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Research article

Bayesian and Non-Bayesian Inference for The Generalized Power Akshaya Distribution with Application in Medical

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Abstract: Generalized power Akshaya distribution is a brand-new two-parameter distribution that builds on the Akshaya distribution first introduced by [1]. The lifetime data is intended to be modelled by this distribution. The generalised power Akshaya's parameters are estimated using both the non-Bayesian and Bayesian approaches in this work. The weighted least square estimation (WLSE), least square estimation (LSE), Cramer-von-Mises estimation (CVME), Anderson and Darling (AD) method of estimation, maximum product Space estimators (MPSE), and maximum likelihood estimation (MLE), six non-Bayesian estimation methods, are used to find the model parameters. The parameters of the suggested distribution were also determined using the squared error loss function and Bayesian estimating (BE) under independent gamma priors. The unknown parameters have been estimated using the Bayesian approach using Markov chain Monte Carlo (MCMC). Additionally, the parameters' average width of the confidence intervals and coverage probability are computed. Additionally, the reliable intervals for Bayesian estimates of the unknown parameters calculated.

Keywords: Generalized power Akshaya distribution; Maximum Product Spacing Estimators; Anderson and Darling (AD) method of estimation; Bayesian Markov chain Monte Carlo; Weighted least square estimation; Cramer-von-Mises estimation.
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1. Introduction

Numerous applied sciences, including insurance, finance, the bio-medical field, and engineering sciences, require the statistical analysis and modelling of lifetime data,see [2], [3], [4], [5], [6], [7], and [8]. As a result, a lot of lifetime distributions have been introduced recently. Al-Kutubi et al. [9] proposed a new extended two-parameter distribution's properties, estimate techniques, and geological and medical applications. Sobhi and Mashail [10] discussed moments of dual generalized order statistics and characterization for transmuted exponential model. Additionally, a thorough instructional overview of Markov chain Monte Carlo (MCMC) techniques was provided by Brooks [11], who also covered various implementation-related concerns. Shafq et al. [12] studied the extension of Lindley distribution with statistical properties, estimation and simulation. Mahmood et al. [13] and Muse et al. [14] explored areas of application of an extended Cosine generalized family of distributions for reliability modeling and modelling the COVID-19 mortality rate with a new versatile modification of the log-logistic distribution.

The Weibull distribution was estimated using the E-Bayesian method by Okasha and Mustafa [15] using adaptive type-1 progressive hybrid censored competing risks. Rama A new one-parameter Akash distribution that combines exponential (θ) and gamma $(3, \theta)$ distributions was introduced by Shanker in [16, 17]. For modelling lifetime data, he also proposed the Akshaya distribution [18], which had just one parameter and performed better than the Lindley and traditional exponential distributions. The links between the exponential distributions of Akash, Shanker, Lindley, and others, as well as comparative examinations of these distributions, were presented by Shanker et al. in their study [19].

Other articles discussed methods for estimating parameters in addition to the lifespan distributions discussed in the earlier articles. Finally, Smith et al. [20] produced and contrasted the Bayesian and maximum likelihood estimators for the three-parameter Weibull distribution.
The probability density function (pdf) of the Akshaya distribution, according to Shanker [18] and Tolba[21], is given by

$$f(x;\theta) = \frac{\theta^4}{\theta^3 + 3\theta^2 + 6\theta + 6}(1+x)^3 e^{-\theta x}, x, \theta > 0, \tag{1.1}$$

the formula for the cumulative distribution function (CDF) is

$$F(x;\theta) = 1 - \{1 + \frac{\theta^3 x^3 + 3\theta^2(\theta+1)x^2 + 3\theta(\theta^2 + 2\theta + 2)x}{\theta^3 + 3\theta^2 + 6\theta + 6}\}e^{-\theta x}, x, \theta > 0, \tag{1.2}$$

and is followed by the hazard rate function

$$h(x;\theta) = \frac{\theta^4(1+x)^3}{\theta^3 x^3 + 3\theta^2(\theta+1)x^2 + 3\theta(\theta^2+2\theta+2)x + (\theta^3 + 3\theta^2 + 6\theta + 6)}, x, \theta > 0. \tag{1.3}$$

Equation (1.3)'s rate function is an increasing function of x and θ. However, from a theoretical standpoint, the Akshaya distribution is not appropriate in many circumstances. So, this work introduces a more flexible extension of the Akshaya distribution.

Ghitany et al. [22] created a novel distribution known as the power Lindley distribution using the transformation $X = Y^{\frac{1}{\alpha}}$. A new generalised power Akshaya distribution can be introduced using this technique.

Let $y = x^\alpha \rightarrow x = y^{\frac{1}{\alpha}} \longrightarrow dx = \frac{1}{\alpha} y^{\frac{1}{\alpha}-1} dy$

$$F(y) = F_0(x^\alpha), f(y) = \alpha x^{\alpha-1} f_0(x^\alpha). \tag{1.4}$$

This paper's goal is to estimate the unknown parameters of the generalised power Akshaya distribution with different methods for estimation, including its density and hazard functions as described in Section

2. The non-Bayesian inference methods were all explored in Section 3. Section 4 provides the Bayesian method to estimate the parameters of the GPA distribution. Section 5 presents the simulation study. An application of two different forms of real data sets studies in section 6 to demonstrate the flexibility of the distribution. Section 7 introduces the conclusion part.

2. Generalized Power Akshaya Distribution

2.1. Some basic functions

The cumulative distribution function (CDF) and probability density function (pdf) of the power Akshaya distribution are provided, respectively, by Equations (1.1), (1.2) and (1.4),

$$F(x;\theta,\alpha) = 1 - \{1 + \frac{\theta^3 x^{3\alpha} + 3\theta^2(\theta+1)x^{2\alpha} + 3\theta(\theta^2+2\theta+2)x^{\alpha}}{\theta^3+3\theta^2+6\theta+6}\}e^{-\theta x^{\alpha}}, x,\theta,\alpha > 0, \tag{2.1}$$

$$f(x;\theta,\alpha) = \frac{\alpha x^{\alpha-1}\theta^4}{\theta^3+3\theta^2+6\theta+6}(1+x^{\alpha})^3 e^{-\theta x^{\alpha}}, \theta,\alpha > 0. \tag{2.2}$$

Equation (2.1) yields the Akshaya distribution function for $\alpha = 1$.

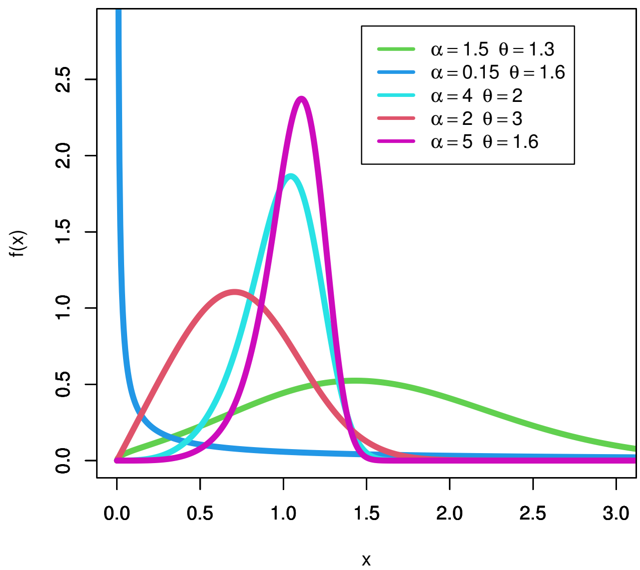

Figure 1. density function with different values of parameters

The GPA distribution is shown in Figure 1 as having a declining, upside-down, left-skewing, and symmetrical shape for certain chosen parameter values.

