347	0.4249	0.3601	0.4088	0.3927	0.4062	0.3513	0.4247	0.4097
	$R_{50\%}^{40}$	0.8894	0.8258	0.768	0.8994	0.8196	0.7677	0.7248	0.89	0.8205	0.692	0.6592
		0.3094	0.1994	0.1232	0.2883	0.1658	0.145	0.0944	0.2535	0.1552	0.1182	0.0768
		0.4383	0.3661	0.2971	0.4311	0.3408	0.253	0.2192	0.411	0.3321	0.255	0.2307
	$U_{70\%}^{40}$	0.2224	0.2375	0.2426	0.2329	0.2456	0.227	0.2397	0.2419	0.2542	0.1954	0.2186
		0.147	0.1394	0.1301	0.1373	0.1281	0.1414	0.1321	0.1309	0.1222	0.1672	0.149
		0.3791	0.3701	0.3574	0.3671	0.3544	0.399	0.3891	0.3581	0.3458	0.4212	0.413
	$L_{70\%}^{40}$	0.4906	0.4941	0.4992	0.5142	0.517	0.4761	0.4831	0.5279	0.5296	0.424	0.4365
		0.0451	0.0403	0.0347	0.041	0.0348	0.0408	0.0355	0.0381	0.0324	0.0528	0.0459
		0.1704	0.1602	0.1474	0.1615	0.1476	0.1757	0.1732	0.1554	0.1424	0.2069	0.205
	$M_{70\%}^{40}$	0.6662	0.6492	0.6326	0.6852	0.6615	0.6185	0.6072	0.6918	0.6697	0.5611	0.5562
		0.1043	0.0776	0.0557	0.1008	0.0686	0.0626	0.047	0.0934	0.0658	0.0584	0.0447
		0.241	0.2156	0.1878	0.2406	0.2064	0.2132	0.2009	0.2352	0.2038	0.227	0.2179
	$R_{70\%}^{40}$	0.8165	0.7875	0.7559	0.8319	0.7917	0.7493	0.7244	0.834	0.7963	0.6915	0.6729
		0.137	0.104	0.0739	0.1379	0.0952	0.08	0.0582	0.1311	0.0929	0.0632	0.0458
		0.2945	0.2614	0.2244	0.299	0.2538	0.1835	0.163	0.2946	0.2525	0.1879	0.1733
	$U_{90\%}^{40}$	0.2209	0.2299	0.258	0.2429	0.2634	0.2325	0.2529	0.2577	0.277	0.2052	0.2189
		0.1461	0.1358	0.1213	0.1323	0.118	0.1392	0.1245	0.1221	0.1091	0.1578	0.1472
		0.3776	0.3625	0.342	0.3571	0.3366	0.3539	0.3489	0.3423	0.323	0.3762	0.3704
	$L_{90\%}^{40}$	0.6186	0.6142	0.6093	0.6368	0.6293	0.5937	0.591	0.6463	0.6384	0.5482	0.5497
		0.0337	0.0284	0.0227	0.0341	0.0266	0.0246	0.0201	0.0333	0.0263	0.0252	0.021
		0.1439	0.1336	0.1209	0.1443	0.1296	0.1205	0.1169	0.143	0.1289	0.14	0.1385
	$M_{90\%}^{40}$	0.6773	0.6679	0.6571	0.6945	0.6802	0.644	0.6362	0.7022	0.6881	0.5973	0.5937
		0.0492	0.04	0.0306	0.0505	0.0379	0.0324	0.0253	0.0494	0.0376	0.0285	0.0225
		0.1659	0.1522	0.1357	0.1683	0.1488	0.1257	0.1177	0.1672	0.1487	0.1376	0.1324
	$R_{90\%}^{40}$	0.6797	0.6724	0.6635	0.697	0.6853	0.65	0.6436	0.7053	0.6934	0.6048	0.6024
		0.0308	0.0257	0.0202	0.033	0.0256	0.0204	0.0162	0.0334	0.0262	0.0169	0.0136
		0.1288	0.119	0.1068	0.1348	0.1204	0.1363	0.135	0.137	0.1229	0.1633	0.1631

Table 8. The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_2 when $N = 100$ using Set-1.

(T_1, T_2)	CS^{N}_{FI}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(2,4)	$U^{100}_{50\%}$	0.2383	0.2505	0.2672	0.2551	0.2718	0.2461	0.2628	0.2673	0.2832	0.2155	0.2341
		0.1339	0.1252	0.1138	0.1223	0.1109	0.1281	0.1166	0.1141	0.1037	0.1323	0.1314
		0.3617	0.3495	0.3328	0.3449	0.3282	0.3667	0.3574	0.3327	0.3168	0.366	0.364
	$L^{100}_{50\%}$	0.3818	0.3802	0.3839	0.4056	0.4033	0.3598	0.3672	0.4173	0.4157	0.302	0.3173
		0.1574	0.1302	0.1121	0.1434	0.1161	0.125	0.1107	0.1315	0.1102	0.1416	0.1253
		0.3288	0.3064	0.2829	0.3171	0.2887	0.237	0.218	0.3066	0.2812	0.2553	0.2406
	$M^{100}_{50\%}$	0.9829	0.9782	0.9209	1.0273	0.9598	0.9356	0.8865	1.0162	0.9544	0.9007	0.8528
		0.4705	0.3275	0.1947	0.4434	0.2516	0.2417	0.1591	0.3966	0.2278	0.2206	0.1466
		0.5273	0.4701	0.3839	0.5182	0.4165	0.3914	0.3612	0.5023	0.3968	0.4071	0.3469
	$R^{40}_{50\%}$	0.9456	0.9134	0.8748	0.9486	0.9044	0.8818	0.8478	0.9442	0.9026	0.8508	0.8182
		0.3071	0.2556	0.2003	0.2999	0.1897	0.1673	0.1065	0.2939	0.1762	0.1368	0.0862
		0.4445	0.4096	0.367	0.4404	0.3614	0.2775	0.2385	0.4385	0.3505	0.2787	0.2462
	$U^{40}_{70\%}$	0.2604	0.2761	0.2969	0.2833	0.3041	0.2694	0.2902	0.2995	0.3188	0.2272	0.2515
		0.1216	0.111	0.1014	0.1074	0.099	0.115	0.1014	0.1011	0.0933	0.1439	0.1262
		0.3396	0.3268	0.3136	0.3227	0.3095	0.3435	0.3372	0.3127	0.3001	0.373	0.3651
	$L^{100}_{70\%}$	0.7241	0.7106	0.695	0.741	0.721	0.6834	0.6714	0.7477	0.7281	0.6343	0.6269
		0.0594	0.0476	0.0358	0.062	0.0457	0.0372	0.0284	0.061	0.0458	0.0392	0.0328
		0.1848	0.1681	0.1484	0.1899	0.1663	0.1549	0.1407	0.1899	0.1673	0.1637	0.1542
	$M^{100}_{70\%}$	0.8722	0.856	0.835	0.8789	0.8556	0.8347	0.8159	0.8799	0.8572	0.8076	0.7901
		0.1382	0.1214	0.1016	0.1395	0.116	0.1059	0.0891	0.1371	0.1148	0.0933	0.078
		0.3027	0.2856	0.2635	0.3061	0.2817	0.2273	0.2031	0.3051	0.2814	0.2116	0.1886
	$R^{100}_{70\%}$	0.8426	0.8302	0.8137	0.8493	0.8312	0.8121	0.7972	0.8511	0.8333	0.7878	0.774
		0.1119	0.1004	0.0862	0.1137	0.0972	0.0888	0.0766	0.1126	0.0967	0.0785	0.0674
		0.2697	0.2565	0.2391	0.2733	0.2543	0.1543	0.1483	0.2732	0.2546	0.1608	0.1555
	$U^{100}_{90\%}$	0.2641	0.2735	0.2866	0.2777	0.2909	0.2694	0.2826	0.2878	0.3003	0.2441	0.2586
		0.1162	0.1099	0.0977	0.1069	0.0939	0.1124	0.1038	0.0971	0.0855	0.1293	0.1192
		0.3362	0.3239	0.3031	0.3167	0.2959	0.3053	0.3006	0.3005	0.2812	0.3296	0.3242
	$L^{100}_{90\%}$	0.6986	0.6947	0.6893	0.706	0.7	0.6838	0.679	0.7099	0.7038	0.6638	0.6599
		0.0322	0.0298	0.0267	0.0334	0.0299	0.0266	0.0239	0.0338	0.0303	0.0231	0.0208
		0.1468	0.1418	0.1348	0.1494	0.1419	0.1063	0.098	0.1501	0.1427	0.1002	0.0932
	$M^{100}_{90\%}$	0.7842	0.7767	0.7666	0.7908	0.7797	0.7633	0.754	0.7935	0.7825	0.7428	0.7343
		0.0648	0.0594	0.0525	0.0667	0.0587	0.0529	0.0469	0.0668	0.059	0.0462	0.0408
		0.1997	0.1918	0.1811	0.2037	0.1921	0.149	0.1357	0.2049	0.1935	0.1383	0.1261
	$R^{100}_{90\%}$	0.7068	0.7033	0.6985	0.7139	0.7086	0.6931	0.6888	0.7178	0.7124	0.6741	0.6706
		0.0215	0.02	0.0182	0.0229	0.0207	0.0175	0.0159	0.0236	0.0214	0.0142	0.0128
		0.1162	0.1125	0.1074	0.1214	0.1158	0.108	0.1099	0.1241	0.1185	0.1225	0.1244

Table 9. Continue: The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_2 when $N = 100$ using Set-1

