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Research article

On the Extension of the Burr XII Distribution: Applications and Regression

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Abstract: In this paper, we introduce a new four-parameter mixture distribution called the Harmonic Mixture Burr XII distribution. The proposed model can be used to model data which exhibit bimodal shapes or are heavy-tailed. Specific properties like non-central and incomplete moments, quantile function, entropy, mean and median deviation, mean residual life, moment generating function, and stress-strength reliability are derived. Maximum likelihood estimation, ordinary least squares estimation, weighted least squares estimation, Cramér-von Mises estimation, and Anderson-Darling estimation methods were used to estimate the parameters of the distribution. Simulation studies was performed to assess the estimators and the maximum likelihood estimation was adjudged the best estimator. Using three sets of lifetime data, the empirical importance of the new distribution was determined. When compared to nine (9) extensions of the Burr XII distribution, it was clear that the proposed distribution fit the data better. Using the proposed model, a log-linear regression model called the log-harmonic mixture Burr XII is proposed.

Keywords: Burr XII distribution, heavy-tailed distribution, simulation, applications, and regression
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1. Introduction

Analysing the nature of data sets is an important step in statistical analysis. Data sets have intrinsic statistical characteristics such as skewness, unimodality, bimodality, kurtosis, the nature of failure rates, etc. A statistician would undoubtedly have a better chance of selecting an appropriate model if they correctly identify the types of data sets being used. The dynamic nature of data sets has made it necessary in the last decade to propose modifications to existing distributions in order to increase their flexibility.

The Burr XII distribution, since its introduction by [1] has been widely used in areas such as reliability studies, actuarial science, health, and agriculture. Several works in the literature have been done to improve the flexibility of the Burr XII distribution. The Marshall-Olkin exponentiated Burr XII distribution [2], the Marshall-Olkin Generalised Burr XII distribution [3], the Lindley-Burr XII distribution [4], the Kumaraswamy exponentiated Burr XII distribution [5], the Weibull Burr XII distribution [6], the Kumaraswamy Burr XII distribution [7], order statistics of inverse Pareto distribution [8], the exponentiated exponential Burr XII distribution [10], the Garhy-Burr XII [11], the equilibrium renewal Burr XII distribution [12], the Gompertz-modified Burr XII distribution [13], and the odd exponentiated half-logistic Burr XII distribution [14].

The maximum likelihood estimation, ordinary least square estimation, weighted least square estimation, Cramér-von Mises estimation, and Anderson-Darling estimation have been discussed by recently papers as: [15], [16], [17], [18] and etc.

In this study, we incorporate the Burr XII distribution into the Harmonic Mixture-G (HM-G) family proposed by [19]. By applying the technique of introducing two extra parameters thus one shape and one scale parameter, a new distribution named the Harmonic Mixture Burr XII distribution (HMBXII) was developed.

The study of the HMBXII distribution is motivated by the following:

- To develop a distribution that is unimodal, bimodal and right-skewed with varying kurtosis.
- To introduce a new distribution that can be used as a better substitute for other heavy-tailed distributions or other extensions of the Burr XII distribution in modelling lifetime data sets.
- To propose the log-Harmonic Mixture Burr XII regression model in location-scale form.

The other parts of the paper are organised as follows: Section 2 presents the formulation of the HMBXII distribution. The statistical properties of the developed distribution is presented in section 3. The maximum likelihood estimation, ordinary least square estimation, weighted least square estimation, Cramér-von Mises estimation, and Anderson-Darling estimation are presented as estimators in Section 4. Through a simulation studies we assess the estimators and the results are shown in Section 5. In Section 6, the applicability of the developed distribution is demonstrated using three lifetime data sets. The log-Harmonic Mixture Burr XII regression model (LHMBXII) is proposed and applied to a survival data set in Section 7 while Section 8 provides the study's conclusions.

2. Formulation of the HMBXII Distribution

If $g(x;\eta)$ and $\bar{G}(x;\eta)$ are the density and survival functions of a baseline distribution with parameter vector η, then according to [19] the probability density function (PDF), survival function (SF) and hazard rate function (HRF) of the HM-G family respectively are

$$f_{Hm}(x;\eta) = g(x;\eta)\bar{G}^{\alpha-1}(x;\eta)\frac{\alpha(1-\theta)+\theta\bar{G}^{\alpha-1}(x;\eta)}{\left[1-\theta\left(1-\bar{G}^{\alpha-1}(x;\eta)\right)\right]^2}, y \in R, \alpha \geq 0, 0 \leq \theta \leq 1, \tag{2.1}$$

$$S_{Hm}(y) = \frac{\bar{G}^{\alpha}(y)}{1-\theta\left(1-\bar{G}^{\alpha-1}(y)\right)}, y \in R, \alpha \geq 0, 0 \leq \theta \leq 1 \tag{2.2}$$

and

$$h_{Hm}(x;\eta) = \frac{g(x;\eta)}{\bar{G}(x;\eta)}\frac{\alpha(1-\theta)+\theta\bar{G}^{\alpha-1}(x;\eta)}{1-\theta\left(1-\bar{G}^{\alpha-1}(x;\eta)\right)}. \tag{2.3}$$

Substituting the PDF and SF of the Burr XII distribution given respectively as

$$f_{Br}(x;\zeta,\upsilon) = \zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon-1}, x>0, \zeta>0, \upsilon>0, \tag{2.4}$$

and

$$S_{Br}(x;\zeta,\upsilon) = \left(1+x^{\zeta}\right)^{-\upsilon}, x>0, \zeta>0, \upsilon>0, \tag{2.5}$$

into equations (2.1) and (2.3), we obtain the PDF and HRF of the HMBXII distribution given respectively as

$$f(x) = \frac{\alpha(1-\theta)\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\alpha\upsilon-1} + \theta\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(2\alpha-1)-1}}{\left[1-\theta\left(1-(1+x^{\zeta})^{-\upsilon(\alpha-1)}\right)\right]^{2}}, \tag{2.6}$$

and

$$h(x) = \frac{\alpha(1-\theta)\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\alpha\upsilon-1} + \theta\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(2\alpha-1)-1}}{(1+x^{\zeta})^{-\alpha\upsilon}\left[1-\theta\left(1-(1+x^{\zeta})^{-\upsilon(\alpha-1)}\right)\right]}, x>0. \tag{2.7}$$

where $\zeta > 0$, $\upsilon > 0$, $\alpha > 0$ are shape parameters, $0 < \theta < 1$ is a scale parameter, and $x > 0$.

Figure 1 shows the density plots of the HMBXII distribution. While varying the values of the parameters, the density exhibited shapes such as decreasing, right-skewed, unimodal, or bimodal.

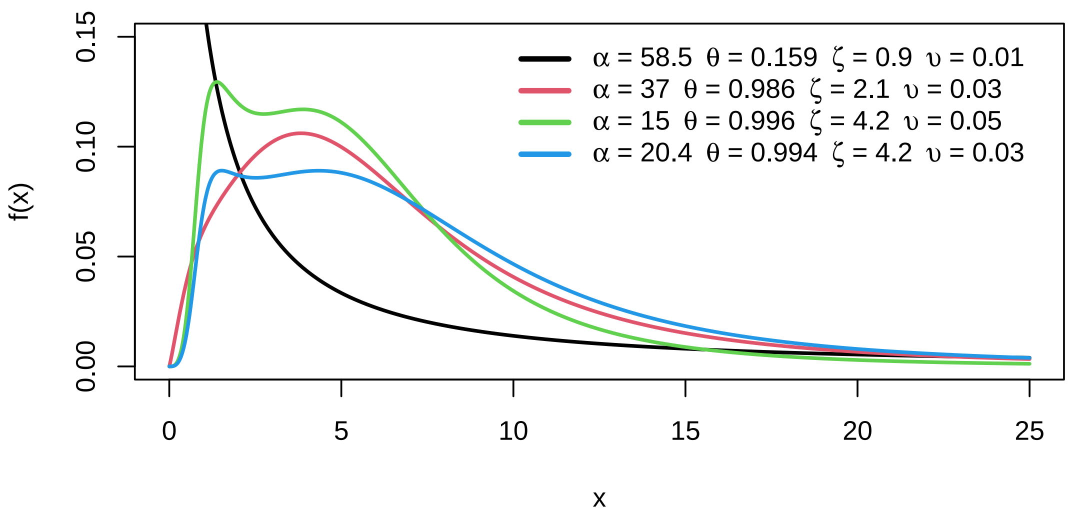

Figure 1. The PDF plot of the HMBXII distribution

Figure 2 shows the plots of the HRF of the HMBXII distribution. While varying the value of some parameters, the hazard rate plots exhibited decreasing or upside down bathtub shapes.

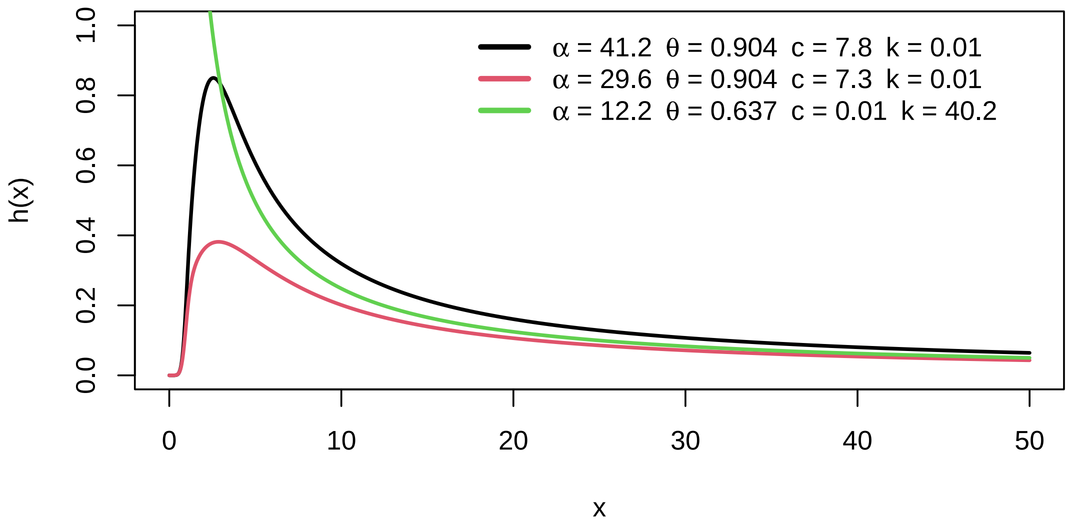

Figure 2. The Hazard Rate Function plots of the HMBXII

Lemma 1. The PDF of the HMBXII distribution, expressed as linear mixture of Burr XII densities is given by

$$f(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\left[\alpha(1-\theta)\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+1)-j)-1} + \theta\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+2)-(j+1))-1}\right], \qquad (2.8)$$

where $\Psi_{ij} = (-1)^j(i+1)\binom{i}{j}\binom{l}{k}\theta^i$, $x > 0, \zeta > 0, \upsilon > 0, \alpha > 0$ and $0 < \theta < 1$.