The generalised power Akshaya distribution's survival function, $S(x)$, and hazard function, $H(x)$, are given as follows, respectively:

$$
\begin{aligned}
S(x;\theta,\alpha) &= 1 - F(x;\theta,\alpha) \\
&= \{1 + \frac{\theta^3 x^{3\alpha} + 3\theta^2(\theta+1)x^{2\alpha} + 3\theta(\theta^2+2\theta+2)x^{\alpha}}{\theta^3+3\theta^2+6\theta+6}\}e^{-\theta x^{\alpha}} \quad x,\theta,\alpha > 0,
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
H(x;\theta,\alpha) &= \frac{f(x;\theta,\alpha)}{S(x;\theta,\alpha)} \\
&= \frac{\alpha\theta^4 x^{\alpha-1}(1+x^{\alpha})^3}{\theta^3(1+x^{3\alpha}) + 3\theta^2(1+(\theta+1)x^{2\alpha}) + 3\theta(2+(\theta^2+2\theta+2)x^{\alpha}) + 6},
\end{aligned} \tag{2.4}
$$

where $x,\theta,\alpha > 0$. We can see from Equation (2.4), that the behaviour of $H(x;\theta,\alpha)$ at $x = 0$ is identical to the behaviour of $f(x;\theta,\alpha)$ at $x = 0$.

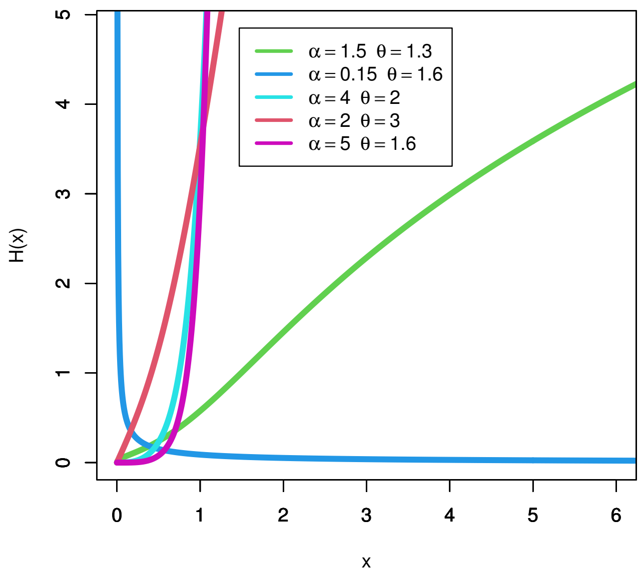

Figure 2. hazard rate function with different values of parameters

The HFs of the GPA distribution are shown in Figure 2. With varied values for the other two parameters, it can be seen that the distribution has decreasing HFs if $\alpha \leq 1$ and increasing HFs if $\alpha \geq 1$.

Some important statistical properties of the GPA distribution studied by Ramadan et al. [1] such as moments, incomplete moments and related measures, quantile function, Bowley Skewness and Moors Kurtosis, mean residual lifetime and mean time to failure. We will study these properties numerically using R program, see Table 1.

3. Non-Bayesian Inference Methods

The parameters of the generalised power Akshaya distribution are estimated in this section using the maximum likelihood estimation (MLE) approach and the Bayesian estimation method, see [23, 24, 25].

Table 1. Summarized of GPA distribution

α	θ	minimum	Q1	Median	mean	SD	Q3	maximum	Sk	Kt
0.5	0.1	20.4635	595.9214	1275.9918	1915.0826	2014.7516	2500.2222	13893.9078	2.4278	11.1983
	0.5	0.0801	16.7645	40.3548	64.8835	74.5261	84.7683	518.8841	2.5605	12.0533
	0.9	0.0027	3.5166	9.7085	16.9615	21.1948	22.0424	149.5139	2.7107	13.0645
	1.3	0.0003	1.1419	3.6270	6.9451	9.3561	8.9168	67.0433	2.8693	14.1900
	1.7	0.0001	0.4641	1.6736	3.5165	5.0520	4.4362	36.8097	3.0264	15.3661
	2.1	0.0001	0.2205	0.8828	2.0252	3.0707	2.5024	22.7444	3.1750	16.5362
	2.5	0.0000	0.1182	0.5132	1.2741	2.0205	1.5403	15.1984	3.3115	17.6608
	2.9	0.0000	0.0697	0.3217	0.8555	1.4082	1.0122	10.7426	3.4346	18.7163
1	0.1	4.5237	24.4113	35.7208	38.9802	19.9403	50.0020	117.8724	0.9408	4.0671
	0.5	0.2829	4.0944	6.3525	7.0085	3.9805	9.2069	22.7790	0.9485	4.0763
	0.9	0.0526	1.8752	3.1158	3.4921	2.1887	4.6949	12.2276	0.9858	4.1323
	1.3	0.0182	1.0686	1.9044	2.1795	1.4852	2.9861	8.1880	1.0487	4.2530
	1.7	0.0089	0.6813	1.2936	1.5166	1.1056	2.1062	6.0671	1.1229	4.4246
	2.1	0.0053	0.4696	0.9396	1.1291	0.8684	1.5819	4.7691	1.1976	4.6246
	2.5	0.0035	0.3438	0.7164	0.8813	0.7071	1.2411	3.8985	1.2677	4.8342
	2.9	0.0026	0.2640	0.5671	0.7126	0.5912	1.0061	3.2776	1.3309	5.0409
2	0.1	2.1269	4.9408	5.9767	6.0416	1.5785	7.0712	10.8569	0.2319	2.9130
	0.5	0.5320	2.0234	2.5204	2.5382	0.7543	3.0343	4.7727	0.1289	2.9274
	0.9	0.2292	1.3694	1.7652	1.7722	0.5942	2.1668	3.4968	0.0827	2.8762
	1.3	0.1348	1.0337	1.3800	1.3853	0.5117	1.7280	2.8615	0.0994	2.7902
	1.7	0.0941	0.8254	1.1374	1.1447	0.4553	1.4513	2.4632	0.1444	2.7269
	2.1	0.0726	0.6853	0.9693	0.9799	0.4119	1.2577	2.1838	0.1968	2.6960
	2.5	0.0595	0.5863	0.8464	0.8602	0.3770	1.1140	1.9745	0.2471	2.6902
	2.9	0.0508	0.5138	0.7531	0.7694	0.3481	1.0030	1.8104	0.2921	2.7003
3	0.1	1.6538	2.9009	3.2934	3.2915	0.5817	3.6841	4.9031	-0.0180	2.8787
	0.5	0.6565	1.5998	1.8520	1.8414	0.3753	2.0959	2.8347	-0.1886	3.0727
	0.9	0.3745	1.2331	1.4606	1.4446	0.3366	1.6745	2.3038	-0.2864	3.1202
	1.3	0.2628	1.0223	1.2395	1.2218	0.3162	1.4400	2.0155	-0.2901	3.0057
	1.7	0.2070	0.8799	1.0896	1.0729	0.3000	1.2818	1.8239	-0.2526	2.8745
	2.1	0.1740	0.7773	0.9794	0.9651	0.2856	1.1652	1.6832	-0.2040	2.7736
	2.5	0.1524	0.7005	0.8948	0.8831	0.2725	1.0746	1.5739	-0.1567	2.7052
	2.9	0.1372	0.6415	0.8277	0.8186	0.2606	1.0020	1.4854	-0.1146	2.6614