(T_1, T_2)	CS_{FI}^{N}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(3,5)	$U_{50\%}^{100}$	0.1894	0.202	0.2176	0.2049	0.2207	0.1991	0.2147	0.2156	0.2307	0.1827	0.1896
		0.1693	0.1602	0.1481	0.158	0.1458	0.1624	0.1502	0.1498	0.1384	0.1748	0.1691
		0.4106	0.398	0.3824	0.3951	0.3793	0.3972	0.3901	0.3844	0.3693	0.4094	0.4062
	$L_{50\%}^{100}$	0.2907	0.2996	0.3124	0.3145	0.3251	0.2869	0.3012	0.3289	0.3384	0.2379	0.2582
		0.1495	0.1378	0.1261	0.1383	0.125	0.1392	0.1279	0.1305	0.1191	0.1618	0.1469
		0.3364	0.3222	0.3047	0.3196	0.3012	0.2598	0.2554	0.3083	0.2914	0.2991	0.2939
	$M_{50\%}^{100}$	0.7039	0.6908	0.6752	0.7142	0.6955	0.6695	0.6566	0.7177	0.6998	0.6359	0.6252
		0.1195	0.1014	0.0822	0.1183	0.0942	0.0882	0.0726	0.114	0.0918	0.0811	0.0669
		0.2774	0.2596	0.2376	0.2781	0.2531	0.2041	0.1904	0.2751	0.2512	0.2029	0.1908
	$R_{50\%}^{40}$	0.9284	0.8981	0.8614	0.932	0.8901	0.8674	0.8352	0.9283	0.8888	0.8364	0.8057
		0.2849	0.2391	0.1893	0.2797	0.2187	0.2059	0.1649	0.2675	0.2115	0.1878	0.1494
		0.4435	0.4101	0.3692	0.4402	0.3942	0.2755	0.2535	0.4318	0.3884	0.2735	0.2525
	$U_{70\%}^{40}$	0.2138	0.2175	0.2229	0.2185	0.2239	0.2164	0.2218	0.2222	0.2276	0.2078	0.2134
		0.1497	0.1469	0.1428	0.1461	0.142	0.1477	0.1436	0.1433	0.1393	0.1543	0.15
		0.3862	0.3825	0.3771	0.3815	0.3761	0.3839	0.3814	0.3778	0.3724	0.3922	0.3896
	$L_{70\%}^{100}$	0.0764	0.0788	0.0824	0.079	0.0826	0.0786	0.0822	0.0811	0.0847	0.0742	0.0778
		0.2747	0.2722	0.2685	0.272	0.2683	0.2724	0.2687	0.2698	0.2662	0.277	0.2733
		0.5206	0.5174	0.5126	0.5168	0.5121	0.3936	0.3919	0.5138	0.5091	0.4058	0.4037
	$M_{70\%}^{100}$	0.6886	0.6833	0.6764	0.6978	0.6898	0.6697	0.6637	0.7024	0.6944	0.6446	0.64
		0.0453	0.041	0.0358	0.0464	0.0403	0.0365	0.032	0.0464	0.0404	0.0326	0.0287
		0.1722	0.1649	0.155	0.175	0.1642	0.1206	0.1146	0.1754	0.1648	0.1218	0.117
	$R_{70\%}^{100}$	0.8264	0.8148	0.7995	0.8333	0.8164	0.7974	0.7836	0.8354	0.8188	0.7732	0.7605
		0.1022	0.092	0.0795	0.1041	0.0896	0.0815	0.0707	0.1034	0.0893	0.0718	0.0621
		0.2654	0.2528	0.2362	0.2687	0.2505	0.1643	0.1567	0.2684	0.2507	0.1677	0.1609
	$U_{90\%}^{100}$	0.2221	0.2285	0.2377	0.2305	0.2398	0.2265	0.2357	0.237	0.246	0.2112	0.2208
		0.1448	0.14	0.1333	0.1386	0.1318	0.1415	0.1347	0.1339	0.1274	0.153	0.1456
		0.3779	0.3715	0.3623	0.3695	0.3602	0.3985	0.3941	0.363	0.354	0.4126	0.4079
	$L_{90\%}^{100}$	0.5979	0.5972	0.5963	0.6056	0.6043	0.5891	0.5886	0.6103	0.6089	0.5707	0.5709
		0.0179	0.0169	0.0156	0.0179	0.0165	0.0162	0.015	0.0178	0.0164	0.0163	0.0152
		0.1131	0.1101	0.1059	0.1135	0.1091	0.0977	0.0972	0.1133	0.1089	0.1055	0.1054
	$M_{90\%}^{100}$	0.659	0.6564	0.6528	0.6664	0.6623	0.6467	0.6437	0.6706	0.6664	0.6277	0.6254
		0.0263	0.0245	0.0221	0.0271	0.0244	0.0223	0.0202	0.0272	0.0246	0.0203	0.0184
		0.1321	0.1279	0.1221	0.1334	0.1272	0.099	0.0961	0.1335	0.1274	0.1022	0.1001
	$R_{90\%}^{100}$	0.6639	0.6617	0.6587	0.6711	0.6677	0.6526	0.65	0.6753	0.6717	0.6342	0.6323
		0.0155	0.0145	0.0132	0.0164	0.0149	0.0129	0.0118	0.0168	0.0153	0.0111	0.0101
		0.0962	0.0933	0.0892	0.099	0.0947	0.1337	0.1343	0.1005	0.0962	0.1463	0.147

Table 10. The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_1 when $N = 40$ using Set-1.

(T_1, T_2)	CS_{FI}^{N}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(2,4)	$U_{50\%}^{40}$	0.676	0.7168	0.7709	0.7361	0.7901	0.6989	0.753	0.7419	0.7942	0.6658	0.7237
		0.7857	0.7131	0.6439	0.6932	0.6249	0.7323	0.6623	0.684	0.6177	0.7739	0.6987
		0.8705	0.8398	0.7976	0.8276	0.7853	0.8514	0.8093	0.8221	0.7808	0.8756	0.8316
	$L_{50\%}^{40}$	0.261	0.2794	0.3058	0.2818	0.3083	0.277	0.3033	0.2869	0.3131	0.264	0.2908
		1.5452	1.5005	1.4373	1.4953	1.4318	1.5057	1.4426	1.4828	1.4203	1.537	1.4721
		1.239	1.2206	1.1942	1.2182	1.1917	1.223	1.1967	1.2131	1.1869	1.236	1.2092
	$M_{50\%}^{40}$	2.6163	2.3367	2.109	2.6091	2.2813	2.1349	1.9696	2.42	2.1739	2.1669	1.9772
		2.9206	1.7079	0.92	2.7419	1.333	1.1393	0.6556	1.9219	1.0345	1.3224	0.7163
		1.3425	1.046	0.7976	1.2957	0.9491	0.8681	0.6799	1.1098	0.8451	0.9217	0.7057
	$R_{50\%}^{40}$	2.386	2.1925	2.0185	2.4058	2.1645	2.0265	1.8976	2.2649	2.0766	2.0452	1.9005
		2.0951	1.3264	0.7888	1.9758	1.0992	0.931	0.5816	1.4835	0.8798	1.0442	0.6272
		1.1494	0.9371	0.7397	1.1228	0.8605	0.8	0.6453	0.9862	0.777	0.8437	0.6695
	$U_{70\%}^{40}$	0.6295	0.6602	0.7024	0.6724	0.7147	0.6486	0.6907	0.6779	0.7192	0.6244	0.6684
		0.6996	0.6386	0.5863	0.6219	0.5703	0.6602	0.6019	0.6156	0.5652	0.714	0.6301
		0.824	0.7956	0.762	0.7848	0.7512	0.8059	0.7724	0.7809	0.7479	0.8342	0.7905
	$L_{70\%}^{40}$	0.3294	0.3537	0.3868	0.3629	0.3957	0.3452	0.3785	0.3672	0.3994	0.3261	0.3612
		1.4693	1.4195	1.3523	1.4082	1.3419	1.4304	1.3625	1.3968	1.3318	1.4678	1.3956
		1.1711	1.1468	1.1136	1.138	1.105	1.155	1.1217	1.1335	1.1012	1.174	1.139
	$M_{70\%}^{40}$	2.2341	2.1337	2.0265	2.2739	2.1359	2.0153	1.9312	2.1899	2.0742	2.0197	1.9298
		0.9592	0.7018	0.4776	0.9815	0.6481	0.5089	0.3543	0.7908	0.5388	0.542	0.3685
		0.8001	0.6952	0.5824	0.8229	0.6804	0.591	0.5001	0.7419	0.6215	0.6049	0.5069
	$R_{70\%}^{40}$	2.1198	2.0421	1.9565	2.162	2.0529	1.939	1.8716	2.0929	2.0003	1.9392	1.8679
		0.7599	0.5775	0.4075	0.7875	0.5427	0.4286	0.3082	0.6476	0.4572	0.4524	0.3193
		0.6919	0.6094	0.5177	0.714	0.6001	0.5248	0.4497	0.6486	0.551	0.5366	0.4563
	$U_{90\%}^{40}$	0.6803	0.7044	0.738	0.7152	0.7488	0.6941	0.7276	0.7192	0.7521	0.6747	0.7095
		0.6775	0.6333	0.5504	0.6052	0.5245	0.6549	0.5755	0.5956	0.5178	0.6864	0.6202
		0.8197	0.7832	0.7291	0.7639	0.7099	0.8011	0.747	0.7581	0.7058	0.8253	0.7763
	$L_{90\%}^{40}$	1.8315	1.8007	1.7638	1.8757	1.8287	1.7334	1.705	1.8383	1.7976	1.7245	1.6957
		0.362	0.2154	0.166	0.2881	0.2192	0.1622	0.1265	0.2446	0.1886	0.1652	0.1276
		0.4637	0.3688	0.3267	0.4248	0.3739	0.3235	0.2881	0.3926	0.3476	0.3268	0.2899
	$M_{90\%}^{40}$	1.9818	1.9627	1.8452	1.9882	1.999	1.7925	1.7201	2.0549	1.9167	1.774	1.6993
		0.758	0.4252	0.2292	0.807	0.4006	0.233	0.1323	0.5383	0.2919	0.2491	0.134
		0.6554	0.5094	0.3844	0.7117	0.5187	0.3729	0.288	0.5841	0.4413	0.3796	0.2871
	$R_{90\%}^{40}$	1.8522	1.7829	1.7187	1.9552	1.8459	1.6468	1.6128	1.8643	1.7838	1.6161	1.5859
		0.2942	0.1861	0.1098	0.3604	0.2015	0.0999	0.0615	0.2476	0.1468	0.1028	0.0613
		0.4085	0.3321	0.26	0.4766	0.3639	0.241	0.1924	0.3925	0.3076	0.2439	0.1921

Table 11. Continue: The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_1 when $N = 40$ using Set-2.