Proof. For any real non-integer $\beta > 0$, the series expansions for $(1-x)^{-\gamma}$ for $|x| < 1$ is $(1-x)^{-\gamma} = \sum_{i=0}^{\infty}\binom{\gamma+i-1}{i}(x)^i$. Since $0 < \left(1+x^{\zeta}\right)^{-\upsilon} < 1$, using the series expansion equation twice, we obtain

$$\left[1-\theta\left(1-\left(1+x^{\zeta}\right)^{-\upsilon(\alpha-1)}\right)\right]^{-2} = \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j(i+1)\binom{i}{j}\theta^i\left(1+x^{\zeta}\right)^{-\upsilon(\alpha-1)j}.$$

We then obtain

$$\begin{aligned} f(x) &= \alpha(1-\theta)\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j(i+1)\binom{i}{j}\theta^i\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+1)-j)-1} \\ &+ \theta\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j(i+1)\binom{i}{j}\theta^i\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+2)-(j+1))-1}. \end{aligned} \qquad (2.9)$$

The proof is complete. □

3. Statistical Properties

3.1. Quantile Function

We obtain the quantile function of the HMBXII distribution by inverting the CDF. The HMBXII distribution's quantile function can therefore be expressed as

$$(1-q)\left[1-\theta\left(1-\left(1+x_q^{\zeta}\right)^{-\upsilon(\alpha-1)}\right)\right]-\left(1+x_q^{\zeta}\right)^{-\alpha\upsilon}=0, \tag{3.1}$$

where $q \in (0,1)$ and $Q(q) = x_q$ is the quantile function.

3.2. Moments

In this section, we derive the r^{th} moments of X for the HMBXII distribution. Getting the moments of a distribution is important since they help with statistical analysis.

By definition, the r^{th} moments of X is given by

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x)\mathrm{d}x. \tag{3.2}$$

Substituting equation (2.8) into equation (3.2), we obtain

$$\mu_r' = \zeta\upsilon\sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\Big[\alpha(1-\theta)\int_0^\infty x^{r+\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+1)-j)-1}\mathrm{d}x$$
$$+\theta\int_0^\infty x^{r+\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+2)-(j+1))-1}\mathrm{d}x\Big].$$

Letting $u = x^{\zeta}$, which implies $x = u^{1/\zeta}$ and $dx = \frac{1}{c}u^{1/\zeta-1}du$, we obtain

$$\mu_r' = \upsilon\sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\Big[\alpha(1-\theta)\int_0^\infty u^{r/\zeta}\left(1+u\right)^{-\upsilon(\alpha(j+1)-j)-1}\mathrm{d}u$$
$$+\theta\int_0^\infty u^{r/\zeta}\left(1+u\right)^{-\upsilon(\alpha(j+2)-(j+1))-1}\mathrm{d}u\Big].$$

Using the identity (see [6])

$$\mathcal{B}(a,b) = \int_0^\infty u^{a-1}(1+u)^{-(a+b)}\mathrm{d}u, a>0, b>0,$$

we obtain

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\Big[\alpha(1-\theta)\upsilon\mathcal{B}\left(\frac{r}{\zeta}+1,\upsilon\left(\alpha(j+1)-j\right)-\frac{r}{\zeta}\right)$$
$$+\theta\upsilon\mathcal{B}\left(\frac{r}{\zeta}+1,\upsilon\left(\alpha(j+2)-(j+1)\right)-\frac{r}{\zeta}\right)\Big].$$

The r^{th} non-central moment of the HMBXII distribution is given as

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\Bigg[\alpha(1-\theta)\upsilon\mathcal{B}\left(\frac{r}{\zeta}+1,\upsilon\left(\alpha(j+1)-j\right)-\frac{r}{\zeta}\right) + \theta\upsilon\mathcal{B}\left(\frac{r}{\zeta}+1,\upsilon\left(\alpha(j+2)-(j+1)\right)-\frac{r}{\zeta}\right)\Bigg]. \tag{3.3}$$

where $\mathcal{B}(\cdot,\cdot)$ is a beta function and r=1,2,....

The variance (σ^2), coefficient of variation (CV), skewness (CS) and kurtosis (CK) for the HMBXII distribution using the non-central moments are shown in the Table 1 for some selected parameter values. The HMBXII distribution is positively skewed.
Additionally, the HMBXII distribution exhibits either a platykurtic or leptokurtic nature depending on the parameter values.

Table 1. HMBXII Distribution First Five Moments for Some Parameter Values

r	α=8.50, θ=0.20, ζ=2.90, υ=10.50	α=28.50,θ=0.30, ζ=1.90,υ=15.00	α=8.50, θ=0.80, ζ=0.90,υ=15.50	α=10.50, θ=0.50, ζ=1.20, υ=20.50	α=10.50, θ=0.55, ζ=1.90, υ=8.50
$\mu_1^{'}$	0.1983	4.0612×10^{-2}	9.4016×10^{-3}	1.4057×10^{-2}	1.0367×10^{-1}
$\mu_2^{'}$	0.0445	2.0809×10^{-3}	1.4754×10^{-4}	3.0313×10^{-4}	1.3227×10^{-2}
$\mu_3^{'}$	0.0110	1.2456×10^{-4}	3.0515×10^{-6}	8.5515×10^{-6}	1.9379×10^{-3}
$\mu_4^{'}$	0.0029	8.4398×10^{-6}	8.3154×10^{-8}	2.9567×10^{-7}	3.1563×10^{-4}
$\mu_5^{'}$	0.0008	6.2010×10^{-7}	2.8202×10^{-9}	1.1774×10^{-8}	5.6099×10^{-5}
σ^2	0.0052	0.0004	5.9146×10^{-5}	0.0001	0.0025
CV	0.3640	0.5115	0.8180	0.7308	0.4802
CS	0.1470	0.5577	1.2141	1.2206	0.4276
CK	2.7379	3.4170	6.6332	5.1259	2.9921

3.3. *Incomplete Moments*

In this section, we derive the incomplete moments of the HMBXII distribution. The mean deviation, the median deviation, the Lorenz curve, and the Bonferroni curve can all be computed using the incomplete moments.

By definition, the incomplete moment is obtained using the equation

$$m_r(y) = E(X^r|X\le y) = \int_0^y x^r f(x)\mathrm{d}x. \tag{3.4}$$

Substituting equation (2.8) into equation (3.4), we have

$$m_r(y) = \int_0^y x^r \sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\Bigg[\alpha(1-\theta)\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+1)-j)-1} + \theta\zeta\upsilon x^{\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+2)-(j+1))-1}\Bigg]\mathrm{d}x.$$

We then obtain

$$m_r(y) = \zeta\upsilon \sum_{i=0}^{\infty}\sum_{j=0}^{i} \Psi_{ij}\Big[\alpha(1-\theta)\int_0^y x^{r+\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+1)-j)-1}\mathrm{d}x + \theta\int_0^y x^{r+\zeta-1}\left(1+x^{\zeta}\right)^{-\upsilon(\alpha(j+2)-(j+1))-1}\mathrm{d}x\Big].$$

Letting $u = x^{\zeta}$, which implies $x = u^{1/\zeta}$ and $dx = \frac{1}{\zeta}u^{1/\zeta-1}du$, we obtain

$$m_r(y) = \upsilon \sum_{i=0}^{\infty}\sum_{j=0}^{i} \Psi_{ij}\Big[\alpha(1-\theta)\int_0^{y^{\zeta}} u^{r/\zeta}\,(1+u)^{-\upsilon(\alpha(j+1)-j)-1}\mathrm{d}u + \theta\int_0^{y^{\zeta}} u^{r/\zeta}\,(1+u)^{-\upsilon(\alpha(j+2)-(j+1))-1}\mathrm{d}u\Big].$$

Using the identity

$$\mathcal{B}(y:a,b) = \int_0^y u^{a-1}(1+u)^{-(a+b)}\mathrm{d}u, a>0, b>0,$$

we have

$$m_r(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} \Psi_{ij}\Big[\alpha(1-\theta)\upsilon\mathcal{B}\left(y^{\zeta}:\frac{r}{\zeta}+1,\upsilon\left(\alpha(j+1)-j\right)-\frac{r}{\zeta}\right) + \theta\upsilon\mathcal{B}\left(y^{\zeta}:\frac{r}{\zeta}+1,\upsilon\left(\alpha(j+2)-(j+1)\right)-\frac{r}{\zeta}\right)\Big].$$

The r^{th} incomplete moment of the HMBXII distribution is given as

$$m_r(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} \Psi_{ij}\Big[\alpha(1-\theta)\upsilon\mathcal{B}\left(y^{\zeta}:\frac{r}{\zeta}+1,\upsilon\left(\alpha(j+1)-j\right)-\frac{r}{\zeta}\right) + \theta\upsilon\mathcal{B}\left(y^{\zeta}:\frac{r}{\zeta}+1,\upsilon\left(\alpha(j+2)-(j+1)\right)-\frac{r}{\zeta}\right)\Big]. \tag{3.5}$$

where $\mathcal{B}(\cdot:\cdot,\cdot)$ is an incomplete beta function and $r = 1,2,3,....$

3.4. Inequality Measures

The Lorenz curve and the Bonfenorri curve enable researchers to compare income distributions across countries over time.

By definition, the Lorenz curve and Bonfenorri curve are respectively given by

$$L_F(y) = \frac{1}{\mu}\int_0^y xf(x)\mathrm{d}x$$

and

$$L_F(y) = \frac{1}{\mu F(y)}\int_0^y xf(x)\mathrm{d}x,$$

where $\int_0^y xf(x)\mathrm{d}x$ can be obtained using the first incomplete moment.
The Lorenz curve of the HMBXII distribution is given as

$$\begin{aligned} L(y) =&\frac{1}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\Bigg[\alpha(1-\theta)\upsilon\mathcal{B}\left(y^{\zeta}:\frac{1}{\zeta}+1,\upsilon\left(\alpha(j+1)-j\right)-\frac{1}{\zeta}\right) \\ &+\theta\upsilon\mathcal{B}\left(y^{\zeta}:\frac{1}{\zeta}+1,\upsilon\left(\alpha(j+2)-(j+1)\right)-\frac{1}{\zeta}\right)\Bigg]. \end{aligned} \tag{3.6}$$

The Bonferroni curve of the HMBXII distribution is given as

$$\begin{aligned} B(y) =&\frac{1}{\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\Bigg[\alpha(1-\theta)\upsilon\mathcal{B}\left(y^{\zeta}:\frac{1}{\zeta}+1,\upsilon\left(\alpha(j+1)-j\right)-\frac{1}{\zeta}\right) \\ &+\theta\upsilon\mathcal{B}\left(y^{\zeta}:\frac{1}{\zeta}+1,\upsilon\left(\alpha(j+2)-(j+1)\right)-\frac{1}{\zeta}\right)\Bigg]. \end{aligned} \tag{3.7}$$

3.5. Mean Deviation and Median Deviation

With the help of the mean and median deviations, the total variation existing in distributions can be measured.