3.1. Maximum likelihood estimation method

Let $(x_1, x_2, ..., x_n)$ be a random sample from generalized power Akshaya distribution, then the likelihood estimation function, L can be given as follows

$$L = \prod_{i=1}^{n} f(x;\theta,\alpha) = \frac{\alpha^n \theta^{4n}}{(\theta^3 + 3\theta^2 + 6\theta + 6)^n} e^{-\theta \sum_{i=1}^{n} x_i^{\alpha}} \prod_{i=1}^{n} x_i^{\alpha-1}(1 + x_i^{\alpha})^3, \tag{3.1}$$

and the natural log likelihood function is given by

$$ln(L) = n\{ln(\alpha) + 4ln(\theta) - ln(\theta^3 + 3\theta^2 + 6\theta + 6)\} - \theta \sum_{i=1}^{n} x_i^{\alpha} + \sum_{i=1}^{n}\{(\alpha - 1)ln(x_i) + 3ln(1 + x_i^{\alpha})\}.$$

With regard to θ, α, the natural log likelihood function's first derivatives are given by

$$\frac{\partial}{\partial\theta} ln(L) = \frac{4n}{\theta} - \frac{3\theta^2 + 6\theta + 6}{\theta^3 + 3\theta^2 + 6\theta + 6} - \sum_{i=1}^{n} x_i^{\alpha}, \tag{3.2}$$

$$\frac{\partial}{\partial\alpha} ln(L) = \frac{n}{\alpha} - \theta \sum_{i=1}^{n} x_i^{\alpha} ln(x_i) + \sum_{i=1}^{n}\{ln(x_i)(1 + \frac{3x_i^{\alpha}}{1 + x_i^{\alpha}})\}. \tag{3.3}$$

Numerical approaches are utilised to provide solutions to the equations (3.2) and (3.3), which have no analytic closed form when equating by zero. With regard to θ, α, the second derivatives of the natural log likelihood function can be given by

$$\frac{\partial^2}{\partial\theta^2} ln(L) = \frac{-4n}{\theta^2} - \frac{6(\theta + 1)(\theta^3 + 3\theta^2 + 6\theta + 6) - (3\theta^2 + 6\theta + 6)^2}{(\theta^3 + 3\theta^2 + 6\theta + 6)^2}, \tag{3.4}$$

$$\frac{\partial^2}{\partial\theta\partial\alpha} ln(L) = \sum_{i=1}^{n} x^{\alpha} ln(x_i), \tag{3.5}$$

$$\frac{\partial^2}{\partial\alpha\partial\theta} ln(L) = \sum_{i=1}^{n} x^{\alpha} ln(x_i), \tag{3.6}$$

$$\frac{\partial^2}{\partial\alpha^2} ln(L) = \frac{-n}{\alpha^2} - \theta \sum_{i=1}^{n} x_i^{\alpha}(ln(x_i))^2 + \sum_{i=1}^{n}\{ln(x_i)(1 + \frac{3x_i^{\alpha} ln(x_i)}{(1 + x_i^{\alpha})^2})\}. \tag{3.7}$$

The $(1 - \zeta)100\%$ confidence interval for the parameters θ and α can be written as
$(\hat{\theta_L}, \hat{\theta_U}) = \hat{\theta} \mp z_{1-\frac{\zeta}{2}} \sqrt{var(\hat{\theta})}$, $\qquad (\hat{\alpha_L}, \hat{\alpha_U}) = \hat{\alpha} \mp z_{1-\frac{\zeta}{2}} \sqrt{var(\hat{\alpha})}$,
where $\hat{\theta}$ and $\hat{\alpha}$ are the maximum likelihood estimates of θ and α, $z_{1-\frac{\zeta}{2}}$ is the percentile of the standard normal distribution and $var(\hat{\theta}), var(\hat{\alpha})$ are the asymptotic variances of maximum likelihood estimates calculated using the inverse of the information matrix as follows

$$F^{-1} = \begin{bmatrix} \frac{-\partial^2}{\partial\theta^2} lnL & \frac{-\partial^2}{\partial\theta\partial\alpha} lnL \\ \frac{-\partial^2}{\partial\alpha^2} lnL & \frac{-\partial^2}{\partial\alpha\partial\theta} lnL \end{bmatrix}^{-1} = \begin{bmatrix} var(\hat{\theta}) & cov(\hat{\theta}, \hat{\alpha}) \\ cov(\hat{\alpha}, \hat{\theta}) & var(\hat{\alpha}) \end{bmatrix}^{-1}. \tag{3.8}$$

Set the normal equations (3.2) and (3.3) equal to zero to obtain the likelihood estimates of the model parameters. Since the aforementioned equations are non-linear, the Newton-Raphson approach in R is used to estimate the model parameters.

3.2. Maximum product space estimators (MPSE)

A good substitute for the greatest likelihood approach is the maximum product spacing method, which approximates the Kullback-Leibler information measure. Let us now suppose that the data are ordered in an increasing manner. Then, the maximum product spacing for the GPA is given as follows

$$Gs(\alpha, \theta | data) = \left(\prod_{i=1}^{n+1} D_l(x_i, \alpha, \theta) \right)^{\frac{1}{n+1}}, \tag{3.9}$$

where $D_l(x_i, \alpha, \theta) = F(x_i; \alpha, \theta) - F(x_{i-1}; \alpha, \theta)$, $i = 1, 2, 3, ..., n$

Similarly, one can also choose to maximize the function

$$H(\alpha, \theta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln D_i(\alpha, \theta). \tag{3.10}$$

By taking the first derivative of the function $H(\vartheta)$ with respect to α, and θ, and solving the resulting nonlinear equations, at $\frac{\partial H(\phi)}{\partial \alpha} = 0$, and $\frac{\partial H(\phi)}{\partial \theta} = 0$, where $\phi = (\alpha, \theta)$, we obtain the value of the parameter estimates.

3.3. Anderson and Darling (AD) method of estimation

The function with respect to the model parameters α, and θ is minimised to get the Anderson and Darling estimates, which are written as

$$AD(\alpha, \theta) = -n - \frac{1}{n} \sum_{k=1}^{n} (2k-1)(lnF(x_k) + ln\bar{F}(x_{n+1-k})),$$

where $\bar{F}(x) = 1 - F(x)$.