(T_1, T_2)	CS^{N}_{FI}	MLE	SE		LINEX				GE			
					$c=-2$		$c=2$		$c=-3$		$c=3$	
Prior			I	II	I	II	I	II	I	II	I	II
(3,5)	$U^{40}_{50\%}$	0.4813	0.5092	0.5482	0.5164	0.5556	0.5022	0.541	0.5228	0.5612	0.4815	0.5216
		1.042	0.9861	0.9105	0.9721	0.8966	0.9998	0.924	0.9595	0.8859	1.0413	0.9613
		1.0187	0.9908	0.9518	0.9836	0.9444	0.9978	0.959	0.9772	0.9388	1.0185	0.9784
	$L^{40}_{50\%}$	0.2029	0.218	0.2399	0.2195	0.2415	0.2165	0.2383	0.2239	0.2457	0.206	0.2282
		1.6884	1.65	1.5948	1.6465	1.5912	1.6534	1.5984	1.6353	1.5806	1.6801	1.6238
		1.2971	1.282	1.2601	1.2805	1.2585	1.2835	1.2617	1.2761	1.2543	1.294	1.2718
	$M^{40}_{50\%}$	1.156	1.0691	1.0494	1.2515	1.1524	0.9529	0.9705	1.1483	1.1078	0.9045	0.9292
		1.4769	0.9194	0.696	1.2912	0.7784	0.8159	0.6695	0.9652	0.7005	0.8888	0.7174
		1.0187	0.8435	0.7179	0.9796	0.7797	0.7776	0.6839	0.8685	0.73	0.8147	0.7098
	$R^{40}_{50\%}$	1.9871	1.9099	1.7372	1.983	1.9459	1.6641	1.5811	1.9745	1.8305	1.6302	1.5454
		1.9134	0.754	0.3402	1.7345	0.6089	0.4075	0.2174	0.9552	0.4195	0.4676	0.2359
		1.0949	0.7356	0.5084	1.0772	0.674	0.5457	0.3999	0.8251	0.5651	0.5776	0.4117
	$U^{40}_{70\%}$	0.5588	0.598	0.651	0.6114	0.6646	0.5854	0.6381	0.6189	0.6707	0.5555	0.611
		0.9164	0.8737	0.815	0.861	0.8025	0.8861	0.8272	0.8519	0.7946	0.9187	0.8569
		0.9555	0.9328	0.9007	0.9258	0.8936	0.9395	0.9076	0.9209	0.8893	0.9567	0.9238
	$L^{40}_{70\%}$	0.243	0.2642	0.2936	0.2694	0.2989	0.2593	0.2886	0.2743	0.3032	0.2436	0.2742
		1.6355	1.5889	1.5256	1.5802	1.5173	1.5972	1.5336	1.5688	1.5072	1.631	1.5639
		1.257	1.2358	1.2064	1.2306	1.2011	1.2407	1.2114	1.2257	1.1968	1.2564	1.2258
	$M^{40}_{70\%}$	1.5694	1.5106	1.4753	1.6922	1.5994	1.3768	1.3752	1.5985	1.5433	1.3295	1.336
		0.5366	0.3189	0.1989	0.5272	0.2738	0.2449	0.1715	0.3648	0.2178	0.2797	0.1917
		0.5654	0.4539	0.3591	0.5697	0.4265	0.3953	0.3243	0.4852	0.3798	0.423	0.343
	$R^{40}_{70\%}$	1.9813	1.9963	1.9178	2.1111	2.0106	1.8974	1.8359	2.046	1.9607	1.8957	1.8309
		0.9273	0.5251	0.376	0.7131	0.4986	0.3917	0.286	0.5893	0.4217	0.4111	0.2952
		0.773	0.6055	0.5177	0.7026	0.5944	0.526	0.4538	0.6414	0.5482	0.5374	0.4603
	$U^{40}_{90\%}$	0.5445	0.5672	0.5993	0.5742	0.6064	0.5605	0.5924	0.5791	0.6107	0.5433	0.5762
		0.899	0.8268	0.7339	0.8038	0.712	0.8488	0.7551	0.7901	0.7014	0.9042	0.8023
		0.9412	0.902	0.849	0.8886	0.8354	0.9146	0.8619	0.8811	0.8293	0.9445	0.889
	$L^{40}_{90\%}$	1.6784	1.6317	1.5925	1.7844	1.7062	1.511	1.4976	1.7083	1.6543	1.4747	1.4666
		0.2895	0.1812	0.1082	0.3187	0.1738	0.121	0.0784	0.2225	0.1309	0.1356	0.0862
		0.403	0.3306	0.2625	0.4248	0.3258	0.2791	0.2278	0.3623	0.2861	0.2964	0.2394
	$M^{40}_{90\%}$	1.6784	1.6317	1.5925	1.7844	1.7062	1.511	1.4976	1.7083	1.6543	1.4747	1.4666
		0.2895	0.1812	0.1082	0.3187	0.1738	0.121	0.0784	0.2225	0.1309	0.1356	0.0862
		0.403	0.3306	0.2625	0.4248	0.3258	0.2791	0.2278	0.3623	0.2861	0.2964	0.2394
	$R^{40}_{90\%}$	1.7281	1.6812	1.6381	1.8344	1.7538	1.5588	1.5411	1.7581	1.7003	1.5237	1.5114
		0.2141	0.1387	0.084	0.2623	0.1493	0.0825	0.0531	0.1817	0.1097	0.0889	0.0561
		0.3477	0.2865	0.2276	0.3957	0.3058	0.2247	0.1827	0.3304	0.2618	0.2341	0.1884

Table 12. The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_1 when $N = 100$ using Set-2.

(T_1, T_2)	CS^{N}_{FI}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(2,4)		0.6717	0.7637	0.8682	0.814	0.9154	0.7211	0.8271	0.8202	0.9164	0.6462	0.7689
	$U^{100}_{50\%}$	0.7724	0.6526	0.52	0.6052	0.4785	0.6969	0.5594	0.5901	0.4704	0.7929	0.6301
		0.8672	0.7955	0.7082	0.7628	0.6761	0.8246	0.7371	0.754	0.6713	0.8813	0.7838
		1.0874	1.0741	1.0973	1.2339	1.199	0.9663	1.0176	1.1596	1.1615	0.8959	0.9648
	$L^{100}_{50\%}$	0.9728	0.6018	0.4611	0.7703	0.4884	0.583	0.4686	0.6034	0.4483	0.6626	0.5199
		0.7702	0.6464	0.5473	0.7233	0.5784	0.6323	0.5484	0.6498	0.5444	0.6876	0.5881
		1.7853	1.4461	1.3219	1.7917	1.4793	1.2524	1.2066	1.5529	1.3956	1.2238	1.1704
	$M^{100}_{50\%}$	1.9338	1.2042	0.6633	1.5013	0.9	0.8351	0.5621	1.3759	0.7098	0.959	0.613
		1.3948	0.9642	0.7328	1.2628	0.848	0.8183	0.6607	1.0159	0.7597	0.8726	0.6889
		1.9813	1.9565	1.7629	1.9768	1.9792	1.6983	1.6021	1.9472	1.8575	1.6706	1.5685
	$R^{40}_{50\%}$	1.3494	0.8758	0.3781	1.0741	0.6816	0.4661	0.2378	1.1026	0.4653	0.5432	0.2596
		1.1963	0.7843	0.5333	1.1479	0.7018	0.5871	0.4246	0.8723	0.5878	0.6295	0.4413
		0.6328	0.7045	0.7918	0.7372	0.8239	0.6754	0.7629	0.746	0.8287	0.6187	0.7162
	$U^{40}_{70\%}$	0.7285	0.5809	0.4687	0.5313	0.4341	0.639	0.5022	0.5207	0.4282	0.7606	0.563
		0.8283	0.7402	0.6696	0.7125	0.6424	0.7789	0.6952	0.7057	0.6384	0.8538	0.7342
		1.8756	1.7908	1.7172	1.977	1.8504	1.6467	1.6075	1.8756	1.7843	1.617	1.5805
	$L^{100}_{70\%}$	0.4217	0.2511	0.1418	0.4782	0.2503	0.1413	0.0848	0.3232	0.1826	0.1505	0.1322
		0.4874	0.3887	0.2998	0.5346	0.3989	0.3251	0.2653	0.4435	0.3417	0.3531	0.2851
		1.992	1.9054	1.7595	1.9086	1.9381	1.6997	1.6207	2.0162	1.8406	1.677	1.5934
	$M^{100}_{70\%}$	1.549	0.6606	0.3176	1.3879	0.5447	0.3704	0.2036	0.8187	0.3881	0.4223	0.2173
		0.9004	0.6428	0.4631	0.9011	0.6036	0.4893	0.369	0.7166	0.5137	0.5156	0.3789
		2.1206	1.9388	1.8033	2.2135	1.9767	1.743	1.6665	2.0454	1.8829	1.7195	1.6406
	$R^{100}_{70\%}$	1.0536	0.5434	0.2756	1.0451	0.481	0.3026	0.1677	0.6828	0.3444	0.3339	0.1761
		0.8122	0.6051	0.4422	0.8351	0.5779	0.4594	0.3483	0.6765	0.4927	0.4799	0.3559
		0.7078	0.7633	0.8332	0.7943	0.8629	0.7356	0.8061	0.7994	0.8657	0.6893	0.7669
	$U^{100}_{90\%}$	0.6577	0.5701	0.4325	0.5183	0.3815	0.6074	0.4813	0.5047	0.3764	0.6798	0.558
		0.7964	0.7363	0.6318	0.6864	0.585	0.7661	0.6729	0.6799	0.5837	0.812	0.7311
		1.5408	1.5036	1.4814	1.6781	1.6046	1.3729	1.3814	1.5931	1.5509	1.3191	1.339
	$L^{100}_{90\%}$	0.3316	0.2073	0.1281	0.3459	0.1823	0.1657	0.1137	0.2397	0.1417	0.1956	0.0872
		0.4605	0.3718	0.2929	0.4765	0.3564	0.2962	0.2344	0.4025	0.3136	0.3048	0.2377
		2.0527	1.9989	1.9356	2.0914	2.0136	1.9169	1.8654	2.0404	1.9724	1.915	1.8614
	$M^{100}_{90\%}$	0.5152	0.4137	0.3105	0.5478	0.4052	0.3134	0.2379	0.4636	0.3484	0.3238	0.2425
		0.5718	0.5169	0.4522	0.6021	0.5234	0.4454	0.3917	0.553	0.484	0.4494	0.393
		1.7546	1.7345	1.7098	1.7997	1.7674	1.6754	1.6571	1.7695	1.7414	1.664	1.6461
	$R^{100}_{90\%}$	0.1296	0.1093	0.0868	0.1525	0.1201	0.0783	0.0626	0.1288	0.1024	0.0773	0.0614
		0.2807	0.2591	0.2325	0.314	0.2805	0.2148	0.1934	0.2865	0.2571	0.2117	0.1899

Table 13. Continue: The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_1 when $N = 100$ using Set-2.