By the definition of mean deviation,

$$\Delta_1(x) = \int_0^{\infty}|x-\mu|f(x)\mathrm{d}x = 2\mu F(\mu) - 2\int_0^{\mu} xf(x)\mathrm{d}x,$$

where $\int_0^{\mu} xf(x)\mathrm{d}x$ can be obtained using the first incomplete moment.
The mean deviation of the HMBXII distribution is given by

$$\begin{aligned} \Delta_1(x) = 2\mu F(\mu) - 2&\sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\Bigg[\alpha(1-\theta)\upsilon\mathcal{B}\left(\mu^{\zeta}:\frac{1}{\zeta}+1,\upsilon\left(\alpha(j+1)-j\right)-\frac{1}{\zeta}\right) \\ &+\theta\upsilon\mathcal{B}\left(\mu^{\zeta}:\frac{1}{\zeta}+1,\upsilon\left(\alpha(j+2)-(j+1)\right)-\frac{1}{\zeta}\right)\Bigg]. \end{aligned} \tag{3.8}$$

By the definition of median deviation,

$$\Delta_2(x) = \int_0^{\infty}|x-D|f(x)\mathrm{d}x = \mu - 2\int_0^{D} xf(x)\mathrm{d}x,$$

where D is the median and $\int_0^{D} xf(x)\mathrm{d}x$ can be obtained using the first incomplete moment.
The median deviation of the HMBXII distribution is given by

$$\begin{aligned} \Delta_2(x) = \mu - 2&\sum_{i=0}^{\infty}\sum_{j=0}^{i}\Psi_{ij}\Bigg[\alpha(1-\theta)\upsilon\mathcal{B}\left(D^{\zeta}:\frac{1}{\zeta}+1,\upsilon\left(\alpha(j+1)-j\right)-\frac{1}{\zeta}\right) \\ &+\theta\upsilon\mathcal{B}\left(D^{\zeta}:\frac{1}{\zeta}+1,\upsilon\left(\alpha(j+2)-(j+1)\right)-\frac{1}{\zeta}\right)\Bigg]. \end{aligned} \tag{3.9}$$

3.6. Mean Residuals

The mean residual life function at time t measures the expected added lifetime that a unit has survived until the time t.

For a non-negative random variable X, the mean residual life is given by

$$m(t) = E(X - t|X > t) = \frac{1}{S(t)} \int_t^{\infty} (x - t) f(x) \mathrm{d}x = \frac{1}{S(t)} \left[\mu - \int_0^t (x) f(x) \mathrm{d}x \right] - t, t \geq 0,$$

where $\int_0^t x f(x) \mathrm{d}x$ can be obtained from the first incomplete moment.
The mean residual life function of the HMBXII distribution is given as

$$\begin{aligned} m(t) = \frac{1}{S(t)} \Bigg[\mu - \sum_{i=0}^{\infty} \sum_{j=0}^{i} \Psi_{ij} \Bigg[\alpha(1-\theta)\upsilon\mathcal{B}\left(t^{\zeta} : \frac{1}{\zeta} + 1, \upsilon\left(\alpha(j+1) - j\right) - \frac{1}{\zeta} \right) \\ + \theta\upsilon\mathcal{B}\left(t^{\zeta} : \frac{1}{\zeta} + 1, \upsilon\left(\alpha(j+2) - (j+1)\right) - \frac{1}{\zeta} \right) \Bigg] \Bigg] - t. \end{aligned} \tag{3.10}$$

3.7. Moment Generating Function

The moments of a distribution are derived using the moment generating function, if one exists. The moment generating function is defined by

$$M(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r E(X^r)}{r!} = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'}. \tag{3.11}$$

Substituting equation (3.2) into equation (3.11), we obtain the moment generating function of the HMBXII distribution. Hence, the moment generating function of the HMBXII distribution is given by

$$\begin{aligned} M(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{r=0}^{\infty} \Psi_{ij} \frac{t^r}{r!} \Bigg[\alpha(1-\theta)\upsilon\mathcal{B}\left(\frac{r}{\zeta} + 1, \upsilon\left(\alpha(j+1) - j\right) - \frac{r}{\zeta} \right) \\ + \theta\upsilon\mathcal{B}\left(\frac{r}{\zeta} + 1, \upsilon\left(\alpha(j+2) - (j+1)\right) - \frac{r}{\zeta} \right) \Bigg]. \end{aligned} \tag{3.12}$$

3.8. Entropy

Entropy can be used to calculate the random variable's variation or uncertainty. Less uncertainty is implied by a lower entropy, and vice versa. By definition, the Rényi entropy is given by

$$I_R(\lambda) = \frac{1}{1-\lambda} \log \int_0^{\infty} f^{\lambda}(x) \mathrm{d}x, \lambda \neq 1. \tag{3.13}$$

Rearranging and raising the power of the PDF of HMBXII to λ, we obtain

$$f^{\lambda}(x) = \frac{(\zeta\upsilon)^{\lambda} x^{\lambda(\zeta-1)} \left(1 + x^{\zeta}\right)^{-\lambda(\alpha\upsilon+1)} (\alpha(1-\theta))^{\lambda} \left(1 + \frac{\theta\left(1+x^{\zeta}\right)^{-\upsilon(\alpha-1)}}{\alpha(1-\theta)}\right)^{\lambda}}{\left[1 - \theta\left(1 - (1 + x^{\zeta})^{-\upsilon(\alpha-1)}\right)\right]^{2\lambda}}.$$

Using the series expansion and simplifying, we obtain

$$f^{\lambda}(x) = (\zeta\upsilon)^{\lambda} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{l=0}^{\infty} \psi^{*}_{ijl} x^{\lambda(\zeta-1)} \left(1 + x^{\zeta}\right)^{-\upsilon(\alpha\lambda+(\alpha-1)(j-l))}, \tag{3.14}$$

where $\psi^{*}_{ijl} = (-1)^{j} \binom{2\lambda+i-1}{i} \binom{i}{j} \binom{l}{\lambda} \theta^{i+l} (\alpha(1-\theta))^{\lambda-l}$.
Letting $u = x^{\zeta}$, which implies $x = u^{1/\zeta}$ and $dx = \frac{1}{c} u^{1/\zeta-1} du$, we obtain

$$\int_{0}^{\infty} f^{\lambda}(x)\mathrm{d}x = \frac{(\zeta\upsilon)^{\lambda}}{\zeta} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{l=0}^{\infty} \psi^{*}_{ijl} u^{\lambda-(\frac{\lambda-1}{\zeta})-1} \left(1 + u\right)^{-\upsilon(\alpha\lambda+(\alpha-1)(j-l))} \mathrm{d}u.$$

But $\mathcal{B}(a, b) = \int_{0}^{\infty} u^{a-1}(1 + u)^{-(a+b)}\mathrm{d}u$, a¿0, b¿0.
We obtain the Rényi entropy of the HMBXII distribution given as

$$I_R(\lambda) = \frac{(\zeta\upsilon)^{\lambda}}{\zeta(1-\lambda)} \log \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{l=0}^{\infty} \psi^{*}_{ijl} \mathcal{B}\left(\upsilon\left(\alpha\lambda + (\alpha-1)(j-l)\right) - \frac{(\lambda-1)}{\zeta}, \lambda - \frac{(\lambda-1)}{c}\right), \lambda \neq 1. \tag{3.15}$$

3.9. Stress-Strength Reliability

The stress-strength reliability can be used to judge how well a system is able to withstand stress. By definition, the stress-strength reliability is given as

$$R_{ss} = \int_{0}^{\infty} f(x) \cdot F(x)\mathrm{d}x = 1 - \int_{0}^{\infty} f(x) \cdot S(x)\mathrm{d}x. \tag{3.16}$$

Finding the product of the density and survival functions of the HMBXII distribution, we obtain

$$f(x) \cdot S(x) = \frac{\alpha(1-\theta)\zeta\upsilon x^{\zeta-1}\left(1 + x^{\zeta}\right)^{-2\alpha\upsilon-1} + \theta\zeta\upsilon x^{\zeta-1}\left(1 + x^{\zeta}\right)^{-\upsilon(3\alpha-1)-1}}{\left[1 - \theta\left(1 - (1 + x^{\zeta})^{-\upsilon(\alpha-1)}\right)\right]^{3}}. \tag{3.17}$$

Simplifying equation(3.17) using the series expansion, we obtain

$$\begin{aligned} f(x) \cdot S(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \delta_{ij} \Big[& \alpha(1-\theta)\zeta\upsilon x^{\zeta-1}\left(1 + x^{\zeta}\right)^{-\upsilon(\alpha(j+2)-j)-1} \\ & + \theta\zeta\upsilon x^{\zeta-1}\left(1 + x^{\zeta}\right)^{-\upsilon(\alpha(j+3)-(j+1))-1} \Big]. \end{aligned} \tag{3.18}$$

Letting $u = x^{\zeta}$, which implies $x = u^{1/\zeta}$ and $dx = \frac{1}{\zeta} u^{1/\zeta-1} du$, we obtain

$$\begin{aligned} R_{ss} = 1 - \Bigg[\sum_{i=0}^{\infty} \sum_{j=0}^{i} \delta_{ij} \upsilon \int_{0}^{\infty} \Big[& \alpha(1-\theta)\left(1 + u\right)^{-\upsilon(\alpha(j+2)-j)-1} \\ & + \theta\left(1 + u\right)^{-\upsilon(\alpha(j+3)-(j+1))-1} \Big] \mathrm{d}u \Bigg]. \end{aligned}$$

Simplifying further we obtain,

$$R_{ss} = 1 - \left[\sum_{i=0}^{\infty}\sum_{j=0}^{i}\delta_{ij}\left(\frac{\alpha(1-\theta)}{(\alpha(j+2)-j)} + \frac{\theta}{(\alpha(j+3)-(j+1))}\right)\right],$$

where $\delta_{ij} = (-1)^j\binom{i+2}{2}\binom{i}{j}\theta^i$.

4. Estimation of Parameters

In this section, the estimates of the parameters of the HMBXII distribution are obtained using five estimation methods, thus the maximum likelihood estimation(MLE), the ordinary least squares method (OLS), the weighted least squares method (WLS), the Cramér-von Mises estimation (CVM) and the Anderson-Darling estimation (ADE).

4.1. Maximum Likelihood Estimation

The MLE is used to obtain the estimates of the parameters through the maximisation of the likelihood function. The likelihood function of the HMBXII distribution is given as

$$L(x, \alpha, \theta, \zeta, \upsilon) = \prod_{k=1}^{n} f(x_k, \alpha, \theta, \zeta, \upsilon). \tag{4.1}$$

The log-likelihood function is obtained by substituting equation (2.6) into (4.1) and thereafter taking the logarithm of the equation obtained. Hence,

$$\begin{aligned} l(x_k, \alpha, \theta, \zeta, \upsilon) = {} & n\ln(\zeta\upsilon) + (\zeta-1)\sum_{k=1}^{n}\ln x_k + \sum_{k=1}^{n}\ln\left[\alpha(1-\theta) + \theta\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}\right] \\ & - 2\sum_{k=1}^{n}\ln\left[1-\theta\left(1-\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}\right)\right]. \end{aligned} \tag{4.2}$$

We derive the MLE of the parameters by differentiating equation (4.2) with respect to parameters and equating the functions obtained to zero. The functions obtained thereafter are

$$\frac{\partial l}{\partial \theta} = \sum_{k=0}^{n}\frac{\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)-\alpha}}{\alpha(1-\theta)+\theta\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}} - \sum_{k=0}^{n}\frac{2\left(-1+\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}\right)}{\left[1-\theta+\theta\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}\right]},$$

$$\begin{aligned} \frac{\partial l}{\partial \alpha} = \sum_{k=0}^{n}\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}\Bigg[& \frac{2\theta\upsilon\ln(1+x_k^{\zeta})}{1-\theta+\theta\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}} \\ & - \frac{(\theta-1)\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)} + \theta\upsilon\ln(1+x_k^{\zeta})}{\alpha(1-\theta)+\theta\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}}\Bigg], \end{aligned}$$

$$\frac{\partial l}{\partial \zeta} = \frac{n}{\zeta} + \sum_{k=0}^{n} \ln x_k + \sum_{k=0}^{n} \upsilon\theta(\alpha - 1) x_k^{\zeta} \ln x_k \left(1 + x_k^{\zeta}\right)^{-\upsilon(\alpha-1)-1} \times$$
$$\left[\frac{2}{1-\theta+\theta\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}} - \frac{1}{\alpha(1-\theta)+\theta\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}}\right]$$

and

$$\frac{\partial l}{\partial \upsilon} = \frac{n}{\upsilon} + \sum_{k=0}^{n} \theta(\alpha - 1) \ln(1 + x_k^{\zeta}) \left(1 + x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}$$
$$\left[\frac{2}{1-\theta+\theta\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}} - \frac{1}{\alpha(1-\theta)+\theta\left(1+x_k^{\zeta}\right)^{-\upsilon(\alpha-1)}}\right].$$

To determine the parameters' maximum likelihood estimates, we equal these functions to zero and simultaneously solve them using numerical techniques.