3.4. Cramer-von-Mises (CVM) method of estimation

Another significant estimating method Macdonald [26] comments is the Cramer-von Mises. By minimising the function $C(\alpha, \theta)$ with respect to the unknown parameters α, and θ, the parameters in the Cramer-von Mises estimation technique can be estimated.

$$C(\alpha, \theta) = \frac{1}{12} + \sum_{k=1}^{n} \left[F(x_k) - \frac{2k-1}{2n} \right]^2$$

$$= \frac{1}{12} + \sum_{k=1}^{n} \left[1 - \{1 + \frac{\theta^3 x_k^{3\alpha} + 3\theta^2(\theta+1)x_k^{2\alpha} + 3\theta(\theta^2 + 2\theta + 2)x_k^{\alpha}}{\theta^3 + 3\theta^2 + 6\theta + 6} \} e^{-\theta x_k^{\alpha}} - \frac{2k-1}{2n} \right]^2.$$

3.5. Least square estimation (LSE) and weighted least square estimation (WLSE)

To estimate the parameters of the beta distribution, Swain et al. [27] offer the least square and weighted least square techniques of estimation. The least squares function $LS(\alpha,\theta)$ with respect to the unknown parameters can be minimised in the LSE approach to provide estimates of the parameters of the proposed model, where

$$LS(\alpha,\theta)=\sum_{k=1}^{n}\left[F(x_k)-\frac{k}{n+1}\right]^2$$

$$=\sum_{k=1}^{n}\left[1-\{1+\frac{\theta^3x_k^{3\alpha}+3\theta^2(\theta+1)x_k^{2\alpha}+3\theta(\theta^2+2\theta+2)x_k^{\alpha}}{\theta^3+3\theta^2+6\theta+6}\}e^{-\theta x_k^{\alpha}}-\frac{k}{n+1}\right]^2.$$

Similar to this, the weighted least square function $WLS(\alpha,\theta)$ is minimised to determine the WLSE of the unknown parameters:

$$WLS(\alpha,\theta)=\sum_{k=1}^{n}\frac{(n+1)^2(n+2)}{k(n-k+1)}\left[F(x_k)-\frac{k}{n+1}\right]^2$$

$$=\sum_{k=1}^{n}\frac{(n+1)^2(n+2)}{k(n-k+1)}\left[1-\{1+\frac{\theta^3x_k^{3\alpha}+3\theta^2(\theta+1)x_k^{2\alpha}+3\theta(\theta^2+2\theta+2)x_k^{\alpha}}{\theta^3+3\theta^2+6\theta+6}\}e^{-\theta x_k^{\alpha}}-\frac{k}{n+1}\right]^2.$$

4. Bayesian estimation method

The parameters θ and α, which are assumed to be independent and follow the gamma prior distribution with parameters a and b, are estimated using the Bayesian estimation (BE) approach in this section, see [28].

The form of the gamma prior density function is

$$g(u;a,b)=\frac{b^a}{\Gamma(a)}u^{a-1}e^{-ub},\quad u,a,b>0. \tag{4.1}$$

Then, the joint prior density of θ and α is given by

$$g(\theta,\alpha)=\prod_{i=1}^{n}g(\theta)g(\alpha)\propto(\theta\alpha)^{a-1}e^{-(\theta+\alpha)b}. \tag{4.2}$$

According to the Bayesian method, the joint posterior distribution function is provided by

$$g(\theta,\alpha|\underline{x})=\frac{g(\theta,\alpha)L(\underline{x})}{\int g(\theta,\alpha)L(\underline{x})}\propto g(\theta,\alpha)L(\underline{x}). \tag{4.3}$$

Substituting from Equations (4.2) and (3.1) into Equation (4.3) we get

$$g(\theta, \alpha|\underline{x}) \propto (\theta\alpha)^{a-1} e^{-(\theta(\sum_{i=1}^{n} x_i^{\alpha}+b)+\alpha b)} \prod_{i=1}^{n} x_i^{\alpha-1}(1+x_i^{\alpha})^3. \tag{4.4}$$

Without determining the normalised constant, the posterior distribution is computationally summarised using the Markov Chain Monte Carlo (MCMC) method [11].

Markov chain Monte Carlo Method

One of the most successful methods in modern Bayesian statistics is the Markov Chain Monte Carlo (MCMC) technique. MCMC method is an algorithm to summarize the posterior distribution without calculating the normalized constant. MCMC techniques have been extensively used to become among the main computational tools in the Bayesian statistical inference [29]. The Metropolis-Hasting sampler is a modified version of the MCMC method. One of the main ideas in MCMC is to find a suitable distribution function, called as 'proposal' that satisfies two conditions: (1) easy to simulate from, and (2) it mimics the posterior distribution function of interest. Once we determine such proposal, we get random draws from it, we apply the acceptance-rejection rule to get random draws from our target posterior distribution.

The following describes the steps of Metropolis—Hasting algorithm to simulate random draws from the posterior distribution $g(\theta|.)$:

1. Set starting point of the chain, say $\theta^{(0)}$.
2. Set a size of trails we get for the random draws, say M.
3. For $i = 1, \cdots, M$ repeat the following steps:
   (a) Set $\theta = \theta^{(i-1)}$.
   (b) Generate a candidate θ^* from a proposal distribution $p(\theta^*|\theta)$.
   (c) Calculate the acceptance probability $\hbar$ as $\hbar = \min\{1, R\}$, where $R = \frac{g(\theta^*|.)p(\theta|\theta^*)}{g(\theta|.)p(\theta^*|\theta)}$.
   (d) Set $\theta^{(i)} = \theta^*$ with probability $\hbar$ or otherwise set $\theta^{(i)} = \theta$.

Under some regularity conditions on the proposal density $p(\theta^*|\theta)$, the sequence of the simulated draws $\{\theta^{(i)}\}_{i=1}^{M}$ will converge to random draws that follow the posterior density $g(\theta|.)$.

5. Simulation

In this section, the parameters are estimated using a Monte Carlo simulation using the MLE, LS, WLS, AD, CVM, MPS, and Bayesian approaches. Using the following and R package: Simulation methodology Data x is distributed as a GPA distribution for different parameters (α, θ) with varying actual values of parameter from 0.5 to 3, and for different samples sizes $n = 50, 100$, and 150. Monte Carlo experiments were performed based on a 10000 random sample for the following data generated from the GPA distribution by using numerical analysis. The best course of action is one that minimises the mean squared error and relative bias (RB) of the estimator (MSE).

The following conclusions can be concluded from Tables 2, 3 and 4; Figures 3, 4, and 5 resectively:

- Because the range of RB, and MSE for two of the parameters of the GPA distribution is quite small, the results of Tables 16 and 17 demonstrate that the GPA distribution is stable.

- In some cases, we see a drop in the RB and MSE for all estimates as the sample size rises.
- This demonstrates that multiple estimating strategies produce accurate RB and MSE findings for large sample sizes.
- The Bayesian estimation approach is the most accurate way to estimate the GPA distribution parameter.
- Better metrics than the MLE, LS, WLS, CVM, and AD approaches are provided by MPS estimation methods.