(T_1, T_2)	CS_{FI}^{N}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(3,5)	$U_{50\%}^{100}$	0.4835	0.5497	0.634	0.5693	0.6543	0.5319	0.6153	0.5821	0.6636	0.4828	0.5735
		1.0444	0.9146	0.7617	0.8792	0.7284	0.9476	0.7932	0.855	0.7123	1.0442	0.8686
		1.0185	0.9503	0.866	0.9307	0.8457	0.9681	0.8847	0.9179	0.8364	1.0172	0.9265
	$L_{50\%}^{100}$	0.7758	0.8327	0.9001	0.9221	0.9682	0.7657	0.8445	0.899	0.9528	0.6945	0.7914
		0.8111	0.6947	0.579	0.681	0.5581	0.7313	0.6097	0.6504	0.5442	0.8254	0.673
		0.7811	0.7065	0.6268	0.6917	0.6065	0.7455	0.6632	0.6716	0.5966	0.8131	0.7136
	$M_{50\%}^{100}$	1.6094	1.5677	1.5307	1.6803	1.6181	1.4749	1.4557	1.6238	1.5779	1.4535	1.4349
		0.513	0.3765	0.2687	0.5111	0.3389	0.3014	0.2276	0.4125	0.2894	0.3254	0.242
		0.5785	0.5074	0.4328	0.5878	0.4907	0.4496	0.3901	0.5337	0.4534	0.4636	0.3996
	$R_{50\%}^{40}$	2.3279	2.1492	1.9858	2.3533	2.1274	1.9892	1.8682	2.2203	2.043	2.0047	1.8697
		1.8716	1.2197	0.7478	1.7969	1.033	0.8642	0.5566	1.3663	0.834	0.9574	0.5953
		1.1436	0.9445	0.7553	1.1244	0.8745	0.809	0.6603	0.9939	0.793	0.8481	0.6821
	$U_{70\%}^{40}$	0.5546	0.6442	0.75	0.6796	0.785	0.6134	0.7189	0.6919	0.7917	0.545	0.6642
		0.9222	0.7898	0.6711	0.7578	0.6412	0.8203	0.6998	0.7408	0.6296	0.9347	0.7604
		0.9454	0.8783	0.8084	0.8586	0.7885	0.8966	0.8269	0.8492	0.7817	0.955	0.8631
	$L_{70\%}^{100}$	1.5694	1.5106	1.4753	1.6922	1.5994	1.3768	1.3752	1.5985	1.5433	1.3295	1.336
		0.5366	0.3189	0.1989	0.5272	0.2738	0.2449	0.1715	0.3648	0.2178	0.2797	0.1917
		0.5654	0.4539	0.3591	0.5697	0.4265	0.3953	0.3243	0.4852	0.3798	0.423	0.343
	$M_{70\%}^{100}$	1.6461	1.6245	1.6008	1.7039	1.6682	1.5547	1.5404	1.6673	1.6385	1.5378	1.5244
		0.2252	0.1823	0.1391	0.2404	0.1782	0.1453	0.114	0.203	0.1538	0.1523	0.1186
		0.3862	0.3505	0.3085	0.4026	0.3508	0.3106	0.2759	0.3713	0.3262	0.3161	0.2797
	$R_{70\%}^{100}$	2.0672	1.9142	1.7876	1.9798	1.9574	1.7235	1.6531	1.9955	1.8665	1.6974	1.6261
		0.6822	0.49	0.254	0.9681	0.4443	0.2729	0.1549	0.6199	0.319	0.2978	0.1616
		0.6835	0.582	0.4291	0.8096	0.5667	0.4355	0.3325	0.6566	0.4825	0.4498	0.3364
	$U_{90\%}^{100}$	0.5683	0.6222	0.6921	0.6426	0.7125	0.6035	0.6732	0.6516	0.719	0.5619	0.637
		0.8879	0.7604	0.5888	0.7065	0.5418	0.8099	0.6329	0.6839	0.5301	0.896	0.7208
		0.9323	0.8558	0.75	0.8206	0.7151	0.8866	0.7811	0.8082	0.7084	0.9382	0.8358
	$L_{90\%}^{100}$	1.4843	1.4811	1.4779	1.5315	1.5232	1.435	1.4361	1.5121	1.5061	1.4185	1.4208
		0.1076	0.0941	0.0784	0.1079	0.0882	0.0871	0.0737	0.0977	0.0809	0.0927	0.0784
		0.279	0.2614	0.239	0.2795	0.254	0.2493	0.2292	0.2668	0.2435	0.2562	0.2352
	$M_{90\%}^{100}$	1.6284	1.6146	1.5983	1.6744	1.6512	1.5604	1.5499	1.6481	1.6287	1.5468	1.537
		0.1514	0.1277	0.1018	0.1629	0.1276	0.1038	0.0841	0.1412	0.1121	0.1075	0.0867
		0.3174	0.2938	0.2647	0.3249	0.2907	0.2697	0.2444	0.3055	0.2749	0.2747	0.2483
	$R_{90\%}^{100}$	1.6837	1.6682	1.6493	1.729	1.7032	1.6129	1.5998	1.7018	1.6798	1.6003	1.5878
		0.1189	0.1003	0.0799	0.1356	0.1067	0.0758	0.061	0.1156	0.0919	0.0766	0.0612
		0.2661	0.2462	0.2215	0.2868	0.2567	0.2151	0.1944	0.2649	0.2382	0.216	0.1947

Table 14. The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_2 when $N = 40$ using Set-2.

(T_1, T_2)	CS^{N}_{FI}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(2,4)	$U^{40}_{50\%}$	0.6381	0.6965	0.7641	0.7406	0.8042	0.659	0.729	0.7506	0.8105	0.5838	0.6682
		0.3579	0.2901	0.2201	0.2559	0.1923	0.3234	0.2477	0.2423	0.185	0.4094	0.3083
		0.5634	0.5047	0.4431	0.4651	0.414	0.5247	0.4374	0.456	0.4075	0.5813	0.484
	$L^{40}_{50\%}$	0.331	0.3439	0.3625	0.3483	0.3668	0.3396	0.3583	0.3527	0.371	0.3259	0.3452
		0.8056	0.7825	0.7496	0.7786	0.7457	0.7864	0.7536	0.77	0.7377	0.8084	0.7743
		0.8725	0.8594	0.8405	0.8563	0.8373	0.9231	0.8968	0.8515	0.8328	0.936	0.9092
	$M^{40}_{50\%}$	2.5359	2.2322	1.9756	2.4783	2.1296	2.048	1.85	2.3099	2.0368	2.0739	1.8512
		2.775	2.1107	1.1565	2.5808	1.587	1.4905	0.866	2.3456	1.2871	1.6767	0.9173
		1.4567	1.1458	0.88	1.3797	1.0228	1.0344	0.8503	1.211	0.93	1.0812	0.8696
	$R^{40}_{50\%}$	1.8208	1.6848	1.5615	1.8332	1.6648	1.5677	1.475	1.7479	1.6131	1.5562	1.4564
		1.4224	0.9149	0.553	1.3122	0.7439	0.667	0.4229	1.0262	0.6181	0.7177	0.4394
		1.1816	0.7288	0.584	0.8527	0.6648	0.933	0.7832	0.7661	0.6124	0.9687	0.7997
	$U^{40}_{70\%}$	0.6556	0.6744	0.7002	0.6867	0.7122	0.6627	0.6886	0.6917	0.7167	0.6391	0.6667
		0.3193	0.285	0.2581	0.2728	0.2467	0.2969	0.2693	0.2673	0.2422	0.3226	0.2921
		0.5444	0.5256	0.4998	0.5133	0.4853	0.5061	0.4725	0.5083	0.4797	0.5642	0.4906
	$L^{40}_{70\%}$	0.5935	0.6014	0.6133	0.6237	0.633	0.5815	0.5955	0.622	0.6321	0.5594	0.5753
		0.7847	0.6109	0.5803	0.6142	0.5814	0.6107	0.5814	0.6038	0.573	0.6299	0.5986
		0.6968	0.6792	0.6561	0.6818	0.6574	0.8578	0.8251	0.674	0.6509	0.8762	0.8417
	$M^{40}_{70\%}$	2.1713	2.0509	1.9188	2.1784	2.0178	1.9423	1.8321	2.1044	1.9645	1.9426	1.8262
		1.3322	0.9995	0.6977	1.304	0.8871	0.7763	0.5526	1.1065	0.7733	0.8011	0.5579
		0.9985	0.8643	0.7311	0.9886	0.8271	0.8281	0.7407	0.9149	0.7742	0.8358	0.7417
	$R^{40}_{70\%}$	1.7859	1.6182	1.4917	1.838	1.6297	1.4604	1.382	1.7185	1.5673	1.4112	1.3364
		0.946	0.4904	0.2496	0.907	0.4099	0.2892	0.1596	0.6192	0.3141	0.2945	0.1554
		0.7445	0.5576	0.4097	0.7437	0.5189	0.6071	0.5064	0.6251	0.4579	0.609	0.4968
	$U^{40}_{90\%}$	0.6916	0.7063	0.7268	0.7171	0.7374	0.696	0.7167	0.7209	0.7409	0.6767	0.6985
		0.2655	0.2505	0.2303	0.2405	0.2208	0.2604	0.2397	0.2365	0.2175	0.2801	0.2575
		0.5094	0.4946	0.474	0.484	0.4636	0.5012	0.447	0.4801	0.4601	0.5255	0.4764
	$L^{40}_{90\%}$	1.8208	1.7704	1.7088	1.8416	1.7697	1.7064	1.6533	1.8071	1.7418	1.6963	1.6423
		0.517	0.4254	0.3337	0.529	0.4107	0.3428	0.2712	0.4727	0.3711	0.3384	0.2649
		0.6212	0.5708	0.5092	0.6417	0.5699	0.5352	0.5059	0.6073	0.542	0.5275	0.4974
	$M^{40}_{90\%}$	1.9447	1.9178	1.8337	1.9725	1.905	1.8419	1.7692	1.9578	1.8692	1.8369	1.7618
		1.1227	0.6605	0.506	1.0375	0.6187	0.5357	0.4147	0.7956	0.5563	0.5393	0.4123
		0.85	0.718	0.6339	0.9687	0.7051	0.7171	0.6655	0.758	0.6694	0.7153	0.6616
	$R^{40}_{90\%}$	1.553	1.4909	1.4307	1.6202	1.5279	1.3866	1.3487	1.5646	1.4907	1.3399	1.3081
		0.255	0.1679	0.1029	0.2931	0.1718	0.0977	0.0618	0.2236	0.1379	0.0887	0.0543
		0.3799	0.3148	0.2511	0.4301	0.3363	0.4503	0.4141	0.3759	0.3004	0.4241	0.3889

Table 15. Continue: The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_2 when $N = 40$ using Set-2.