4.2. Ordinary Least Squares

The OLS estimates of the parameters of the HMBXII distribution, where $X_{(1)}, X_{(2)}, ...X_{(n)}$ are the order statistics of the random sample of size n, are obtained through the minimisation of the function

$$LS(\alpha, \theta, \zeta, \upsilon) = \sum_{k=1}^{n} \left\{ (F(X_{(k)})) - \frac{k}{n+1} \right\}^2 . \tag{4.3}$$

4.3. Weighted Least Squares

The WLS estimates of the parameters of the HMBXII distribution, where $X_{(1)}, X_{(2)}, ...X_{(n)}$ are the order statistics of the random sample of size n, are obtained through the minimisation of the function

$$WLS(\alpha, \theta, \zeta, \upsilon) = \sum_{k=1}^{n} \frac{(n+1)^2(n+2)}{k(n-k+1)} \left\{ (F(X_{(k)})) - \frac{k}{n+1} \right\}^2 . \tag{4.4}$$

4.4. Cramér-Von Mises Estimation

The CVM estimates of the parameters of the HMBXII distribution, where $X_{(1)}, X_{(2)}, ...X_{(n)}$ are the order statistics of the random sample of size n, are obtained through the minimisation of the function

$$CME(\alpha, \theta, \zeta, \upsilon) = \frac{1}{12n} + \sum_{k=1}^{n} \left\{ (F(X_{(k)})) - \frac{2k-1}{2n} \right\}^2 . \tag{4.5}$$

4.5. Anderson-Darling Estimation

The ADE estimates of the parameters of the HMBXII distribution, where $X_{(1)}, X_{(2)}, ...X_{(n)}$ are the order statistics of the random sample of size n, are obtained through the minimisation of the function

$$ADE(\alpha, \theta, \zeta, \upsilon) = -n - \frac{1}{n} \sum_{k=1}^{n} (2k-1) \cdot \left\{ (\log F(X_{(k)})) + \log\left(1 - F(X_{(n+1-k)})\right) \right\} . \tag{4.6}$$

5. Monte Carlo Simulations

In this section, an assessment is made through a simulation study to ascertain the capabilities of the estimators of the HMBXII. Three sets of parameter values are used. We generate 1000 samples for each sample size $n = 30, 80, 200, 500, 1000$. We compute the absolute average biases (AB) and their corresponding mean square errors (MSE) for the MLE, OLS, WLS, CVM and ADE using the relations;

$$AB = \frac{1}{1000}\sum_{i=1}^{1000}|\hat{V}_i - V|,$$

and

$$MSE = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{V}_i - V\right)^2.$$

In this study, the estimates are obtained numerically using R (optim).
Tables 2, 3 and 4 show the ABs and MSEs of the estimators for $(\alpha, \theta, \zeta, \upsilon) = (0.50, 0.20, 2.60, 1.20)$, $(\alpha, \theta, \zeta, \upsilon) = (0.90, 0.50, 2.60, 1.02)$ and $(\alpha, \theta, \zeta, \upsilon) = (0.45, 0.30, 2.05, 1.20)$ respectively. The ABs and MSEs of the estimators for the various parameters tend to decrease as the sample sizes are increased despite some fluctuations. The MLE estimators produced the lowest ABs and MSEs and thus could be considered the best estimator.

Table 2. Results of Simulations for $(\alpha, \theta, \zeta, \upsilon) = (0.50, 0.20, 2.60, 1.20)$

Parameter	N	AB					MSE				
		MLE	OLS	WLS	CVM	ADE	MLE	OLS	WLS	CVM	ADE
α	30	0.2095	0.3210	0.1973	0.3039	0.2254	0.1193	0.3842	0.1157	0.3334	0.1568
	80	0.0837	0.1185	0.0970	0.0963	0.0720	0.1374	0.2800	0.1928	0.2141	0.1146
	200	0.0173	0.0354	0.0346	0.0387	0.0442	0.0562	0.1565	0.1487	0.1836	0.2660
	500	0.0058	0.0118	0.0145	0.0116	0.0213	0.0394	0.1129	0.1964	0.1435	0.3456
	1000	0.0031	0.0060	0.0071	0.0061	0.0114	0.0473	0.1279	0.1872	0.1408	0.3778
θ	30	0.1049	0.1138	0.1917	0.1312	0.0982	0.0282	0.0327	0.1171	0.0538	0.0322
	80	0.0474	0.0407	0.0594	0.0387	0.0593	0.0446	0.0361	0.1100	0.0327	0.0977
	200	0.0192	0.0174	0.0140	0.0097	0.0274	0.0508	0.0392	0.0436	0.0163	0.1183
	500	0.0067	0.0068	0.0066	0.0070	0.0135	0.0351	0.0489	0.0579	0.0522	0.1461
	1000	0.0022	0.0031	0.0039	0.0038	0.0053	0.0280	0.0311	0.0775	0.0638	0.1117
ζ	30	0.6061	2.5971	2.5128	2.5805	2.5668	0.5243	6.7452	6.3828	6.6663	6.5950
	80	0.2929	2.5662	2.5650	2.5744	2.5588	0.1264	6.5958	6.6027	6.6398	6.5543
	200	0.2449	2.5906	2.5777	2.5942	2.5999	0.0800	6.7126	6.6533	6.7301	6.7592
	500	0.1202	2.5892	2.5664	2.5968	2.5911	0.0239	6.7054	6.6078	6.7435	6.7147
	1000	0.0975	2.5981	2.5398	2.5995	2.5968	0.0179	6.7502	6.4968	6.7572	6.7434
υ	30	1.5589	0.3573	0.2025	0.3194	0.1875	1.5863	2.1662	2.0779	2.1360	2.0263
	80	0.7502	0.2424	0.1754	0.2555	0.1475	1.3973	2.3678	2.1691	2.1124	2.0728
	200	0.4360	0.2324	0.2308	0.1947	0.2007	1.3507	1.9656	2.0370	2.1451	2.3965
	500	0.3919	0.2024	0.2766	0.3416	0.2118	1.3666	1.7739	2.0492	1.5906	2.6104
	1000	0.1257	0.3361	0.2325	0.1993	0.2189	1.6405	1.7688	2.0521	1.9289	2.5952

Table 3. Results of Simulations for $(\alpha, \theta, \zeta, \upsilon) = (0.90, 0.50, 2.60, 1.02)$

Parameter	N	AB					MSE				
		MLE	OLS	WLS	CVM	ADE	MLE	OLS	WLS	CVM	ADE
α	30	0.3864	0.1206	0.1146	0.2967	0.2300	0.4450	0.0599	0.0471	1.3988	0.2195
	80	0.1226	0.0441	0.0470	0.0622	0.0340	0.3550	0.0456	0.0531	0.1649	0.0343
	200	0.0468	0.0206	0.0427	0.0246	0.0271	0.3151	0.0922	0.6318	0.1157	0.2055
	500	0.0160	0.0203	0.0133	0.0158	0.0167	0.1979	0.6006	0.2604	0.7425	0.3018
	1000	0.0037	0.0049	0.0047	0.0084	0.0056	0.0970	0.1725	0.1007	0.6187	0.2482
θ	30	0.1827	0.1280	0.0969	0.1391	0.1454	0.1199	0.0608	0.0311	0.0697	0.0634
	80	0.0440	0.0396	0.0525	0.0506	0.0563	0.0653	0.0405	0.0616	0.0581	0.0715
	200	0.0165	0.0196	0.0205	0.0220	0.0221	0.0573	0.0469	0.0631	0.0615	0.0713
	500	0.0137	0.0089	0.0096	0.0092	0.0083	0.1579	0.0643	0.0792	0.0728	0.0652
	1000	0.0038	0.0039	0.0052	0.0044	0.0047	0.0625	0.0658	0.0870	0.0740	0.0815
ζ	30	0.6399	2.5304	2.3095	2.5051	2.5391	0.7009	6.4324	5.6597	6.3179	6.4772
	80	0.3121	2.6000	2.5844	2.5993	2.5745	0.1874	6.7600	6.6838	6.7562	6.6340
	200	0.1854	2.5987	2.5359	2.6000	2.6000	0.0619	6.7531	6.4688	6.7600	6.7600
	500	0.0866	2.5875	2.5461	2.5999	2.5938	0.0103	6.6959	6.5373	6.7593	6.7281
	1000	0.1151	2.5985	2.6000	2.5988	2.5955	0.0200	6.7524	6.7600	6.7535	6.7365
υ	30	2.1154	0.4273	0.0746	0.0546	0.1623	6.0270	4.1687	2.4208	2.6104	2.4577
	80	0.8363	0.0510	0.0433	0.1000	0.0574	1.6054	2.5507	2.6102	2.4850	2.5018
	200	0.5378	0.0705	0.0799	0.0909	0.0793	1.6920	2.5944	2.4326	2.5093	2.5998
	500	0.7724	0.1394	0.0815	0.0801	0.1187	1.1552	2.9596	2.5572	2.6029	2.3718
	1000	0.2979	0.0628	0.0541	0.1396	0.1190	2.0218	2.5780	2.4886	2.6408	2.3721

Table 4. Results of Simulations for $(\alpha, \theta, \zeta, \upsilon) = (0.45, 0.30, 2.05, 1.20)$