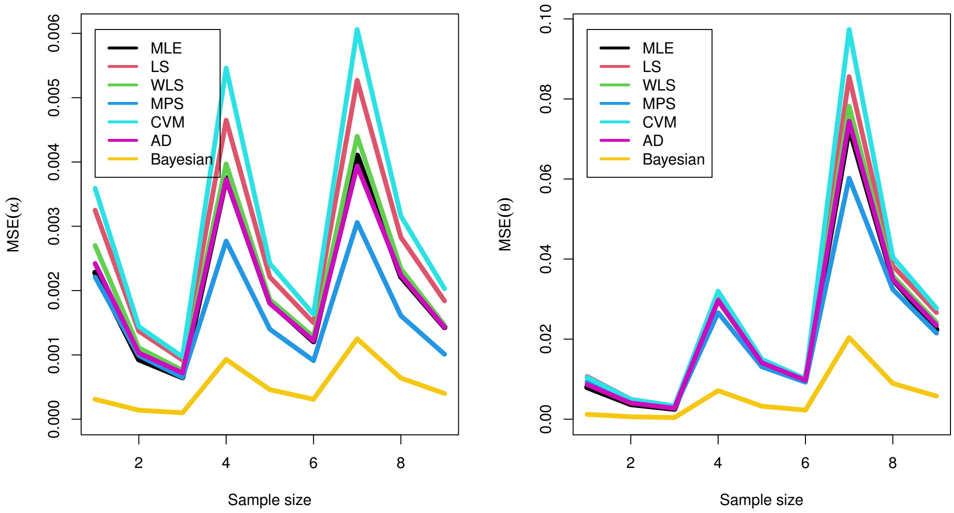

Figure 3. MSE for results in Table 2

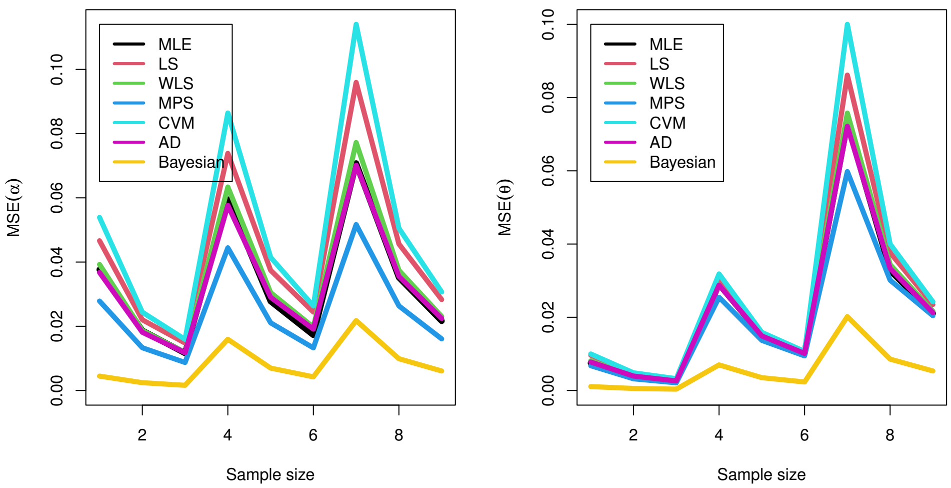

Figure 4. MSE for results in Table 3

Table 2. Bayesian and Non-Bayesian Inference for parameters of GPA distribution when $\alpha = 0.5$

$\alpha = 0.5$			MLE		LS		WLS		MPS		CVM		AD		Bayesian	
θ	n		RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE
0.5	50	α	-0.03392	0.00228	-0.00438	0.00325	0.00303	0.00270	-0.03388	0.00221	0.02056	0.00359	0.00506	0.00242	0.00115	0.00031
		θ	0.07040	0.00798	0.02602	0.01066	0.01139	0.00915	0.07030	0.00940	-0.01312	0.01046	0.00698	0.00866	0.01222	0.00122
	100	α	-0.02298	0.00093	-0.00262	0.00137	0.00192	0.00111	-0.02300	0.00099	0.00966	0.00144	0.00193	0.00103	0.00076	0.00014
		θ	0.04421	0.00375	0.01201	0.00496	0.00340	0.00421	0.04423	0.00431	-0.00761	0.00493	0.00317	0.00405	0.00507	0.00061
	150	α	-0.01371	0.00065	0.00194	0.00092	0.00544	0.00076	-0.01376	0.00065	0.01013	0.00097	0.00500	0.00072	0.00098	0.00010
		θ	0.02314	0.00257	-0.00079	0.00327	-0.00736	0.00283	0.02327	0.00270	-0.01386	0.00331	-0.00682	0.00274	0.00054	0.00040
2	50	α	-0.01589	0.00375	0.01528	0.00465	0.02571	0.00397	-0.01581	0.00277	0.04643	0.00546	0.02953	0.00373	0.02464	0.00093
		θ	-0.00422	0.03020	-0.00335	0.03009	-0.00527	0.02956	-0.00420	0.02662	-0.00462	0.03198	-0.00551	0.02969	0.00161	0.00714
	100	α	-0.00369	0.00182	0.00969	0.00221	0.02053	0.00186	-0.00367	0.00140	0.02490	0.00242	0.02059	0.00180	0.01673	0.00046
		θ	-0.00709	0.01428	-0.00466	0.01452	-0.00686	0.01408	-0.00719	0.01311	-0.00536	0.01499	-0.00672	0.01411	-0.00129	0.00325
	150	α	0.00402	0.00121	0.01307	0.00150	0.02211	0.00128	0.00408	0.00091	0.02323	0.00163	0.02072	0.00122	0.01482	0.00031
		θ	-0.01130	0.00999	-0.00769	0.00986	-0.01033	0.00971	-0.01133	0.00925	-0.00818	0.01008	-0.01002	0.00973	-0.00375	0.00227
3	50	α	-0.02357	0.00410	0.00079	0.00527	0.01425	0.00440	-0.02362	0.00306	0.03477	0.00606	0.01925	0.00394	0.03147	0.00125
		θ	-0.01235	0.07300	-0.00005	0.08556	0.00259	0.07819	-0.01242	0.06021	0.01255	0.09735	0.00421	0.07446	0.00951	0.02038
	100	α	-0.00212	0.00222	0.01523	0.00283	0.02568	0.00234	-0.00218	0.00161	0.03226	0.00316	0.02483	0.00224	0.02374	0.00064
		θ	-0.01141	0.03500	-0.00218	0.03832	-0.00033	0.03579	-0.01144	0.03243	0.00387	0.04047	-0.00044	0.03495	0.00237	0.00894
	150	α	0.00538	0.00143	0.01825	0.00184	0.02591	0.00146	0.00547	0.00101	0.02961	0.00203	0.02462	0.00143	0.02033	0.00040
		θ	-0.01004	0.02240	-0.00048	0.02665	-0.00031	0.02418	-0.00995	0.02150	0.00357	0.02773	-0.00067	0.02345	0.00059	0.00578

Table 3. Bayesian and Non-Bayesian Inference for parameters of GPA distribution when $\alpha = 2$