(T_1, T_2)	CS^{N}_{FI}	MLE	SE		LINEX				GE			
					$c=-2$		$c=2$		$c=-3$		$c=3$	
Prior			I	II	I	II	I	II	I	II	I	II
(3,5)	$U^{40}_{50\%}$	0.4922	0.5108	0.537	0.5178	0.544	0.5041	0.5301	0.524	0.5496	0.4841	0.5113
		0.5096	0.4796	0.4442	0.4702	0.4351	0.4887	0.4531	0.4618	0.4277	0.5167	0.4785
		0.7078	0.6892	0.663	0.6822	0.656	0.6978	0.659	0.676	0.6504	0.7185	0.6784
	$L^{40}_{50\%}$	0.2532	0.2648	0.2815	0.2674	0.2841	0.2623	0.279	0.2716	0.2881	0.251	0.2681
		0.9236	0.9023	0.8719	0.8991	0.8687	0.9056	0.8751	0.8911	0.8611	0.9255	0.8941
		0.9469	0.9353	0.9186	0.9328	0.916	0.9835	0.9617	0.9285	0.912	0.994	0.9718
	$M^{40}_{50\%}$	0.7784	0.7332	0.726	0.8285	0.7832	0.6684	0.6808	0.7875	0.7687	0.6202	0.638
		1.0666	0.72	0.5608	0.9146	0.6089	0.6508	0.5404	0.7438	0.5601	0.7053	0.5805
		0.8912	0.7796	0.6874	0.8503	0.7213	0.6778	0.5912	0.79	0.6907	0.6997	0.6037
	$R^{40}_{50\%}$	1.8202	1.7028	1.5762	1.8526	1.6814	1.5839	1.4881	1.7666	1.6284	1.5727	1.4702
		1.4984	0.9034	0.563	1.2813	0.7523	0.6634	0.4329	1.0163	0.6304	0.7032	0.4452
		0.9651	0.7789	0.6302	0.908	0.7161	0.9377	0.7928	0.8197	0.6615	0.9711	0.8091
	$U^{40}_{70\%}$	0.5246	0.5531	0.589	0.5648	0.6007	0.542	0.5779	0.5729	0.6077	0.5127	0.551
		0.4834	0.4314	0.3879	0.4172	0.3798	0.445	0.3987	0.4067	0.3743	0.4841	0.4327
		0.6755	0.6469	0.6196	0.6352	0.6129	0.6395	0.6076	0.6295	0.6084	0.6567	0.6239
	$L^{40}_{70\%}$	0.4228	0.4376	0.458	0.4491	0.4689	0.427	0.4479	0.4526	0.472	0.4071	0.4296
		0.7331	0.712	0.6834	0.7051	0.677	0.7192	0.69	0.6983	0.671	0.7424	0.7108
		0.782	0.7668	0.7459	0.7578	0.7372	0.9407	0.9114	0.7535	0.7335	0.9564	0.9259
	$M^{40}_{70\%}$	1.3551	1.2895	1.2388	1.4242	1.333	1.1866	1.1615	1.3649	1.2987	1.134	1.1161
		0.4831	0.3118	0.2064	0.4781	0.2771	0.232	0.169	0.3655	0.2337	0.2433	0.1757
		0.5412	0.4483	0.365	0.5484	0.4287	0.416	0.359	0.4869	0.3932	0.417	0.3536
	$R^{40}_{70\%}$	1.8007	1.6329	1.6028	1.8546	1.6761	1.5962	1.5379	1.7342	1.6429	1.5813	1.5216
		0.9009	0.4767	0.3464	0.8821	0.4422	0.3718	0.2736	0.6069	0.3884	0.3752	0.2718
		0.7556	0.5698	0.482	0.7627	0.5435	0.7243	0.6567	0.6426	0.511	0.7282	0.6561
	$U^{40}_{90\%}$	0.5439	0.559	0.5805	0.5657	0.5872	0.5525	0.574	0.5706	0.5917	0.5356	0.5579
		0.4346	0.415	0.3858	0.4066	0.3724	0.4231	0.3958	0.4004	0.3638	0.4452	0.4161
		0.6562	0.641	0.611	0.6344	0.5993	0.6146	0.5619	0.6271	0.5923	0.6445	0.589
	$L^{40}_{90\%}$	1.538	1.4746	1.4146	1.6005	1.5088	1.3731	1.3351	1.5455	1.4726	1.3291	1.2961
		0.3106	0.201	0.1219	0.3272	0.1936	0.1246	0.0788	0.2568	0.1562	0.1198	0.0736
		0.4098	0.3391	0.2707	0.4417	0.3446	0.3644	0.3293	0.3889	0.3104	0.3528	0.3138
	$M^{40}_{90\%}$	1.538	1.4746	1.4146	1.6005	1.5088	1.3731	1.3351	1.5455	1.4726	1.3291	1.2961
		0.3106	0.201	0.1219	0.337	0.1936	0.1246	0.0788	0.2568	0.1562	0.1198	0.0736
		0.4098	0.3391	0.2707	0.4417	0.3446	0.3644	0.3293	0.3889	0.3104	0.3528	0.3138
	$R^{40}_{90\%}$	1.4808	1.4591	1.4322	1.5094	1.4766	1.4132	1.3913	1.4905	1.4608	1.3956	1.3744
		0.1805	0.1535	0.1233	0.1944	0.1547	0.1218	0.0986	0.1745	0.1401	0.1175	0.0945
		0.3372	0.3129	0.2825	0.3518	0.3163	0.4168	0.4023	0.334	0.3016	0.4059	0.3914

Table 16. The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_2 when $N = 100$ using Set-2.

(T_1, T_2)	CS_{FI}^{N}	MLE	SE		LINEX				GE			
					$c=-2$		$c=2$		$c=-3$		$c=3$	
Prior			I	II	I	II	I	II	I	II	I	II
(2,4)	$U_{50\%}^{100}$	0.637	0.6632	0.6981	0.6801	0.7147	0.6473	0.6826	0.687	0.7203	0.6148	0.6531
		0.3373	0.3075	0.2697	0.2912	0.2548	0.3233	0.2843	0.2834	0.2487	0.3601	0.3156
		0.563	0.5368	0.5019	0.5199	0.4878	0.5514	0.5093	0.513	0.4833	0.5756	0.5316
	$L_{50\%}^{100}$	1.307	1.206	1.1527	1.3797	1.259	1.0867	1.0691	1.296	1.2206	1.0188	1.0129
		0.866	0.4597	0.2645	0.7682	0.4148	0.2945	0.2083	0.5734	0.3319	0.3241	0.2257
		0.6948	0.5577	0.4324	0.7322	0.5531	0.4989	0.43	0.6374	0.4954	0.5235	0.4433
	$M_{50\%}^{100}$	1.3463	1.0826	0.9883	1.3068	1.0893	0.9549	0.9129	1.1627	1.0464	0.9162	0.8684
		1.8706	1.0309	0.5432	1.5429	0.7138	0.7225	0.4613	1.1605	0.5771	0.8336	0.5041
		1.1912	0.8528	0.667	1.0401	0.7336	0.7732	0.6214	0.8773	0.6765	0.8177	0.6402
	$R_{50\%}^{40}$	1.866	1.5726	1.4107	1.8926	1.576	1.3734	1.2867	1.6989	1.4989	1.3091	1.2284
		1.8081	0.7005	0.3108	1.5131	0.5239	0.3972	0.2063	0.8897	0.3844	0.4275	0.2134
		1.029	0.6889	0.4766	0.9615	0.6016	0.6851	0.532	0.7637	0.5227	0.7098	0.5325
	$U_{70\%}^{40}$	0.6623	0.7052	0.7579	0.7377	0.7883	0.6764	0.7305	0.7462	0.7943	0.6206	0.6831
		0.3134	0.2667	0.2144	0.2397	0.1915	0.293	0.237	0.2298	0.1843	0.3538	0.2833
		0.5391	0.496	0.4369	0.464	0.3993	0.4797	0.4035	0.4527	0.3921	0.554	0.441
	$L_{70\%}^{100}$	1.8867	1.7505	1.6262	1.9293	1.7506	1.6116	1.5235	1.834	1.6925	1.5792	1.491
		0.7551	0.3988	0.245	0.6991	0.3274	0.282	0.1688	0.4662	0.2708	0.2726	0.155
		0.6621	0.5146	0.4077	0.6453	0.4766	0.4627	0.4147	0.5553	0.434	0.4455	0.3944
	$M_{70\%}^{100}$	1.9655	1.7164	1.5434	1.9719	1.6923	1.5401	1.4266	1.8167	1.618	1.5097	1.3905
		1.3096	0.8316	0.418	1.0824	0.6478	0.509	0.2856	1.0086	0.5023	0.5413	0.284
		0.9968	0.7324	0.5408	0.8747	0.673	0.6191	0.5223	0.8121	0.5977	0.6264	0.5039
	$R_{70\%}^{100}$	1.7349	1.6721	1.6017	1.7617	1.6745	1.5943	1.5372	1.718	1.6418	1.579	1.5206
		0.6075	0.4672	0.2343	0.6125	0.4289	0.3608	0.2626	0.5243	0.3754	0.3654	0.2612
		0.5926	0.5264	0.4015	0.6046	0.5138	0.7494	0.6789	0.5618	0.482	0.7528	0.6774
	$U_{90\%}^{100}$	0.761	0.7892	0.8257	0.8242	0.8572	0.7584	0.7974	0.827	0.8594	0.7119	0.757
		0.2562	0.2208	0.1808	0.2061	0.1663	0.238	0.1968	0.1962	0.1601	0.2809	0.2307
		0.4845	0.4491	0.4053	0.4313	0.387	0.4733	0.3745	0.4214	0.3799	0.5179	0.4318
	$L_{90\%}^{100}$	1.5996	1.4999	1.4176	1.6669	1.5321	1.3734	1.3242	1.5865	1.4854	1.3214	1.2787
		0.513	0.2945	0.164	0.5255	0.2655	0.1785	0.1069	0.3763	0.2083	0.1774	0.1033
		0.5543	0.4379	0.3364	0.5756	0.4254	0.3533	0.3091	0.4958	0.3797	0.3479	0.2998
	$M_{90\%}^{100}$	1.9881	1.864	1.706	2.003	1.8452	1.7057	1.5925	1.9536	1.776	1.6801	1.5632
		0.8083	0.6431	0.3556	0.8183	0.5531	0.3971	0.2301	0.7892	0.4391	0.3966	0.2164
		0.7883	0.6687	0.5102	0.8032	0.6472	0.5979	0.5088	0.7561	0.5782	0.5848	0.5014
	$R_{90\%}^{100}$	1.4573	1.4393	1.4167	1.4871	1.4593	1.3955	1.3773	1.4702	1.445	1.3768	1.3596
		0.1198	0.1025	0.0829	0.1341	0.1077	0.0781	0.0635	0.12	0.0972	0.073	0.0528
		0.2674	0.249	0.2257	0.2927	0.2645	0.4755	0.4571	0.2763	0.2506	0.4141	0.3642

Table 17. Continue: The AEs (first-line), MSEs (second-line) and MABs (third-line) of θ_2 when $N = 100$ using Set-2.