Parameter	N	AB					MSE				
		MLE	OLS	WLS	CVM	ADE	MLE	OLS	WLS	CVM	ADE
α	30	0.1885	0.4023	0.3328	0.4181	0.3547	0.0941	0.4076	0.3696	0.4413	0.3635
	80	0.0595	0.1333	0.1310	0.1393	0.1087	0.0768	0.3305	0.3364	0.3501	0.2230
	200	0.0168	0.0548	0.0513	0.0522	0.0508	0.0454	0.3529	0.2987	0.3146	0.3082
	500	0.0059	0.0229	0.0196	0.0215	0.0262	0.0408	0.4820	0.2846	0.3351	0.4547
	1000	0.0031	0.0125	0.0122	0.0102	0.0136	0.0530	0.4306	0.4547	0.3176	0.4642
θ	30	0.1609	0.1187	0.1387	0.1241	0.1233	0.0629	0.0502	0.0751	0.0456	0.0563
	80	0.0677	0.0426	0.0465	0.0475	0.0582	0.0775	0.0410	0.0549	0.0451	0.0929
	200	0.0216	0.0225	0.0181	0.0120	0.0190	0.0602	0.0610	0.0547	0.0485	0.0713
	500	0.0092	0.0085	0.0084	0.0072	0.0048	0.0635	0.0682	0.0652	0.0601	0.0308
	1000	0.0026	0.0040	0.0038	0.0046	0.0029	0.0421	0.0677	0.0525	0.0821	0.0363
ζ	30	0.5270	2.0274	1.9810	2.0220	2.0053	0.4315	4.1201	3.9772	4.0987	4.0438
	80	0.2356	2.0500	1.9693	2.0490	2.0466	0.1053	4.2025	3.9146	4.1984	4.1889
	200	0.1421	2.0433	2.0161	2.0497	2.0322	0.0346	4.1759	4.0736	4.2011	4.1324
	500	0.1162	2.0453	2.0088	2.0500	2.0480	0.0238	4.1835	4.0508	4.2024	4.1943
	1000	0.1013	2.0465	2.0153	2.0495	2.0457	0.0143	4.1883	4.0714	4.2006	4.1850
υ	30	2.3595	0.2819	0.2415	0.2515	0.2697	6.7571	1.1769	0.9350	1.2204	1.0038
	80	1.0508	0.2318	0.2163	0.2175	0.1544	0.8475	1.052	1.0525	1.1285	1.0085
	200	0.4050	0.2138	0.1908	0.2273	0.2068	0.3997	1.1003	1.0776	1.0416	1.1099
	500	0.4281	0.3246	0.2487	0.2577	0.2714	0.3679	1.1316	1.0712	1.0932	1.2624
	1000	0.2975	0.3252	0.2545	0.2458	0.2824	0.5682	1.2391	1.2135	1.1125	1.2843

6. Applications

In this section, we ascertain the empirical importance of the HMBXII distribution by using three complete real datasets.

The taxes revenues data was analyzed by [20]. The dataset are 3.7, 3.11, 4.42, 3.28, 3.75, 2.96,3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53,2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67,4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93,3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22,2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

The failure times of kevlar 49/epoxy strands data was analysed by [21]. The dataset are 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

The precipitation (in inches) in Minneapolis/St Paul data was analysed by [22]. The dataset are 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Using the bbmle package in R, we estimated the MLE estimates of the parameters. Also, using the GenSA package in R, the initial values of the parameters of the fitted models used for the optimisation are obtained.

The performance of the HMBXII distribution is compared with nine (9) modifications of the Burr XII distribution which include the Marshall-Olkin exponentiated Burr XII distribution (MOEBXII) [2], the exponentiated Burr XII Poisson distribution (EBXIIP) [22], the Marshall-Olkin generalised Burr XII distribution (MOGBXII) [3], the Weibull Burr XII distribution (WBXII) [6], the Kumaraswamy exponentiated Burr XII distribution (KEBXII) [5], the Kumaraswamy Burr XII distribution (KWBXII) [7], the study on an extension to Lindley distribution [9] the exponentiated Exponential Burr XII distribution (EEBXII) [10], the Gompertz-modified Burr XII distribution (GMBXII) [13] and the odd exponentiated half-logistic Burr XII distribution (OEHLBXII)[14].

The Kolmogorov–Smirnov(K-S), Anderson-Darling(AD) and Cramér-von Mises (CVM) tests were used to evaluate the goodness-of-fit of the distributions fitted to the data. The most appropriate model that fits the dataset will be adjudged using the distribution that produces the least Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). That model should as well produce the least K-S, AD and CVM test statistic.

6.1. *Taxes Revenues*

The taxes revenues dataset is positively skewed (0.3985) and is less peak than the normal distribution curve thus platykurtic (0.2492) as shown in Table 5.

The MLEs for the fitted models and their corresponding standard errors are displayed in Table 6.

The newly proposed HMBXII model provides a better fit as shown in Table 7, where * means

Table 5. Descriptive statistics of taxes revenues dataset

Minimum	Maximum	Mean	Median	CV	CS	CK
0.3900	5.5600	2.6110	2.6750	0.3861	0.3985	0.2492

significant at 5% significance level. The HMBXII model had the greatest log-likelihood value as well as the least values for the AIC, CAIC and BIC. Furthermore, the HMBXII model had the lowest AD, K-S and CVM values.

To give further validation to the results obtained, the fitted PDFs and CDFs of the models compared are respectively presented in Figures 3 and 4. The HMBXII model fits the taxes revenues dataset better.

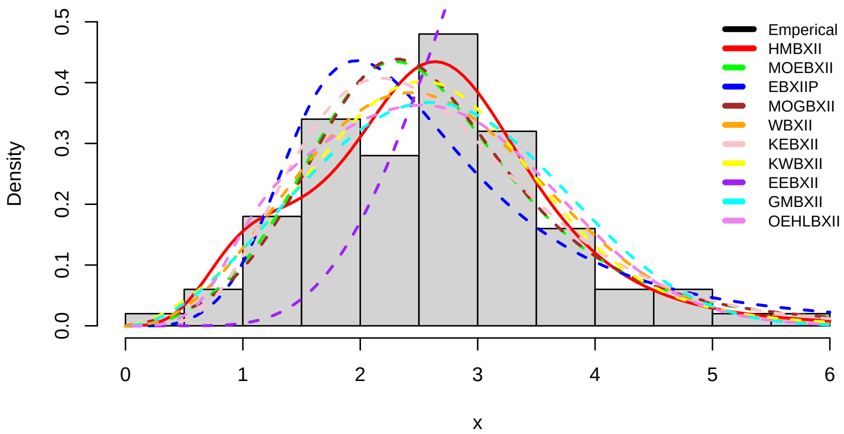

Figure 3. The fitted PDFs for taxes revenues dataset

In Figure 5, the profile log-likelihood plots for the estimated parameter values of the HMBXII distribution for the taxes revenues data set are shown. The plots show that the estimated values correspond to the true maxima.

6.2. Failure times of kevlar 49/epoxy strands

The failure times of kevlar 49/epoxy strands dataset is positively skewed (1.1447) and is less peak than the normal distribution curve thus platykurtic (1.6653) as shown in Table 8.

The MLEs for the fitted models and their corresponding standard errors are displayed in Table 9.

The newly proposed HMBXII model provides a better fit as shown in Table 10, where * means significant at 5% significance level.. The HMBXII model had the greatest log-likelihood value as well as the least values for the AIC, CAIC and BIC. Furthermore, the HMBXII model had the lowest AD, K-S and CVM values.

To give further validation to the results obtained, the fitted PDFs and CDFs of the models compared are respectively presented in Figures 6 and 7. The HMBXII model fits the failure times of kevlar 49/epoxy strands dataset better.

In Figure 8, the profile log-likelihood plots for the estimated parameter values of the HMBXII

Table 6. MLEs for taxes revenues dataset

Model	Parameters	Estimates	Standard Errors	Z-Statistic	P-Value
HMBXII	α	0.0481	0.0159	3.0307	0.0024*
	θ	0.0016	0.0024	0.6636	0.5069
	c	4.1246	1.2244	3.3686	0.0008*
	k	1.5220	0.4241	3.5884	0.0003*
MOEBXII	α	60.0862	46.7044	1.2865	0.1983
	λ	4.2279	5.7793	0.7316	0.4644
	c	1.5179	0.6927	2.1911	0.0284 *
	k	3.4361	1.9796	1.7358	0.0826
EBXIIP	α	7.1759	6.3678	1.1269	0.2598
	θ	-4.8060	1.4890	-3.2276	0.0012 *
	c	1.4015	0.4865	2.8812	0.0040 *
	k	2.5949	1.1402	2.2758	0.0229 *
MOGBXII	α	2.1372	0.6966	3.0681	0.0022 *
	β	0.3726	0.1585	2.3511	0.0187 *
	δ	114.4625	97.2701	1.1767	0.2393
	a	6.0629	2.5787	2.3511	0.0187 *
WBXII	α	1.5004	0.7911	1.8966	0.0579
	β	0.6418	0.2379	2.6975	0.0070 *
	a	0.1656	0.1753	0.9447	0.3448
	b	2.3511	0.9867	2.3829	0.0172 *
KEBXII	a	1.4068	0.3096	4.5433	5.5390×10^{-6}*
	β	10.3220	2.2719	4.5433	5.5390×10^{-6}*
	b	24.3381	10.9175	2.2293	0.0258 *
	c	0.7261	0.2635	2.7550	0.0059 *
	k	1.4149	0.7555	1.8728	0.0611
KWBXII	a	0.5328	0.2340	2.2771	0.0228 *
	b	0.8207	0.6270	1.3089	0.1906
	c	4.8674	1.5509	3.1384	0.0017 *
	k	2.2539	1.7097	1.3184	0.1874
	s	3.5320	0.6912	5.1098	3.2240×10^{-7}*
EEBXII	a	19.7046	11.0208	1.7879	0.0738
	b	178.1639	0.0317	5619.2433	2.2000×10^{-16}*
	c	0.4794	0.1517	3.1603	0.0016 *
	k	1.5022	0.6502	2.3103	0.0209 *
GMBXII	λ	0.1534	0.3612	0.4246	0.6711
	θ	0.8417	0.2649	3.1780	0.0015 *
	c	2.8218	0.7607	3.7097	0.0002*
	d	0.0430	0.0107	4.0070	6.1490×10^{-5}*
OEHLBXII	a	6.0863	3.6184	1.6820	0.0926
	α	0.7758	0.2557	3.0340	0.0024*
	b	0.3994	0.1860	2.1470	0.0318*
	λ	0.0993	0.0940	1.0565	0.2907

Table 7. Log-likelihood and comparison criteria for taxes revenues dataset

Model	ℓ	AIC	CAIC	BIC	K-S	AD	CVM
HMBXII	-140.7242	289.4483	289.8694	299.8690	0.0491	0.3041	0.0471
MOEBXII	-143.0858	294.1716	294.5927	304.5923	0.0816	0.7654	0.1343
EBXIIP	-149.7781	307.5561	307.9772	317.9768	0.1190	1.7729	0.2993
MOGBXII	-143.2508	294.5016	294.9226	304.9223	0.0772	0.7810	0.1303
WBXII	-140.7915	289.5830	290.0040	300.0037	0.0619	0.4455	0.0748
KEBXII	-143.3410	296.6819	297.3202	309.7078	0.0901	0.7952	0.1448
KWBXII	-140.5348	291.0695	291.7078	304.0954	0.0572	0.3577	0.0613
EEBXII	-141.6000	291.2001	291.6211	301.6207	0.0883	0.6362	0.1291
GMBXII	-141.4459	290.8917	291.3128	301.3124	0.0843	0.6907	0.1094
OEHLBXII	-141.3246	290.6492	291.0703	301.0699	0.0641	0.4929	0.0783