$\alpha = 2$			MLE		LS		WLS		MPS		CVM		AD		Bayesian	
θ	n		RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE
0.5	50	α	-0.01361	0.03764	0.00664	0.04663	0.01660	0.03925	-0.01365	0.02785	0.03180	0.05390	0.02023	0.03674	0.00774	0.00447
		θ	0.02238	0.00749	-0.00123	0.00963	-0.02063	0.00809	0.02242	0.00677	-0.04007	0.00997	-0.02722	0.00777	-0.00478	0.00106
	100	α	0.00160	0.01867	0.01000	0.02202	0.01882	0.01872	0.00160	0.01331	0.02245	0.02438	0.01885	0.01814	0.00766	0.00243
		θ	-0.01029	0.00401	-0.01678	0.00458	-0.03338	0.00395	-0.01022	0.00319	-0.03625	0.00482	-0.03354	0.00394	-0.01192	0.00055
	150	α	0.00222	0.01162	0.00761	0.01477	0.01465	0.01177	0.00227	0.00872	0.01586	0.01590	0.01392	0.01173	0.00634	0.00161
		θ	-0.00953	0.00257	-0.01189	0.00311	-0.02549	0.00256	-0.00955	0.00210	-0.02493	0.00322	-0.02420	0.00260	-0.00997	0.00036
2	50	α	-0.01698	0.05952	0.01263	0.07385	0.02340	0.06338	-0.01690	0.04445	0.04377	0.08644	0.02678	0.05767	0.02465	0.01595
		θ	-0.00603	0.02905	-0.00496	0.02988	-0.00705	0.02881	-0.00621	0.02545	-0.00627	0.03183	-0.00739	0.02878	0.00023	0.00699
	100	α	-0.00329	0.02778	0.01394	0.03743	0.02219	0.03049	-0.00335	0.02108	0.02932	0.04147	0.02206	0.02902	0.01610	0.00697
		θ	-0.00935	0.01497	-0.00703	0.01537	-0.00923	0.01492	-0.00942	0.01369	-0.00775	0.01587	-0.00907	0.01489	-0.00247	0.00350
	150	α	0.00020	0.01727	0.01085	0.02443	0.01939	0.01972	0.00014	0.01327	0.02101	0.02630	0.01799	0.01910	0.01272	0.00424
		θ	-0.01053	0.01019	-0.00678	0.01043	-0.00942	0.01011	-0.01047	0.00945	-0.00726	0.01066	-0.00913	0.01012	-0.00365	0.00232
3	50	α	-0.02081	0.07076	0.01406	0.09595	0.02334	0.07723	-0.02095	0.05168	0.04900	0.11404	0.02682	0.07019	0.03280	0.02177
		θ	-0.01002	0.07332	0.00585	0.08615	0.00688	0.07578	-0.00995	0.05979	0.01893	0.10001	0.00796	0.07216	0.01055	0.02018
	100	α	-0.00448	0.03518	0.01295	0.04563	0.02305	0.03742	-0.00455	0.02634	0.02995	0.05051	0.02254	0.03553	0.02213	0.00989
		θ	-0.01007	0.03278	-0.00051	0.03763	0.00112	0.03439	-0.01006	0.03013	0.00558	0.03999	0.00112	0.03331	0.00348	0.00856
	150	α	0.00098	0.02164	0.01209	0.02831	0.02092	0.02306	0.00110	0.01607	0.02334	0.03068	0.01964	0.02243	0.01792	0.00607
		θ	-0.01069	0.02109	-0.00301	0.02347	-0.00187	0.02164	-0.01060	0.02042	0.00096	0.02422	-0.00207	0.02125	0.00045	0.00533

Table 4. Bayesian and Non-Bayesian Inference for parameters of GPA distribution when $\alpha = 3$

$\alpha = 3$			MLE		LS		WLS		MPS		CVM		AD		Bayesian	
θ	n		RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE
0.5	50	α	-0.01521	0.08089	0.00674	0.09854	0.01520	0.08354	-0.01516	0.06068	0.03199	0.11447	0.01950	0.07805	0.00783	0.01100
		θ	0.02939	0.00711	0.00338	0.00918	-0.01369	0.00772	0.02938	0.00664	-0.03568	0.00944	-0.02154	0.00746	-0.00076	0.00105
	100	α	0.00215	0.04632	0.01234	0.05581	0.02040	0.04612	0.00224	0.03349	0.02484	0.06197	0.02069	0.04506	0.00807	0.00550
		θ	-0.00985	0.00449	-0.01921	0.00525	-0.03481	0.00446	-0.00996	0.00368	-0.03866	0.00552	-0.03526	0.00445	-0.01193	0.00059
	150	α	0.00521	0.02776	0.00997	0.03267	0.01753	0.02704	0.00520	0.01997	0.01822	0.03551	0.01679	0.02672	0.00721	0.00359
		θ	-0.01655	0.00270	-0.01695	0.00304	-0.03184	0.00257	-0.01651	0.00211	-0.02993	0.00319	-0.03054	0.00261	-0.01254	0.00035
2	50	α	-0.02049	0.13011	0.00774	0.16884	0.01877	0.14356	-0.02040	0.10044	0.03876	0.19475	0.02259	0.13138	0.02301	0.03546
		θ	-0.00358	0.02717	-0.00286	0.02762	-0.00458	0.02692	-0.00374	0.02393	-0.00415	0.02939	-0.00504	0.02684	0.00120	0.00679
	100	α	-0.00092	0.06274	0.01726	0.08197	0.02552	0.06816	-0.00083	0.04597	0.03267	0.09178	0.02534	0.06469	0.01775	0.01571
		θ	-0.01044	0.01444	-0.00789	0.01424	-0.01019	0.01413	-0.01037	0.01317	-0.00863	0.01470	-0.01004	0.01412	-0.00266	0.00329
	150	α	0.00167	0.04034	0.00952	0.05273	0.01847	0.04309	0.00167	0.03063	0.01965	0.05666	0.01746	0.04156	0.01347	0.01000
		θ	-0.00866	0.00998	-0.00514	0.01005	-0.00750	0.00982	-0.00861	0.00930	-0.00561	0.01027	-0.00731	0.00982	-0.00254	0.00231
3	50	α	-0.01678	0.16613	0.01567	0.20719	0.02714	0.17844	-0.01676	0.11685	0.05034	0.24598	0.03129	0.16411	0.03461	0.05025
		θ	-0.01036	0.07492	0.00447	0.08138	0.00634	0.07629	-0.01039	0.06105	0.01734	0.09323	0.00763	0.07421	0.01092	0.02095
	100	α	-0.00456	0.07367	0.01241	0.09476	0.02190	0.07690	-0.00461	0.05466	0.02939	0.10523	0.02236	0.07457	0.02214	0.02127
		θ	-0.01220	0.03289	-0.00269	0.03718	-0.00133	0.03403	-0.01226	0.03069	0.00334	0.03924	-0.00107	0.03332	0.00217	0.00828
	150	α	0.00091	0.04674	0.01190	0.06474	0.02146	0.05244	0.00102	0.03435	0.02313	0.07011	0.01983	0.04942	0.01779	0.01297
		θ	-0.01065	0.02258	-0.00208	0.02749	-0.00124	0.02483	-0.01071	0.02180	0.00192	0.02848	-0.00157	0.02368	0.00046	0.00570

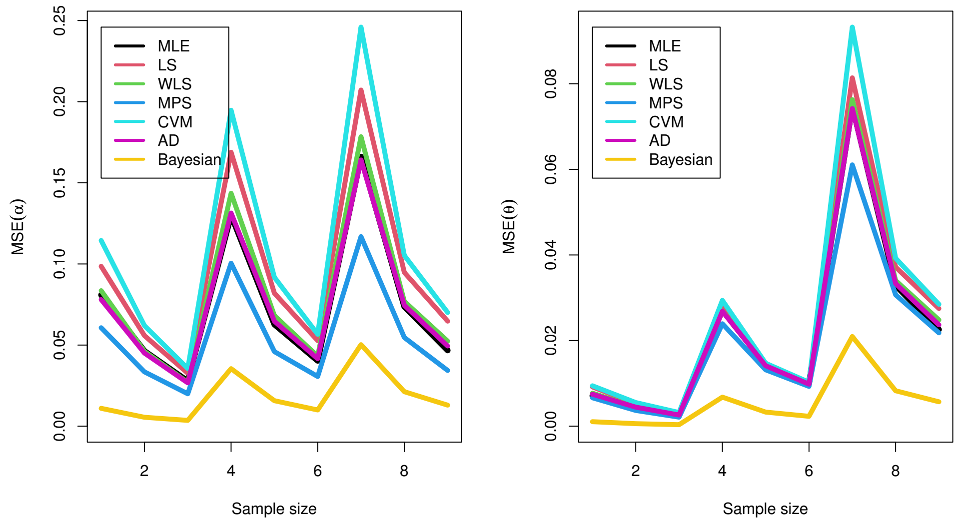

Figure 5. MSE for results in Table 4

6. Application

The GPA distribution is used in this section to model several real data examples from many scientific domains. Different distributions, including Weilbull, Lomax, XGamma Lomax (XGL) Almetwally et al. [30], Inverse Weibul (IW), Inverted Nadarajah-Haghighi (INH), Tahir et al. [31], and Akshaya distribution, are offered for comparison with the GPA distribution.