(T_1, T_2)	CS_{FI}^{N}	MLE	SE		LINEX				GE			
					$c = -2$		$c = 2$		$c = -3$		$c = 3$	
Prior			I	II	I	II	I	II	I	II	I	II
(3,5)	$U_{50\%}^{100}$	0.4961	0.5402	0.5966	0.5589	0.6152	0.5231	0.5795	0.5716	0.6253	0.4754	0.5378
		0.5072	0.4469	0.3751	0.4241	0.3543	0.4684	0.395	0.4074	0.3421	0.5346	0.448
		0.7039	0.6598	0.6034	0.6412	0.5848	0.6681	0.5847	0.6284	0.5747	0.7172	0.6265
	$L_{50\%}^{100}$	0.8895	0.8983	0.9174	0.9869	0.9834	0.8301	0.8628	0.9654	0.9714	0.7585	0.806
		0.4831	0.3118	0.2393	0.4781	0.2771	0.2959	0.2453	0.3655	0.2337	0.3441	0.2801
		0.5412	0.4483	0.3823	0.5484	0.4287	0.5056	0.4414	0.4869	0.3932	0.5505	0.467
	$M_{50\%}^{100}$	1.4279	1.3836	1.3387	1.468	1.4057	1.3121	1.2801	1.432	1.3803	1.2852	1.2542
		0.4371	0.3293	0.2361	0.4371	0.2992	0.2586	0.1927	0.3688	0.2619	0.2655	0.1956
		0.5357	0.4754	0.4097	0.5407	0.4588	0.5171	0.4629	0.5024	0.4317	0.5208	0.4628
	$R_{50\%}^{40}$	1.8391	1.5582	1.4069	1.8637	1.5697	1.3644	1.2842	1.6837	1.495	1.2961	1.2248
		1.361	0.614	0.2837	1.3107	0.4784	0.3493	0.1884	0.7887	0.3541	0.3678	0.1925
		0.9324	0.6605	0.463	0.924	0.59	0.6557	0.5189	0.7399	0.5132	0.6702	0.5149
	$U_{70\%}^{40}$	0.5267	0.5849	0.6571	0.6156	0.6865	0.5581	0.6309	0.6304	0.697	0.4901	0.5746
		0.4665	0.4039	0.3176	0.3717	0.2901	0.434	0.3439	0.3527	0.2777	0.5245	0.4103
		0.6733	0.6152	0.543	0.5848	0.5142	0.5987	0.5296	0.5699	0.5061	0.6551	0.565
	$L_{70\%}^{100}$	1.3551	1.2895	1.2388	1.4242	1.333	1.1866	1.1615	1.3649	1.2987	1.134	1.1161
		0.3643	0.296	0.2064	0.337	0.2461	0.232	0.169	0.2902	0.2301	0.2433	0.1757
		0.4938	0.4382	0.365	0.4654	0.3946	0.416	0.359	0.4352	0.3768	0.417	0.3536
	$M_{70\%}^{100}$	1.5069	1.4763	1.4402	1.5404	1.495	1.4191	1.3907	1.5148	1.4746	1.3982	1.3707
		0.2509	0.2047	0.1568	0.2663	0.2007	0.1592	0.1237	0.2336	0.1788	0.1557	0.1198
		0.4065	0.3708	0.3282	0.4224	0.3713	0.4343	0.4055	0.3973	0.3514	0.4294	0.3993
	$R_{70\%}^{100}$	1.7384	1.6745	1.5039	1.7648	1.6438	1.4732	1.3928	1.7205	1.5803	1.4236	1.3472
		0.6166	0.479	0.2468	0.6246	0.407	0.2791	0.1565	0.5371	0.3129	0.2787	0.1499
		0.6292	0.5603	0.4208	0.6379	0.5359	0.5962	0.5024	0.5941	0.4731	0.5936	0.4906
	$U_{90\%}^{100}$	0.5974	0.6312	0.6752	0.6531	0.6961	0.6113	0.656	0.6613	0.7028	0.5693	0.619
		0.3982	0.3553	0.3031	0.3367	0.286	0.3738	0.3201	0.3249	0.2772	0.4238	0.361
		0.611	0.5761	0.5308	0.5594	0.5137	0.5867	0.4813	0.5493	0.5031	0.6397	0.5359
	$L_{90\%}^{100}$	1.4062	1.3914	1.3728	1.4349	1.4118	1.3514	1.3366	1.4202	1.3993	1.3332	1.3194
		0.1182	0.1018	0.083	0.1289	0.1044	0.0809	0.0665	0.1163	0.0949	0.0777	0.0635
		0.2627	0.2454	0.2234	0.276	0.2503	0.2967	0.2839	0.2626	0.239	0.2927	0.2789
	$M_{90\%}^{100}$	1.5001	1.4775	1.4495	1.5275	1.4936	1.4319	1.4087	1.5084	1.4776	1.4152	1.3926
		0.194	0.1649	0.1324	0.2077	0.1653	0.1313	0.1062	0.1866	0.1499	0.1271	0.102
		0.3387	0.3144	0.284	0.3545	0.3189	0.3838	0.3666	0.3365	0.3039	0.3769	0.3588
	$R_{90\%}^{100}$	1.4961	1.4418	1.3898	1.564	1.4822	1.343	1.3116	1.5131	1.4481	1.2956	1.2706
		0.2359	0.1567	0.0969	0.2684	0.158	0.0952	0.0613	0.2056	0.1273	0.0897	0.0566
		0.3703	0.3083	0.2473	0.406	0.3184	0.3753	0.3495	0.3573	0.2864	0.3518	0.3259

Table 18. The ACLs and CPs of θ_1 and θ_2 using Set-1.

(m,n)			Asymptotic		Credible				Asymptotic		Credible			
Prior			ACL	CP	I		II		ACL	CP	I		II	
					ACL	CP	ACL	CP			ACL	CP	ACL	CP
(T_1,T_2)	$\text{CS}^{\text{N}}_{\text{FI}}$	Par.	(2,4)						(3,5)					
(20,20)	$U^{40}_{50\%}$	θ_1	0.285	0.922	0.28	0.923	0.277	0.929	0.243	0.931	0.241	0.933	0.24	0.936
		θ_2	0.317	0.918	0.317	0.92	0.316	0.926	0.275	0.93	0.272	0.931	0.271	0.933
	$L^{40}_{50\%}$	θ_1	0.646	0.913	0.607	0.918	0.563	0.925	0.515	0.927	0.492	0.931	0.465	0.935
		θ_2	0.676	0.911	0.639	0.916	0.597	0.923	0.578	0.923	0.552	0.929	0.524	0.932
	$M^{40}_{50\%}$	θ_1	0.734	0.905	0.663	0.916	0.603	0.918	0.552	0.924	0.524	0.928	0.492	0.933
		θ_2	0.973	0.898	0.811	0.914	0.725	0.916	0.708	0.91	0.654	0.917	0.603	0.92
	$R^{40}_{50\%}$	θ_1	0.787	0.908	0.692	0.912	0.605	0.918	0.767	0.92	0.679	0.927	0.598	0.93
		θ_2	0.988	0.895	0.941	0.907	0.814	0.911	0.932	0.907	0.906	0.914	0.79	0.918
	$U^{40}_{70\%}$	θ_1	0.278	0.926	0.273	0.93	0.267	0.934	0.226	0.938	0.223	0.941	0.221	0.942
		θ_2	0.258	0.924	0.256	0.933	0.256	0.932	0.21	0.936	0.207	0.939	0.206	0.938
	$L^{40}_{70\%}$	θ_1	0.617	0.918	0.573	0.924	0.53	0.929	0.478	0.93	0.456	0.933	0.432	0.935
		θ_2	0.634	0.916	0.599	0.923	0.561	0.927	0.531	0.928	0.509	0.931	0.485	0.936
	$M^{40}_{70\%}$	θ_1	0.706	0.909	0.657	0.919	0.592	0.925	0.531	0.925	0.498	0.931	0.462	0.934
		θ_2	0.9	0.901	0.805	0.916	0.678	0.922	0.704	0.916	0.617	0.921	0.561	0.925
	$R^{40}_{70\%}$	θ_1	0.657	0.916	0.607	0.922	0.554	0.927	0.665	0.923	0.615	0.93	0.561	0.932
		θ_2	0.834	0.904	0.769	0.917	0.7	0.92	0.814	0.913	0.753	0.918	0.687	0.923
	$U^{40}_{90\%}$	θ_1	0.261	0.931	0.257	0.938	0.254	0.94	0.204	0.941	0.203	0.945	0.202	0.949
		θ_2	0.246	0.928	0.246	0.939	0.243	0.939	0.206	0.94	0.205	0.944	0.201	0.947
	$L^{40}_{90\%}$	θ_1	0.612	0.922	0.561	0.927	0.508	0.932	0.44	0.934	0.416	0.94	0.392	0.943
		θ_2	0.505	0.919	0.468	0.93	0.435	0.933	0.385	0.931	0.369	0.938	0.354	0.942
	$M^{40}_{90\%}$	θ_1	0.516	0.925	0.448	0.93	0.395	0.938	0.331	0.937	0.305	0.942	0.282	0.945
		θ_2	0.747	0.912	0.7	0.925	0.648	0.931	0.627	0.924	0.596	0.931	0.543	0.936
	$R^{40}_{90\%}$	θ_1	0.514	0.924	0.491	0.928	0.464	0.934	0.489	0.932	0.468	0.937	0.444	0.94
		θ_2	0.667	0.915	0.633	0.925	0.594	0.933	0.613	0.927	0.585	0.934	0.554	0.938
(50,50)	$U^{40}_{50\%}$	θ_1	0.182	0.923	0.181	0.924	0.18	0.924	0.156	0.935	0.156	0.936	0.156	0.938
		θ_2	0.202	0.92	0.201	0.922	0.201	0.921	0.174	0.933	0.173	0.934	0.173	0.936
	$L^{40}_{50\%}$	θ_1	0.397	0.918	0.388	0.92	0.376	0.923	0.318	0.934	0.312	0.934	0.305	0.936
		θ_2	0.412	0.916	0.404	0.918	0.393	0.919	0.353	0.93	0.347	0.932	0.34	0.933
	$M^{40}_{50\%}$	θ_1	0.753	0.912	0.694	0.918	0.63	0.92	0.45	0.931	0.432	0.933	0.411	0.934
		θ_2	0.818	0.907	0.757	0.91	0.689	0.915	0.561	0.917	0.535	0.92	0.506	0.922
	$R^{40}_{50\%}$	θ_1	0.487	0.916	0.463	0.92	0.435	0.922	0.499	0.927	0.474	0.93	0.445	0.932
		θ_2	0.698	0.911	0.658	0.914	0.612	0.917	0.685	0.914	0.647	0.917	0.603	0.92
	$U^{40}_{70\%}$	θ_1	0.164	0.93	0.163	0.932	0.162	0.935	0.129	0.941	0.128	0.945	0.128	0.946
		θ_2	0.161	0.923	0.16	0.928	0.16	0.935	0.13	0.935	0.129	0.937	0.129	0.938
	$L^{40}_{70\%}$	θ_1	0.207	0.924	0.203	0.926	0.198	0.931	0.155	0.937	0.153	0.939	0.151	0.947
		θ_2	0.087	0.933	0.086	0.935	0.086	0.937	0.069	0.938	0.068	0.944	0.067	0.945
	$M^{40}_{70\%}$	θ_1	0.539	0.919	0.518	0.921	0.492	0.928	0.392	0.932	0.382	0.937	0.369	0.936
		θ_2	0.583	0.913	0.561	0.918	0.534	0.922	0.46	0.923	0.448	0.927	0.433	0.927
	$R^{40}_{70\%}$	θ_1	0.393	0.922	0.382	0.928	0.369	0.93	0.397	0.93	0.386	0.934	0.372	0.935
		θ_2	0.534	0.916	0.517	0.92	0.496	0.925	0.524	0.92	0.508	0.923	0.487	0.925
	$U^{40}_{90\%}$	θ_1	0.152	0.933	0.152	0.941	0.151	0.942	0.125	0.944	0.124	0.946	0.124	0.947
		θ_2	0.151	0.935	0.151	0.939	0.15	0.94	0.126	0.941	0.126	0.944	0.127	0.945
	$L^{40}_{90\%}$	θ_1	0.091	0.938	0.091	0.945	0.09	0.945	0.073	0.949	0.073	0.948	0.073	0.951
		θ_2	0.066	0.939	0.065	0.948	0.064	0.948	0.052	0.95	0.053	0.951	0.054	0.952
	$M^{40}_{90\%}$	θ_1	0.436	0.927	0.424	0.936	0.409	0.939	0.342	0.94	0.335	0.942	0.327	0.943
		θ_2	0.46	0.923	0.448	0.926	0.434	0.928	0.386	0.931	0.379	0.936	0.37	0.938
	$R^{40}_{90\%}$	θ_1	0.305	0.928	0.3	0.935	0.294	0.937	0.291	0.939	0.287	0.943	0.281	0.945
		θ_2	0.403	0.925	0.396	0.929	0.386	0.932	0.379	0.934	0.372	0.937	0.364	0.941