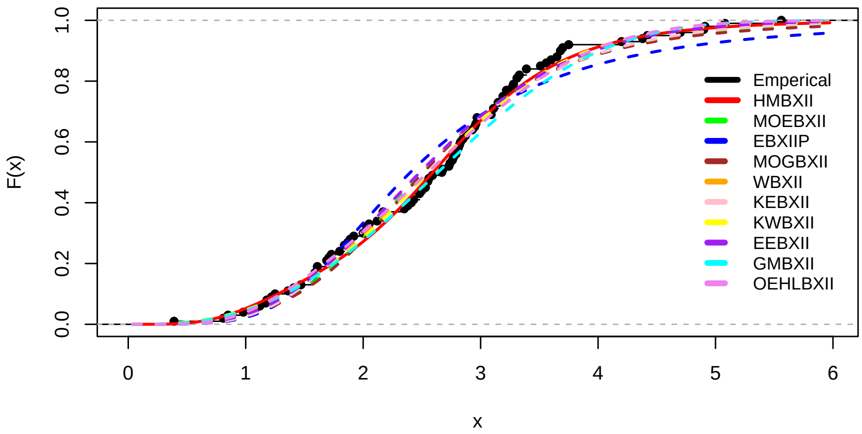

Figure 4. The fitted CDFs for taxes revenues dataset

Table 8. Descriptive statistics of failure times of kevlar 49/epoxy strands dataset

Minimum	Maximum	Mean	Median	CV	CS	CK
0.3200	4.7500	1.6750	1.4700	0.5974	1.1447	1.6653

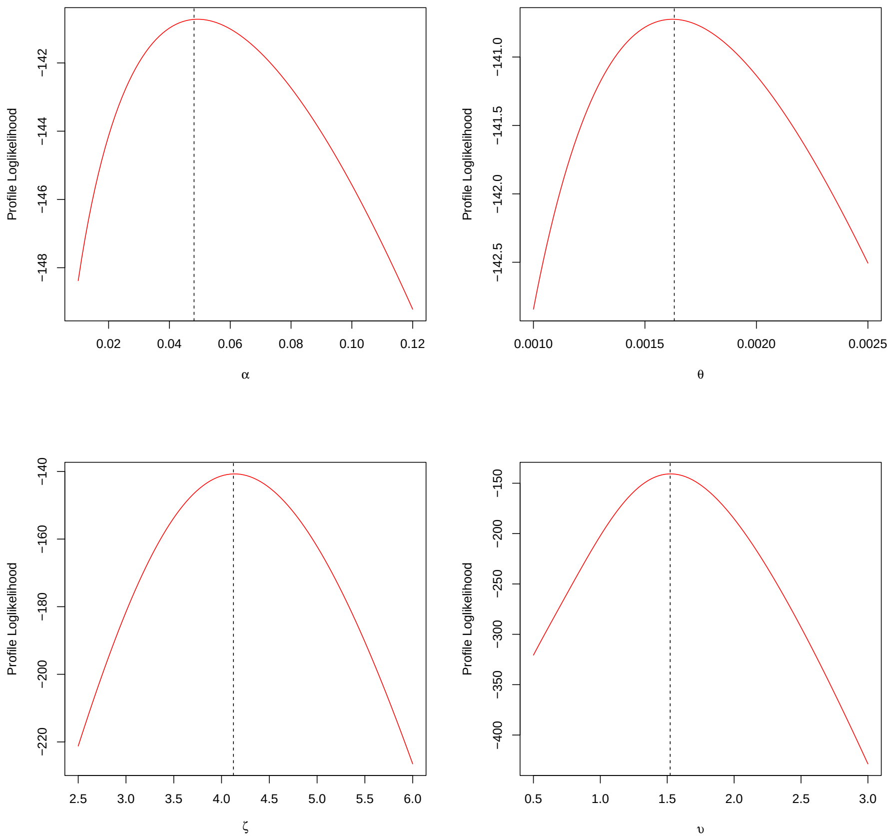

Figure 5. Profile log-likelihood plots for estimated parameters of HMBXII for taxes revenues

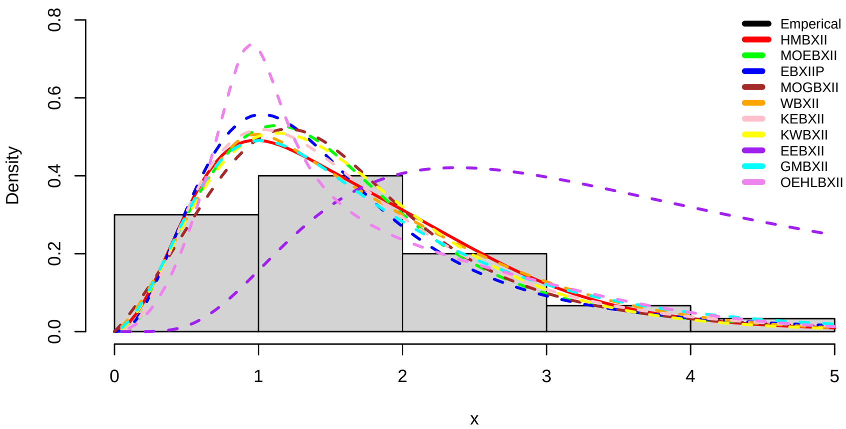

Figure 6. The fitted PDFs for failure times of kevlar 49/epoxy strands dataset

Table 9. MLEs for failure times of kevlar 49/epoxy strands dataset

Model	Parameters	Estimates	Standard Errors	Z-Statistic	P-Value
HMBXII	α	0.2153	0.0932	2.3104	0.0209 *
	θ	0.0138	0.0492	0.2803	0.7792
	ζ	2.7434	0.8156	3.3637	0.0008*
	υ	2.0127	0.9895	2.0341	0.0419 *
MOEBXII	α	8.4755	10.7738	0.7867	0.4315
	λ	11.8805	20.3697	0.5832	0.5597
	c	0.8339	0.4231	1.9708	0.0487 *
	k	5.4476	3.3016	1.6500	0.0989
EBXIIP	α	3.2341	2.8006	1.1548	0.2482
	θ	-2.2210	2.4519	-0.9058	0.3650
	c	1.3024	0.6100	2.1351	0.0328 *
	k	2.5781	1.5266	1.6887	0.09127
MOGBXII	α	2.1205	1.0414	2.0362	0.0417 *
	β	2.9777	0.0793	37.5387	2.0000×10^{-16}*
	δ	6.2105	8.6193	0.7205	0.4712
	a	0.5598	0.4209	1.3302	0.1835
WBXII	α	7.3965	6.6295	1.1157	0.2645
	β	0.6732	0.2726	2.4697	0.0135 *
	a	0.3856	0.1464	2.6346	0.0084 *
	b	0.3221	0.2919	1.1036	0.2698
KEBXII	a	1.3161	0.4666	2.8209	0.0048 *
	β	8.8604	3.1410	2.8209	0.0048 *
	b	10.3081	11.0407	0.9336	0.3505
	c	0.5475	0.5605	0.9768	0.3287
	k	1.9585	2.6041	0.7521	0.4520
KWBXII	a	3.0600	2.3867	1.2821	0.1998
	b	4.6300	1.4464	3.2010	0.0014 *
	c	0.9600	0.3687	2.6034	0.0092 *
	k	4.7500	0.8365	5.6784	1.3600×10^{-8}*
	s	9.2800	8.7099	1.0655	0.2867
EEBXII	a	6.5384	5.1732	0.2063	0.2063
	b	3.8550	3.0890	1.2480	0.2120
	c	0.8343	0.3642	2.2909	0.0220 *
	k	1.6775	0.9982	1.6806	0.0928
GMBXII	λ	0.2538	0.7446	0.3409	0.7332
	θ	0.2980	0.2587	1.1518	0.2494
	c	2.5256	0.7829	3.2258	0.0013 *
	d	0.3509	0.1949	1.8004	0.0718
OEHLBXII	a	11.6035	5.7152	2.0303	0.0423 *
	α	0.2672	0.1145	2.3338	0.0196 *
	b	0.1684	0.0599	2.8123	0.0049 *
	λ	0.1932	0.2563	0.7538	0.4510

Table 10. Log-likelihood and comparison criteria for failure times of kevlar 49/epoxy strands dataset

Model	ℓ	AIC	CAIC	BIC	K-S	AD	CVM
HMBXII	-38.1548	84.3095	85.9095	89.9143	0.0664	0.1024	0.0140
MOEBXII	-38.34203	84.6841	86.2841	90.2889	0.0769	0.1490	0.0222
EBXIIP	-38.7648	85.5295	87.1295	91.1343	0.0932	0.2183	0.0381
MOGBXII	-38.6445	85.2891	86.8891	90.8938	0.0686	0.1355	0.0188
WBXII	-38.2047	84.4094	86.0095	90.0143	0.0678	0.1111	0.0151
KEBXII	-38.2047	86.4095	88.9094	93.4155	0.0712	0.1278	0.0182
KWBXII	-38.1967	86.3934	88.8934	93.3994	0.0700	0.1442	0.0196
EEBXII	-38.5193	85.0386	86.6386	90.6434	0.0880	0.1703	0.0262
GMBXII	-38.1606	84.3212	85.9212	89.9260	0.0701	0.1529	0.0200
OEHLBXII	-39.3450	86.6900	88.2900	92.2948	0.1230	0.3444	0.0576

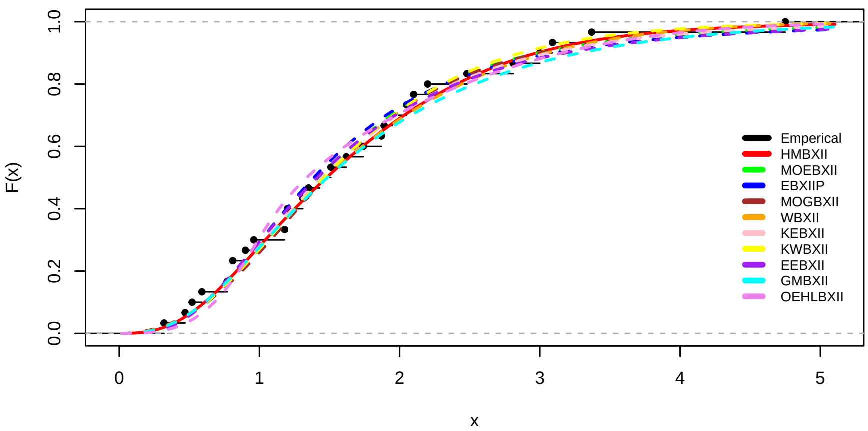

Figure 7. The fitted CDFs for failure times of kevlar 49/epoxy strands dataset

distribution for the failure times of kevlar 49/epoxy strands data set are shown. The plots show that the estimated values correspond to the true maxima.