In Table 5, and 6, the terms "Akaike information criterion (AIC), correct Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC)" were used to analyse MLE with standard error (SE) and various measures (AIC, CAIC, BIC, and HQIC). The Kolmogorov-Smirnov goodness of fit test is used for real data, and the results show that the GPA, Lomax, Weibull, XGL, INH, and IW distributions fit each of the data sets according to the Kolmogorov-Smirnov distance and Kolmogorov-Smirnov p value.

Data set I: The cancer data set are given by Lee and Wang [32] which represent remission times (in months) of a random sample of 128 bladder cancer patients. The data is as follows: "0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69".

Data set II: The data set, which was used by Nassar et al. [33], corresponds to the days between 109 consecutive coal-mining incidents in Great Britain. "1, 4, 4, 7, 11, 13, 15, 15, 17, 18, 19, 19, 20, 20, 22, 23, 28, 29, 31, 32, 36, 37, 47, 48, 49, 50, 54, 54, 55, 59, 59, 61, 61, 66, 72, 72, 75, 78, 78, 81, 93, 96, 99, 108, 113, 114, 120, 120, 120, 123, 124, 129, 131, 137, 145, 151, 156, 171, 176, 182, 188, 189, 195, 203, 208, 215, 217, 217, 217, 224, 228, 233, 255, 271, 275, 275, 275, 286, 291, 312, 312, 312, 315, 326, 326, 329, 330, 336, 338, 345, 348, 354, 361, 364, 369, 378, 390, 457, 467, 498, 517, 566, 644, 745, 871, 1312, 1357, 1613, 1630".

Table 5. MLE with SE and different measures: data set I

		estimates	SE	KS	P-Value	AIC	CAIC	BIC	HQIC
GPA	α	0.6465	0.0341	0.0632	0.6863	828.7689	828.8649	834.4729	831.0864
	θ	0.8342	0.0666						
Weilbull	α	1.047835	0.067577	0.0700	0.5570	832.1738	832.2698	837.8778	834.4913
	θ	9.560699	0.852901						
XGL	α	0.2403	0.0129	0.0724	0.5139	828.9956	828.9891	836.0516	831.5719
	θ	13.5172	0.0027						
	λ	5.7705	0.0027						
Lomax	α	8.3509	4.7050	0.1033	0.1305	831.9923	832.0883	837.6964	834.3099
	θ	69.5592	43.2859						
IW	α	0.7520	0.0424	0.1408	0.0125	892.0015	892.0975	897.7056	894.3191
	θ	2.4305	0.2192						
INH	α	0.5065	0.0476	0.1237	0.0398	866.1182	866.2142	871.8223	868.4358
	θ	10.5993	2.3250						
Akshaya	θ	0.38625	0.017084	0.225396	4.50E-06	926.3736	926.4054	929.2257	927.5324

For comparison with the GPA distribution, Table 5 offers six other distributions. It is evident that the GPA distribution achieves the minimal value for all goodness-of-fit metrics, with a non significant PVKS value ($p > 0.05$) (accept the null hypotheses the data fit of this model). This demonstrates that it is more appropriate and effective than utilising the other competing distributions to simulate the recovery rate of bladder cancer patients. Figure 6 discussed MLE of cdf, and pdf with empirical and histogram, QQ and PP of the GPA model for data set I, which indicate the GDP distribution is fit for this bladder cancer patients data.

For comparison with the GPA distribution, Table 6 offers six other distributions. It is evident that the GPA distribution achieves the minimal value for all goodness-of-fit metrics, with a non significant PVKS value ($p > 0.05$) (accept the null hypotheses the data fit of this model). This demonstrates that it is more appropriate and effective than utilising the other competing distributions to simulate the recovery rate of data set II. Figure 7 discussed MLE of cdf, and pdf with empirical and histogram, QQ and PP of the GPA model for data set II, which indicate the GDP distribution is fit for this data set II. Tables 7 and 8 discussed MLE, LS, WLS, CVM, AD, and Bayesian estimation methods for parameters of GPA distribution. To confirm the MCMC results of parameters of GPA distribution see Figures 8 and 9 foe data set I and II, respectively.

7. Conclusions

For modeling lifetime data, a novel two-parameter lifetime distribution known as the generalized power Akshaya distribution has been developed by Ramadan et al. [1]. Which, Several statistical

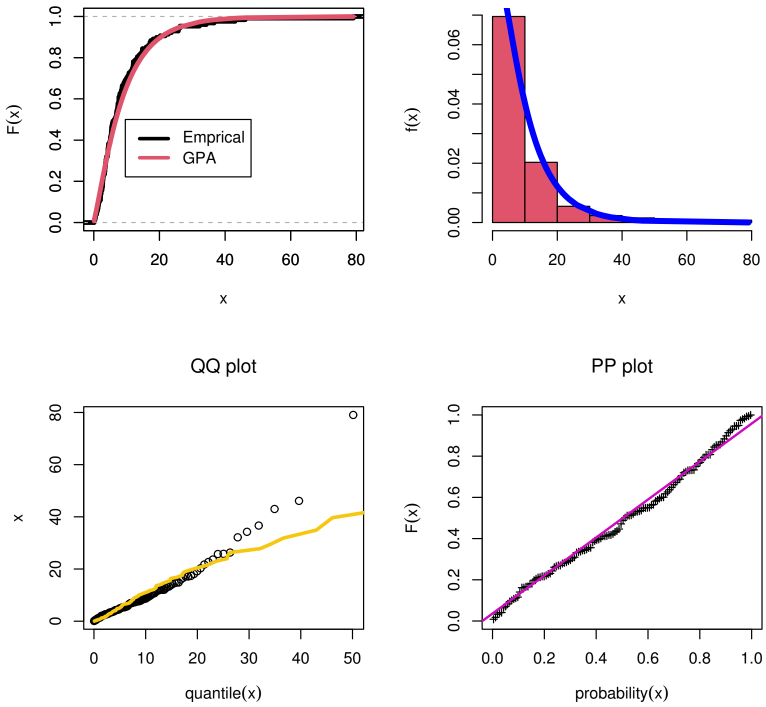

Figure 6. MLE of cdf, and pdf with empirical and histogram, QQ and PP of the GPA model for data set I

Table 6. MLE with SE and different measures: data set II

		estimates	SE	KS	P-Value	AIC	CAIC	BIC	HQIC
GPA	α	0.4599	0.0268	0.0665	0.7212	1405.8852	1405.9984	1411.2679	1408.0680
	θ	0.3466	0.0490						
Weilbull	α	1.758784	0.377937	0.0925	0.3081	1412.4198	1412.5330	1417.8025	1414.6027
	θ	237.0444	68.09407						
XGL	α	0.1672	0.0096	0.1197	0.0882	1423.5023	1423.7309	1431.5764	1426.7767
	θ	8.6186	0.0062						
	λ	24.1269	0.0061						
Lomax	α	0.8849	0.0639	0.0785	0.5135	1407.5448	1407.6580	1412.9275	1409.7277
	θ	218.7039	24.9933						
IW	α	0.6403	0.0407	0.1453	0.0201	1456.6029	1456.7161	1461.9856	1458.7858
	θ	13.4503	2.1748						
INH	α	0.3969	0.0387	0.1257	0.0639	1429.7446	1429.8578	1435.1273	1431.9275
	θ	320.6890	82.5935						
Akshaya	θ	0.0171	0.0008	0.3133	0.0000	1681.0724	1681.1098	1683.7638	1682.1639