Table 19. The ACLs and CPs of θ_1 and θ_2 using Set-2

(m,n)			Asymptotic		Credible				Asymptotic		Credible			
Prior			ACL	CP	I		II		ACL	CP	I		II	
					ACL	CP	ACL	CP			ACL	CP	ACL	CP
(T_1,T_2)	$\text{CS}_{\text{FI}}^{\text{N}}$	Par.	(0.5,1.5)						(1,2)					
(20,20)	$U^{40}_{50\%}$	θ_1	0.822	0.876	0.817	0.879	0.805	0.883	0.695	0.903	0.69	0.91	0.679	0.917
		θ_2	0.782	0.873	0.764	0.877	0.741	0.88	0.643	0.902	0.642	0.908	0.637	0.914
	$L^{40}_{50\%}$	θ_1	1.743	0.867	1.533	0.872	1.334	0.879	1.65	0.899	1.433	0.906	1.245	0.916
		θ_2	1.732	0.865	1.504	0.871	1.29	0.877	1.418	0.895	1.272	0.906	1.126	0.913
	$M^{40}_{50\%}$	θ_1	2.241	0.86	1.807	0.868	1.485	0.872	1.65	0.898	1.433	0.905	1.245	0.914
		θ_2	2.027	0.853	1.632	0.865	1.358	0.87	1.418	0.894	1.272	0.902	1.126	0.912
	$R^{40}_{50\%}$	θ_1	2.733	0.863	1.942	0.866	1.606	0.868	2.625	0.892	1.996	0.904	1.583	0.911
		θ_2	2.329	0.851	1.753	0.862	1.388	0.865	2.279	0.88	1.74	0.891	1.385	0.9
	$U^{40}_{70\%}$	θ_1	0.673	0.889	0.672	0.897	0.671	0.906	0.538	0.919	0.532	0.927	0.528	0.933
		θ_2	0.699	0.887	0.679	0.9	0.656	0.904	0.549	0.917	0.543	0.925	0.536	0.929
	$L^{40}_{70\%}$	θ_1	1.643	0.883	1.443	0.892	1.261	0.901	1.551	0.911	1.39	0.919	1.232	0.925
		θ_2	1.639	0.879	1.418	0.891	1.219	0.899	1.399	0.909	1.227	0.917	1.071	0.927
	$M^{40}_{70\%}$	θ_1	1.998	0.873	1.672	0.89	1.428	0.897	1.551	0.91	1.39	0.918	1.232	0.925
		θ_2	1.885	0.87	1.608	0.884	1.334	0.894	1.399	0.898	1.227	0.907	1.071	0.926
	$R^{40}_{70\%}$	θ_1	1.995	0.876	1.788	0.888	1.489	0.9	1.951	0.905	1.766	0.916	1.476	0.923
		θ_2	1.906	0.865	1.584	0.885	1.321	0.892	1.924	0.895	1.6	0.904	1.332	0.914
	$U^{40}_{90\%}$	θ_1	0.657	0.903	0.653	0.915	0.647	0.921	0.537	0.932	0.525	0.938	0.513	0.944
		θ_2	0.682	0.905	0.669	0.917	0.654	0.92	0.515	0.931	0.513	0.937	0.511	0.943
	$L^{40}_{90\%}$	θ_1	1.398	0.897	1.192	0.909	1.044	0.919	0.998	0.928	0.924	0.934	0.857	0.94
		θ_2	1.564	0.891	1.296	0.907	1.1	0.912	0.997	0.919	0.966	0.927	0.876	0.937
	$M^{40}_{90\%}$	θ_1	1.942	0.894	1.547	0.904	1.227	0.915	1.145	0.925	1.416	0.933	0.974	0.938
		θ_2	1.588	0.887	1.158	0.903	0.943	0.914	0.924	0.922	0.785	0.931	0.692	0.94
	$R^{40}_{90\%}$	θ_1	1.684	0.896	1.499	0.905	1.315	0.913	1.573	0.923	1.414	0.935	1.254	0.935
		θ_2	1.453	0.889	1.303	0.901	1.152	0.91	1.401	0.916	1.261	0.924	1.119	0.933
(50,50)	$U^{40}_{50\%}$	θ_1	0.526	0.891	0.526	0.892	0.526	0.9	0.444	0.904	0.439	0.908	0.435	0.912
		θ_2	0.497	0.898	0.493	0.901	0.489	0.908	0.412	0.913	0.411	0.915	0.411	0.917
	$L^{40}_{50\%}$	θ_1	1.077	0.885	1.023	0.888	0.957	0.897	0.872	0.9	0.841	0.904	0.802	0.909
		θ_2	1.057	0.89	1	0.895	0.931	0.906	0.817	0.907	0.786	0.91	0.748	0.913
	$M^{40}_{50\%}$	θ_1	1.959	0.872	1.733	0.88	1.453	0.891	1.27	0.891	1.164	0.903	1.055	0.906
		θ_2	1.946	0.88	1.636	0.888	1.366	0.897	1.099	0.904	1.016	0.907	0.926	0.911
	$R^{40}_{50\%}$	θ_1	1.798	0.878	1.565	0.885	1.345	0.894	1.756	0.897	1.534	0.9	1.323	0.903
		θ_2	1.454	0.886	1.28	0.892	1.115	0.903	1.47	0.901	1.294	0.904	1.126	0.908
	$U^{40}_{70\%}$	θ_1	0.426	0.907	0.424	0.912	0.423	0.917	0.333	0.923	0.327	0.928	0.323	0.929
		θ_2	0.428	0.91	0.425	0.911	0.422	0.916	0.324	0.923	0.322	0.927	0.321	0.93
	$L^{40}_{70\%}$	θ_1	0.274	0.915	0.271	0.918	0.27	0.924	0.208	0.93	0.203	0.935	0.199	0.937
		θ_2	0.452	0.906	0.437	0.915	0.421	0.918	0.322	0.925	0.318	0.931	0.314	0.933
	$M^{40}_{70\%}$	θ_1	1.487	0.894	1.36	0.898	1.221	0.905	1.096	0.914	1.035	0.92	0.965	0.923
		θ_2	1.425	0.9	1.3	0.907	1.163	0.913	0.989	0.918	0.936	0.924	0.873	0.928
	$R^{40}_{70\%}$	θ_1	1.367	0.899	1.264	0.903	1.149	0.911	1.334	0.91	1.236	0.916	1.126	0.92
		θ_2	1.17	0.902	1.088	0.91	0.994	0.915	1.173	0.913	1.089	0.921	0.995	0.924
	$U^{40}_{90\%}$	θ_1	0.4	0.931	0.4	0.934	0.4	0.934	0.325	0.935	0.322	0.938	0.32	0.942
		θ_2	0.402	0.93	0.399	0.934	0.396	0.936	0.316	0.938	0.316	0.938	0.315	0.942
	$L^{40}_{90\%}$	θ_1	0.186	0.938	0.18	0.941	0.176	0.944	0.146	0.941	0.141	0.943	0.137	0.946
		θ_2	0.218	0.933	0.217	0.938	0.216	0.939	0.17	0.939	0.167	0.942	0.165	0.945
	$M^{40}_{90\%}$	θ_1	1.201	0.925	1.131	0.926	1.047	0.928	0.953	0.932	0.913	0.936	0.864	0.939
		θ_2	1.159	0.92	1.088	0.927	1.003	0.933	0.875	0.93	0.838	0.935	0.793	0.937
	$R^{40}_{90\%}$	θ_1	1.01	0.928	0.966	0.929	0.912	0.931	0.97	0.93	0.93	0.933	0.88	0.937
		θ_2	0.863	0.924	0.828	0.93	0.786	0.935	0.877	0.931	0.84	0.933	0.794	0.936

7. As θ_i, $i = 1, 2$ increase, in most cases, the MSEs, MABs and ACLs for all estimates of θ_1 and θ_2 increase while their CPs decrease.
8. Comparing the proposed censoring schemes U, L, M and R on the basis of smallest MSEs, MABs and ACLs as well as highest CPs, it can be seen in most cases that the proposed (point and interval) estimates of θ_1 and θ_2 based on scheme U (or L) perform satisfactory than those obtained based on other censoring schemes. This result is due to the fact that the expected duration of the experiment using scheme U (where the remaining $N - r$ units are withdrawn uniformly) or using scheme L (where the remaining $N-r$ units are withdrawn at the first stage) is greater than any other scheme.
9. Also, the Bayes estimates developed against the LINEX and GE loss functions show better performances than those obtained under the SE loss function. One of the main properties of the SE loss is that it gives equal weight to underestimation and overestimation due to its symmetrical nature.
10. To sum up, the estimation methodologies proposed in this work provided consistent results in both cases of equal and unequal actual parameter values. As a main result, the Bayesian point and interval estimation of the unknown exponential population parameters is recommended.

6. Numerical Applications

To show the usefulness of the theoretical results and to verify how our estimates work real phenomena, two numerical applications based on simulated and real data sets are analyzed in this section.

6.1. *Application 1: (Simulated data)*

Using the simulation algorithm proposed in Section 5, three different jointly Type-II generalized progressive hybrid censored samples with different choices of T_i, $i = 1, 2$ and R_j, $j = 1, 2, ..., r$ are generated from two different exponential populations (when $(\theta_1, \theta_2) = (0.8, 0.5)$). For fixed $(m, n) = (20, 30)$ and $r = 20$, all simulated samples are reported in Table 20. The Bayes estimates against SE, LINEX (for $\tau(= -2\,, +2)$) and GE (for $c(= -3\,, +3)$) loss functions are obtained. To carried out the Bayesian estimates and credible intervals of θ_1 and θ_2, two informative sets of the hyper-parameter $a_i, b_i, i = 1, 2$ values namely (i) Prior-I: $(a_1, a_2) = (1.6, 1)$ and $b_i = 2$ for $i = 1, 2$ and (ii) Prior-II: $(a_1, a_2) = (4, 2.5)$ and $b_i = 5$ for $i = 1, 2$ are considered. Using each simulated sample, both point and interval estimates using classical and Bayesian approaches of θ_1 and θ_2 are calculated and presented in Tables 21 and 22, respectively.

It is noted, from Table 21, that Bayes estimates obtained based on SE, LINEX or GE loss functions of the unknown parameters θ_1 and θ_2 performed satisfactory in terms of the lowest absolute bias values compared to the classical estimates. Moreover, from Table 22, the credible intervals of θ_1 and θ_2 show good behavior compared to the asymptotic intervals in terms of the shortest confidence length. It is also noted that the calculated estimates proposed here support the same findings established in the simulation study section.

6.2. *Application 2: (Real-life data)*

To demonstrate the usefulness of the proposed estimation methodologies and to show how our estimates work in practice situation, one dataset represents the intervals between failures times (in hours)

Table 20. Three JGPHCS-TII simulated samples

Sample	Scheme	**w** **z** ***s*** ***q***	T_1 $(d_1)T_2(d_2)$	$\sum s$ $\sum q$	R^*_s R^*_q	R^* T^*
1	(3*10,0*10)	0.0119, 0.2584, 0.3880, 0.4322, 0.5990, 0.7474, 0.9843, 1.2206, 1.3943, 1.6119, 2.2698, 2.4658, 2.8515, 2.9003	1(7) 3(14)	11 19	1 5	6 3
		1,1,1,1,0,0,1,1,0,0,1,1,0,0				
		3,3,3,3,0,0,3,3,0,0,0,0,0,0				
		0,0,0,0,3,3,0,0,3,3,0,0,0,0				
2	0*5,3*10,0*5))	0.0119, 0.0816, 0.0937, 0.1509, 0.2584, 0.3303, 0.3996, 0.4494, 0.6163, 0.7614, 1.1747, 1.2781, 1.4157, 1.9448, 2.4658, 3.2938, 3.3278, 3.5598, 4.9222, 9.6946	1(10) 10(25)	13 17	0 0	0 9.6946
		1,0,0,1,1,1,0,0,0,0,1,1,0,0,1,0,0,0,0,0				
		0,0,0,0,0,3,0,0,0,0,3,3,0,0,3,0,0,0,0,0				
		0,0,0,0,0,0,3,3,3,3,0,0,3,3,0,0,0,0,0,0				
3	(0*10,3*10)	0.0119, 0.0816, 0.0937, 0.1509, 0.2584, 0.3303, 0.3309, 0.3464, 0.3880, 0.3996, 0.4154, 0.4938, 0.7211, 0.8003, 1.2171, 1.3271, 1.4785, 2.0792, 2.8515, 3.3278, 3.5598, 4.9222	5(22) 10(23)	11 16	0 1	1 5
		1,0,0,1,1,1,1,0,1,0,1,0,0,0,1,0,1,0,0,0,0,0				
		0,0,0,0,0,0,0,0,0,0,0,3,0,0,0,3,0,3,0,0,0,0,0				
		0,0,0,0,0,0,0,0,0,0,0,3,3,3,0,3,0,3,3,0,0,0				

Table 21. Point estimates with their (absolute biases) under simulated datasets.