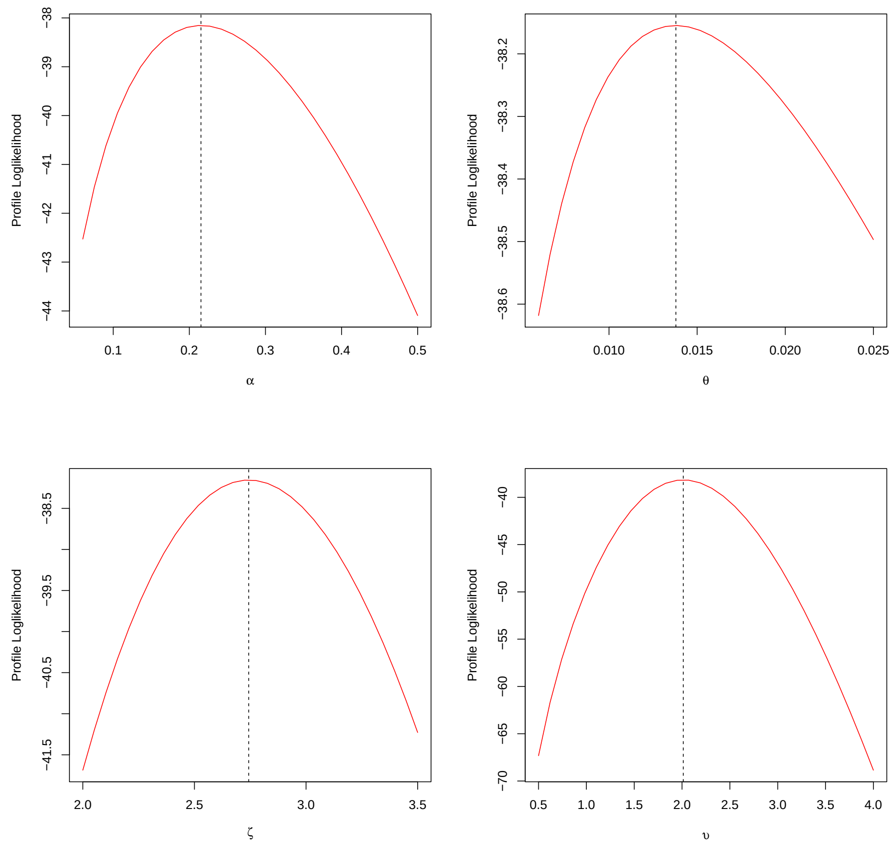

Figure 8. Profile log-likelihood plots for estimated parameters of HMBXII for failure times of kevlar 49/epoxy strands

6.3. Precipitation in Minneapolis/St Paul

The precipitation in Minneapolis/St Paul dataset is positively skewed (3.0471) and is more peak than the normal distribution curve thus leptokurtic (14.4745) as shown in Table 11.

Table 11. Descriptive statistics of precipitation in Minneapolis/St Paul dataset

Minimum	Maximum	Mean	Median	CV	CS	CK
0.0100	7.8900	1.0250	0.8000	1.0922	3.0471	14.4745

The MLEs for the fitted models and their corresponding standard errors are displayed in Table 12.

The newly proposed HMBXII model provides a better fit as shown in Table 13, where * means significant at 5% significance level. The HMBXII model had the greatest log-likelihood value as well as the least values for the AIC, CAIC and BIC. Furthermore, the HMBXII model had the lowest AD, K-S and CVM values.

Table 12. MLEs for precipitation in Minneapolis/St Paul dataset.

Model	Parameters	Estimates	Standard Errors	Z-Statistic	P-Value
HMBXII	α	0.1055	0.0386	2.7360	0.0062 *
	θ	0.0061	0.0081	0.7502	0.4531
	ζ	0.7341	0.1234	5.9469	2.7330×10^{-9} *
	υ	7.0748	1.7098	4.1378	3.5070×10^{-5} *
MOEBXII	α	3.3778	1.7764	1.9016	0.0572
	λ	0.2064	0.1113	1.8541	0.0637
	c	2.5851	1.0257	2.5204	0.0117 *
	k	0.8750	0.5031	1.7391	0.0820
EBXIIP	α	0.2129	0.0845	2.5106	0.0121 *
	θ	-1.5830	0.8542	-1.8533	0.0638
	c	2.9162	0.9253	3.1516	0.0016 *
	k	0.6235	0.3028	2.0588	0.0395 *
MOGBXII	α	0.7905	0.1671	4.7300	2.2450×10^{-6}*
	β	8.4341	0.0068	1235.0971	2.0000×10^{-16}*
	δ	7.5801	6.1041	1.2418	0.2143
	a	0.4541	0.1245	3.6457	0.0003 *
WBXII	α	3.6599	1.6549	2.2115	0.0270 *
	β	1.2229	0.2756	4.4367	9.1350×10^{-6}*
	a	0.9114	0.1480	6.1594	7.3030×10^{-10}*
	b	0.2299	0.1064	2.1608	0.0307 *
KEBXII	a	0.5289	0.0589	8.9870	2×10^{-16}*
	β	5.3352	0.5937	8.9870	2.0000×10^{-16}*
	b	9.3028	4.2286	2.2000	0.0278 *
	c	0.5175	0.2396	2.1598	0.0308 *
	k	0.8782	0.5629	1.5602	0.1187
KWBXII	a	0.1000	0.0378	2.6444	0.0082 *
	b	0.7000	0.4937	1.4177	0.1563
	c	5.8000	0.9572	6.0595	1.365×10^{-9}
	k	0.5600	0.5033	1.1126	0.2659
	s	1.5000	0.3766	3.9831	6.8030×10^{-5}*
EEBXII	a	0.1146	0.0615	1.8636	0.0624
	b	0.3255	0.0981	3.3194	0.0009 *
	c	4.5666	2.0305	2.2490	0.0245 *
	k	1.4252	1.0232	1.3929	0.1636
GMBXII	λ	49.2480	3.7881×10^{-5}	1.3001×10^{6}	2.2000×10^{-16}*
	θ	8.1410×10^{-4}	1.4677×10^{-3}	0.5547	0.5791
	c	0.1877	0.1580	1.1883	0.2347
	d	0.0177	2.2693×10^{-3}	7.8041	5.9950×10^{-15}*
OEHLBXII	a	2.9958	1.3454	2.2267	0.0260 *
	α	0.2603	0.1192	2.1834	0.0290 *
	b	0.2156	0.0900	2.3941	0.0167 *
	λ	2.0530	1.2029	1.7067	0.0879

Table 13. Log-likelihood and comparison criteria for precipitation in Minneapolis/St Paul dataset

Model	ℓ	AIC	CAIC	BIC	K-S	AD	CVM
HMBXII	-99.3723	206.7446	207.6537	217.2051	0.0547	0.3671	0.0450
MOEBXII	-102.2690	212.5380	214.1380	222.9984	0.0820	0.8305	0.1220
EBXIIP	-103.5217	215.0434	216.6434	225.5039	0.0902	1.2939	0.2317
MOGBXII	-103.7589	215.5178	217.1178	225.9782	0.0908	1.3305	0.1920
WBXII	102.6317	213.2634	214.8634	223.7239	0.0876	0.9187	0.1629
KEBXII	-107.9973	225.9946	228.4946	239.0703	0.1249	2.2530	0.3895
KWBXII	-99.2130	208.4261	210.9261	221.5017	0.0559	0.3873	0.0584
EEBXII	-101.1276	210.2552	211.8552	220.7157	0.0736	0.6479	0.0943
GMBXII	-106.5181	221.0363	222.6363	231.4967	0.1065	1.9254	0.2023
OEHLBXII	-101.8226	211.6453	213.2453	222.1058	0.0704	0.8439	0.1335

To give further validation to the results obtained, the fitted PDFs and CDFs of the models compared are respectively presented in Figures 9 and 10. The HMBXII model fits the precipitation in Minneapolis/St Paul dataset better.

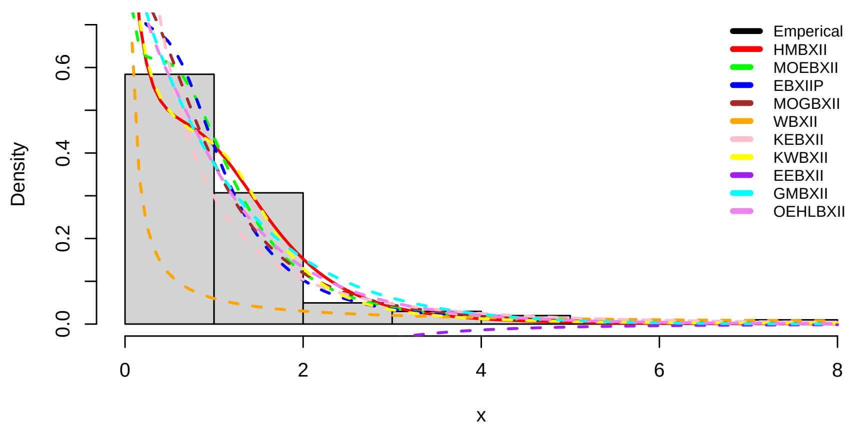

Figure 9. The fitted PDFs for precipitation in Minneapolis/St Paul dataset

In Figure 11, the profile log-likelihood plots for the estimated parameter values of the HMBXII distribution for the precipitation in Minneapolis/St Paul data set are shown. The plots show that the estimated values correspond to the true maxima.

7. The Log-Harmonic Mixture Burr XII Regression Model

If X is a random variable that follows the HMBXII distribution, we can take the log transform of X defined as $Y = \log(\tau X)$, where $\tau > 0$ to obtain a log-linear regression model. The density function of Y is obtained by redefining the parameters as $\zeta = 1/\sigma$ and $\tau = exp(\mu)$.