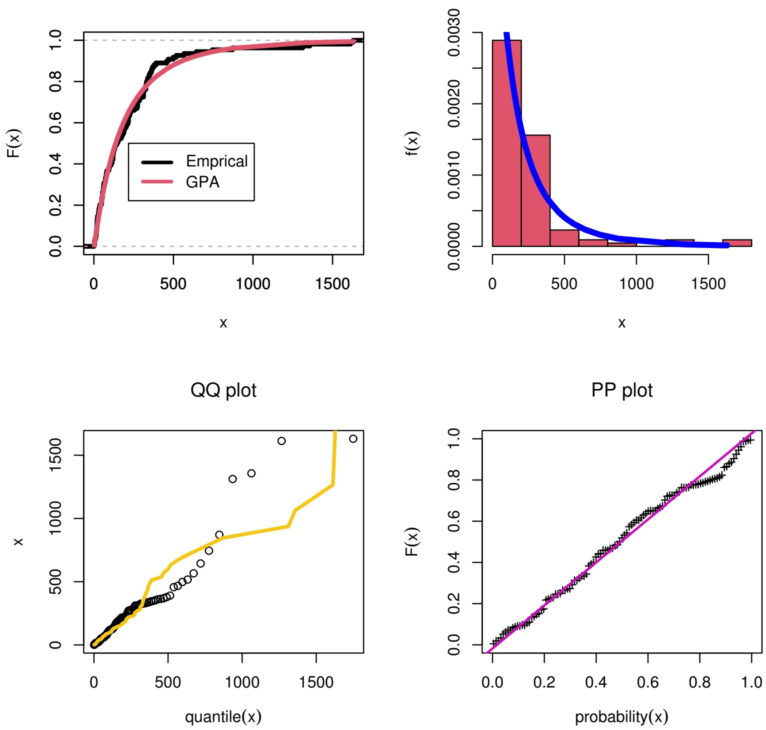

Figure 7. MLE of cdf, and pdf with empirical and histogram, QQ and PP of the GPA model for data set II

Table 7. Estimators and SE for different estimation methods: data set I

method	MLE		Bayesian	
	estimates	SE	Bayes	SEBayes
α	0.6465	0.0341	0.6465	0.0305
θ	0.8342	0.0666	0.8368	0.0587
method	LS		WLS	
	estimates	SE	estimates	SE
α	0.6892	0.2102	0.6744	0.0337
θ	0.8023	0.2686	0.8116	0.0597
method	CVM		AD	
	estimates	SE	estimates	SE
α	0.6960	0.2116	0.6744	0.0368
θ	0.7943	0.2680	0.8116	0.0620

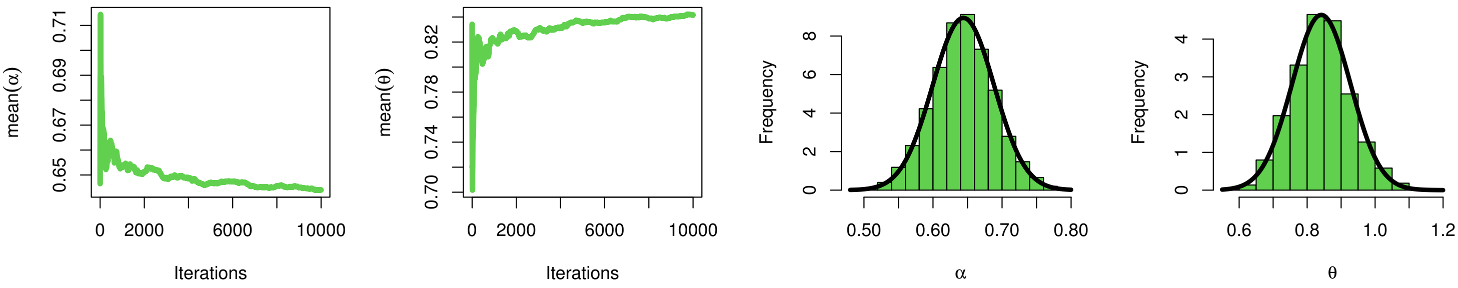

Figure 8. Convergence and histogram of posterior MCMC results of GPA model for data set I

Table 8. Estimators and SE for different estimation methods: data set II

method	MLE		Bayesian	
	estimates	SE	Bayes	SEBayes
α	0.4599	0.0268	0.4596	0.0353
θ	0.3466	0.0490	0.3531	0.0466
method	LS		WLS	
	estimates	SE	estimates	SE
α	0.4528	0.1452	0.4581	0.0050
θ	0.3549	0.2435	0.3497	0.0083
method	CVM		AD	
	estimates	SE	estimates	SE
α	0.4578	0.1466	0.4574	0.0507
θ	0.3468	0.2408	0.3497	0.0841

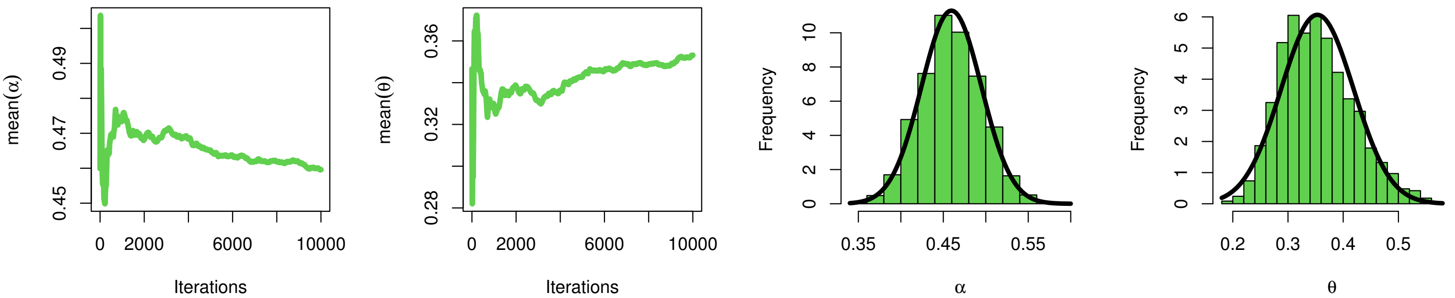

Figure 9. Convergence and histogram of posterior MCMC results of GPA model for data set II

characteristics, including the moments functions, survival, hazard, density, and cumulative distribution are introduced. In this paper, Bayesian and some non-Bayesian methods such as MLE, LS, WLS, AD, CVM, MPS, methods are employed to estimate the distribution parameters. To demonstrate its applicability over Weibull, Akshaya, XGL, Lomax, IW,INH and GPA distributions, the goodness of fit utilising -2 $ln(L)$, Akaike Information Criterion (AIC), Kolmogorov-Samirnov Statistics (K-S), and P-value for real lifespan data has been provided. The mean of the estimated values is then displayed through the use of a simulation study. It is discussed how the maximum likelihood estimators of the model parameters residual bias and mean square error. In addition the confidence intervals for the parameters are calculated.

Conflict of interest

The authors state that they have no financial or other conflicts of interest to disclose with connection to this research.

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Supplementary (if necessary)

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