Sample	Par.	MLE	SE	LINEX		GE	
				$\tau = -2$	$\tau = 2$	$c = -2$	$c = 3$
			Prior – I				
			Prior – II				
1	θ_1	0.8825	0.863	0.8819	0.846	0.8806	0.8446
		-0.0825	-0.063	-0.0819	-0.046	-0.0806	-0.0446
			0.8189	0.8383	0.8116	0.8612	0.8297
			-0.0189	-0.0383	-0.0116	-0.0612	-0.0297
	θ_2	0.5572	0.5496	0.5488	0.5425	0.5982	0.5464
		-0.0572	-0.0496	-0.0488	-0.0425	-0.0482	-0.0464
			0.5443	0.5401	0.5689	0.5228	0.5193
			-0.0443	-0.0401	-0.0189	-0.0228	-0.0193
2	θ_1	0.8269	0.7773	0.823	0.7824	0.8232	0.7797
		-0.0269	-0.0227	-0.023	-0.0176	-0.0232	-0.0203
			0.8164	0.8204	0.7839	0.8185	0.7893
			-0.0164	-0.0204	-0.0161	-0.0185	-0.0107
	θ_2	0.5747	0.5639	0.5702	0.5698	0.5427	0.5237
		-0.0747	-0.0639	-0.0702	-0.0698	-0.0427	-0.0237
			0.5612	0.5681	0.5607	0.5375	0.515
			-0.0612	-0.0681	-0.0607	-0.0375	-0.015
3	θ_1	0.8468	0.8661	0.8708	0.8674	0.8471	0.7679
		-0.0825	-0.0661	-0.0708	-0.0674	-0.0471	-0.0321
			0.8543	0.8664	0.8533	0.8389	0.78
			-0.0543	-0.0664	-0.0553	-0.0389	-0.02
	θ_2	0.5558	0.565	0.5702	0.5601	0.5736	0.5567
		-0.0825	-0.065	-0.0702	-0.0601	-0.0736	-0.0567
			0.5577	0.5658	0.5528	0.5652	0.5414
			-0.0577	-0.0658	-0.0528	-0.0652	-0.0414

Table 22. Interval estimates with their lengths under simulated datasets.

Sample	Par.	Asymptotic			Credible					
Prior →					I			II		
		Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length
1	θ_1	0.1174	1.4475	1.3301	0.1973	1.4250	1.2277	0.2393	1.3594	1.1201
	θ_2	0.0314	0.9830	0.9516	0.0701	0.9252	0.8551	0.0876	0.9197	0.8321
2	θ_1	0.0847	1.5690	1.4843	0.1643	1.4654	1.3011	0.2079	1.3962	1.1883
	θ_2	0.1254	0.9424	0.8170	0.1552	0.9607	0.8055	0.1676	0.9309	0.7633
3	θ_1	0.2242	1.3694	1.1452	0.3275	1.4229	1.0954	0.3660	1.2282	0.8622
	θ_2	0.1168	0.9948	0.8780	0.1449	0.9509	0.8060	0.1571	0.9220	0.7649

Table 23. Failure times of air-conditioning systems in two Boeing airplanes.

Group	Times
X	1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216
Y	3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139, 188, 197, 210

of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes "7913" (group X) and "7914" (group Y). The failure times (in ascending order) corresponding to groups X and Y with sample sized $m = 27$ and $n = 24$, respectively, are listed in Table 23. Proschan [25] who originally presented this dataset and showed that the failure distribution of the air-conditioning system for both groups X and Y is quite approximated by exponential distribution. Recently, this dataset is also analyzed by Abo-Kasem and Elshahhat [1] for jointly progressive hybrid censored Weibull populations.

For our illustrative purpose, based on the complete failure times of the air-conditioning systems datasets, three different artificial samples with $r = 20$ using various choices of T_i, $i = 1, 2$ and R_j, $j = 1, 2, ..., r$ are created and reported in Table 24. Using the generated samples 1, 2 and 3, the maximum likelihood and Bayes estimates are calculated and provided in Table 25. Because no any prior information is available about the unknown parameters θ_i, $i = 1, 2$, the Bayes estimates relative to the SE, LINEX (for $\tau = -5, +5$) and GE (for $c = -2, +2$) loss functions are obtained based on gamma improper priors, i.e., $a_i = b_i = 0$, $i = 1, 2$. Further, 95% two-sided asymptotic/credible interval estimates of θ_1 and θ_2 are also calculated and presented in Table 26.

From Tables 25-26, it can be seen that maximum likelihood and the Bayes estimates are quite close to each other for every parameter as well as the interval estimates obtained by 95% asymptotic/credible intervals are also similar. As $\tau > 0$, using LINEX loss, it is implies that overestimation results in more penalty than underestimation and reverse is true for $\tau < 0$. Similarly, it is to be noted for the GE loss that as $c > 0$ means overestimation is more serious than under estimation and opposite is true for $c < 0$. Using the invariance property of the MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$, from Table 25, it can be seen that the estimated mean lifetime due to group X are ($0.00538^{-1} = 185.87$, $0.00327^{-1} = 305.81$ and $0.00259^{-1} = 386.10$ from three samples 1, 2 and 3 respectively) and due to group Y are ($0.00237^{-1} = 421.94$, $0.00321^{-1} = 311.53$ and $0.00293^{-1} = 341.30$ from three samples 1, 2 and 3 respectively). It is evident that the group Y has longer lifetime as compared to group X using samples 1 and 2 with the group X has longer lifetime as compared to group Y using samples 3. A similar conclusion can be made using the Bayes estimates with different loss functions. Lastly, we can conclude that the proposed methodologies provide a good demonstration of the proposed censoring plan in the presence of simulated and/or real-life data.

7. Conclusions

In this paper, we considered a new progressive hybrid censoring scheme for two samples called the joint Type-II generalized progressively hybrid censoring scheme (JGPHCS-II). The maximum likelihood and Bayesian estimations based on different loss functions for the unknown parameters of two

Table 24. Three JGPHCS-II samples generated from Proschan's datasets.

Sample	Scheme	**w** **z** ***s*** ***q***	$T_1\ (d_1)$ $T_2(d_2)$	$\sum s$ $\sum q$	R_s^* R_q^*	R^* T^*
1	(4*10,0)	1, 11, 18, 23, 39, 50, 72, 88, 111, 188, 216	60(6) 220(11)	21 19	0 0	0 216
		1,1,1,0,1,0,0,0,1,0,1				
		4,4,4,0,4,0,0,0,4,0,0				
		0,0,0,4,0,4,4,4,0,4,0				
2	0*3,8*5,0*3))	1, 3, 4, 5, 18, 39, 72, 106, 206	50(6) 200(9)	21 19	1 1	2 200
		1,0,1,0,1,0,0,1,1				
		0,0,0,0,8,0,0,8,0				
		0,0,0,8,0,8,8,0,0				
3	0,4*10))	1, 3, 13, 18, 24, 39, 51, 77, 97, 139, 191, 197	200(12) 220(15)	18 18	2 1	3 200
		1,0,0,1,1,0,1,1,1,0,1,0				
		0,0,0,4,4,0,4,4,4,0,0,0				
		0,4,4,0,0,4,0,0,0,4,0,0				

Table 25. Point estimates from Proschan's datasets.

Sample	Par.	MLE	SE	LINEX		GE	
				$\tau = -5$	$\tau = +5$	$c = -2$	$c = +2$
1	θ_1	0.00538	0.00537	0.00539	0.00536	0.00581	0.00401
	θ_2	0.00237	0.00238	0.00238	0.00237	0.00260	0.00165
2	θ_1	0.00327	0.00327	0.00328	0.00326	0.00359	0.00227
	θ_2	0.00321	0.00320	0.00321	0.00320	0.00358	0.00196
3	θ_1	0.00259	0.00259	0.00260	0.00259	0.00284	0.00180
	θ_2	0.00293	0.00292	0.00293	0.00292	0.00327	0.00179

Table 26. Interval estimates from Proschan's datasets

Sample	Par.	Asymptotic	Credible
1	θ_1	(0.00107,0.00968)	(0.00197,0.01045)
	θ_2	(0.00029,0.00446)	(0.00077,0.00486)
2	θ_1	(0.00040,0.00614)	(0.00106,0.00671)
	θ_2	(0.00006,0.00635)	(0.00087,0.00703)
3	θ_1	(0.00032,0.00487)	(0.00084,0.00531)
	θ_2	(0.00006,0.00579)	(0.00079,0.00641)

exponential populations have been discussed based on the proposed plan. Using asymptotic normality of the maximum likelihood estimators, the two-sided asymptotic confidence intervals of the unknown population parameters are constructed. Further, the corresponding two-sided Bayes credible intervals of the Bayes estimators are also constructed. To study the performance of the proposed estimates, Monte Carlo simulation experiments have been employed and they showed that the Bayes estimates with associated credible intervals perform quite satisfactorily than the other estimators. A numerical example has also been presented to illustrate all the inferential results established here. We hope that the methodology and results discussed here will be extended to include other distributions (e.g., Weibull or generalized exponential distribution). Finally, it may be of important to consider the problem of predicting the failure times of units placed on the proposed censoring scheme as a future work.

List of Abbreviations

Abbreviations	**Meaning**
ACI	Approximate Confidence Interval
ACLs	Average confidence lengths
BCIs	Bayes credible intervals
AEs	Average estimates
BEs	Bayes estimators
CDF	Cumulative density function
CPs	Coverage probabilities
CSs	Censoring schemes
ER	Estimated Risk
GD	Gamma Distribution
GPHCS	Generalized progressive hybrid censored scheme
GE	General Entropy
HPD	Highest Posterior Density.
i.i.d	Independent Identically Distributed
JPHC-I	Joint progressive hybrid type-I censored
JGPHCS-I	Joint type-I generalized progressive hybrid censored scheme
JGPHCS-II	Joint type-II generalized progressive hybrid censored scheme
K-S	Kolmogorov-Smirnov
LINEX	Linear Exponential
MAB	Mean absolute bias
MLEs	Maximum Likelihood Estimators
MSE	Mean Squared Error
MVUE	Minimum variance unbiased estimator
PDF	Probability Density Function
SE	Squared Error

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