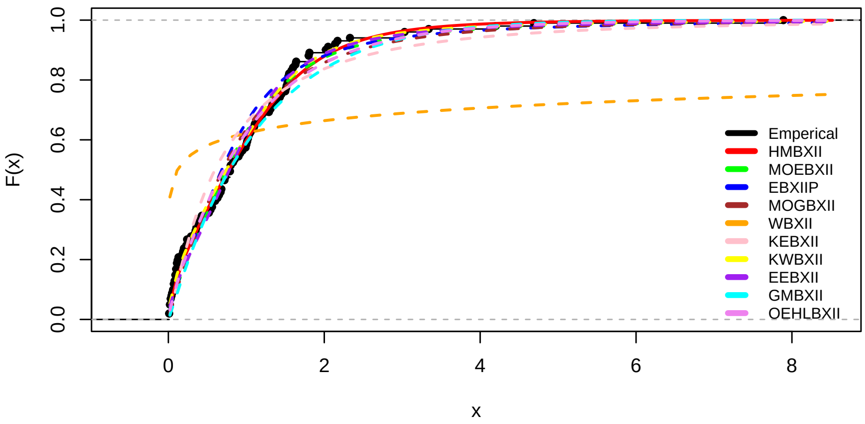

Figure 10. The fitted CDFs for precipitation in Minneapolis/St Paul dataset

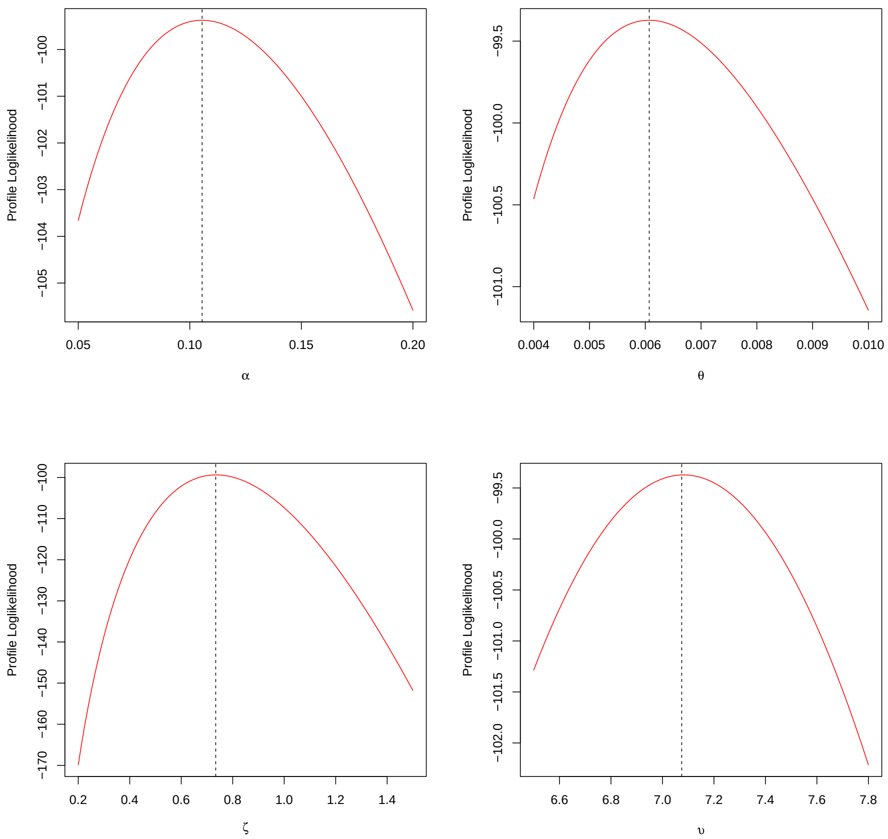

Figure 11. Profile log-likelihood plots for estimated parameters of HMBXII for precipitation in Minneapolis/St Paul

$$f(y)=\frac{\frac{\alpha\upsilon(1-\theta)}{\sigma}\exp(\frac{y-\mu}{\sigma})\left(1+\exp(\frac{y-\mu}{\sigma})\right)^{-\alpha\upsilon-1}+\frac{\theta\upsilon}{\sigma}\exp(\frac{y-\mu}{\sigma})\left(1+\exp(\frac{y-\mu}{\sigma})\right)^{-\upsilon(2\alpha-1)-1}}{\left[1-\theta\left(1-\left(1+\exp(\frac{y-\mu}{\sigma})\right)^{-\upsilon(\alpha-1)}\right)\right]^{2}}, \tag{7.1}$$

where $y > 0, \sigma > 0, \upsilon > 0, \alpha > 0, 0 < \theta < 1$ and $\mu \in \mathbb{R}$.
Equation (7.1) is the log-harmonic mixture Burr XII (LHMBXII) distribution, where μ is the location parameter and σ is the scale parameter.
Mathematically, if $X \sim HMBXII(\alpha, \theta, \zeta, \upsilon)$ then $Y = \log(\tau X) \sim LHMBXII(\alpha, \theta, \upsilon, \sigma, \mu)$.

The survival function of the LHMBXII is given as

$$S(y)=\frac{\left(1+\exp(\frac{y-\mu}{\sigma})\right)^{-\alpha\upsilon}}{\left[1-\theta\left(1-\left(1+\exp(\frac{y-\mu}{\sigma})\right)^{-\upsilon(\alpha-1)}\right)\right]}. \tag{7.2}$$

We propose a log-linear regression model with the response variable y_i and covariates $Z_i^T = (1, z_{i1}, ..., z_{ip})$, where 1 is the intercept term, given as

$$y_i = Z_i^T\beta + \sigma W_i,$$

where $i = 1, 2, ..., n$, $\beta = (\beta_1, \beta_2, ..\beta_p)^T$ are coefficients of the regression of the covariates, σ is a scale parameter, W_i is the random error and $\mu_i = Z_i^T\beta$ is the location parameter of y_i.

The log-likelihood for estimating the parameters $\Omega = (\alpha, \theta, \upsilon, \sigma, \beta^T)^T$ of the model can be expressed as

$$\begin{aligned} l(\Omega) &= n\left(\ln(\upsilon)-\ln(\sigma)\right)+\sum_{i=1}^{n}\frac{y_i-\mu_i}{\sigma} \\ &+\sum_{i=1}^{n}\ln\left[\alpha(1-\theta)+\theta\left(1+e^{\frac{y_i-\mu_i}{\sigma}}\right)^{-\upsilon(\alpha-1)}\right]-2\sum_{i=1}^{n}\ln\left[1-\theta\left(1-\left(1+e^{\frac{y_i-\mu_i}{\sigma}}\right)^{-\upsilon(\alpha-1)}\right)\right]. \end{aligned} \tag{7.3}$$

The maximum likelihood estimates of the regression model is obtained by maximising the likelihood function. To ascertain the adequacy of a model, the Cox-Snell residuals would be assessed and should be found to behave as a standard exponential distribution. The Cox-Snell residuals is the negative natural logarithm of the survival function. The model would then be assessed using the goodness-of-fit measures of the Cox Snell residuals which includes the Cramér-von Mises, Anderson Darling, and Kolmogorov-Smirnov.

7.1. Application of Log-Harmonic Mixture Burr XII Regression Model

The LHMBXII model was applied to a real data set used by [23] and compared to the log-Weibull Burr XII (LWBXII) distribution [6] and the log-Gumbel Burr XII (LGBXII) distribution [24]. The data set was retrieved from `http://www.leg.ufpr.br/doku.php/publications:papercompanions:multquasibeta` (accessed on 13 December, 2022). The response variable y_i

is the percentage of body fat in the arms while the covariates are: age (z_{i1}), body mass index (z_{i2}) and sex (z_{i3}, 0=female, 1=male). The fitted model can be expressed as

$$y_i = \beta_0 + \beta_1 z_{i1} + \beta_2 z_{i2} + \beta_3 z_{i3}.$$

The MLEs, standard errors and P-values of the fitted model are presented in Table 14. The LHMBXII model is the most appropriate model using the selection criteria of least AIC and BIC.
Using the parameter estimates of the LHMBXII model, we obtain

$$\hat{y}_i = -0.1021 + 0.0011 z_{i1} + 0.0156 z_{i2} - 0.1714 z_{i3}.$$

We can deduce that age and body mass index had a significant positive effect on the percentage of body fat in the arms while sex (female=0, male=1) showed a significant negative effect.

In order to evaluate the adequacy of the LHMBXII, LWBXII, and LGBXII models, the Cox-Snell residuals were created. As compared to the residuals of the LWBXII and LGBXII models, the residuals of the LHMBXII model can be seen to be closer to the diagonal line on the probability-probability (P-P) plot shown in Figure 12, demonstrating that the LHMBXII model gives a better fit to the data set.

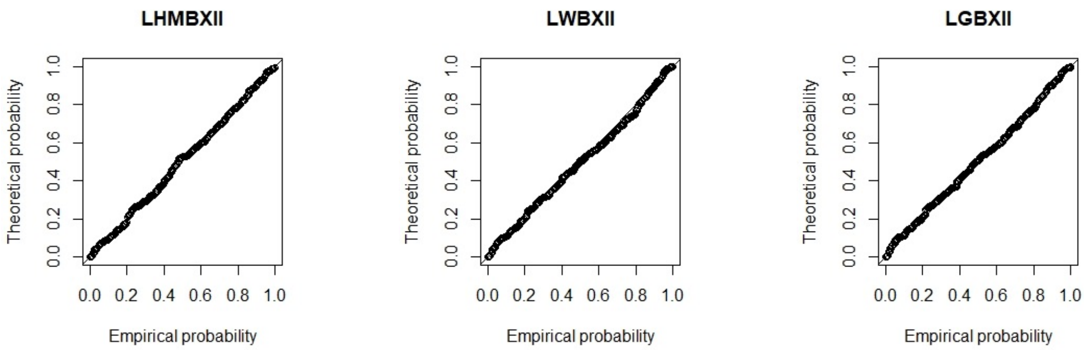

Figure 12. P-P plot of residuals

The diagnosis results shown in Table 15 reveals that the data set can be modelled better using the LHMBXII model.

8. Conclusion

In this study, we introduced a new four-parameter distribution called the Harmonic Mixture Burr XII distribution. Specific properties like non-central and incomplete moments, quantile functions, entropy, mean and median deviation, mean residual life, moment generating function, and stress-strength reliability were derived. The estimators assessed included the maximum likelihood estimation, ordinary least squares estimation, weighted least squares estimation, Cramer-von Mises estimation, and Anderson-Darling estimation. We illustrated the application of the proposed distribution using three sets of lifetime data and was evident that the proposed distribution fit the data sets better when compared to nine (9) other extensions of the Burr XII distribution. The new model is used to create a log-linear regression model known as the log-harmonic mixture Burr XII and applied to a real life data.

Table 14. Parameter estimates and goodness-of-fit statistics for regression models

Model	Parameters	Estimates	P-values	Goodness-of-fit
LHMBXII	β_0	-0.1021 (0.0242)	2.4910×10^{-5}	ℓ= 456.8401 AIC = -897.6803 BIC= -868.1035
	β_1	0.0011 (0.0002)	7.8270×10^{-11}	
	β_2	0.0156 (0.0011)	2.2000×10^{-16}	
	β_3	-0.1714 (0.0061)	2.2000×10^{-16}	
	θ	0.4084 (0.0365)	2.2000×10^{-16}	
	α	0.4323 (0.0423)	2.2000×10^{-16}	
	υ	0.9954 (0.1762)	1.6240×10^{-8}	
	σ	0.0275 (0.0027)	2.2000×10^{-16}	
LWBXII	β_0	-0.1980 (0.0262)	4.5800×10^{-14}	ℓ=453.0533 AIC= -890.1065 BIC= -860.5298
	β_1	0.0013 (0.0002)	4.5800×10^{-13}	
	β_2	0.0163 (0.0011)	2.2000×10^{-16}	
	β_3	-0.1681 (0.0063)	2.2000×10^{-16}	
	β	0.1359 (0.0325)	2.9220×10^{-5}	
	a	4.6862 (0.0005)	2.2000×10^{-16}	
	b	1.8952 (0.0027)	2.2000×10^{-16}	
	σ	0.0399 (0.0071)	2.3710×10^{-8}	
LGBXII	β_0	-0.0880 (0.0247)	4.0000×10^{-3}	ℓ=453.031 AIC= -890.0619 BIC= -860.4852
	β_1	0.0013 (0.0002)	7.7800×10^{-13}	
	β_2	0.0158 (0.0011)	2.2000×10^{-16}	
	β_3	-0.1705 (0.0063)	2.2000×10^{-16}	
	β	3.2727 (0.0003)	2.2000×10^{-16}	
	k	6.0574 (0.0003)	2.2000×10^{-16}	
	σ	2.2967 (0.0009)	2.2000×10^{-16}	
	τ	0.0537 (0.0024)	2.2000×10^{-16}	

Table 15. Goodness-of-fit statistics for residuals

Model	KS		CVM		ADE	
	Statistic	P-value	Statistic	P-value	Statistic	P-value
LHMBXII	0.0343	0.8738	0.0320	0.9694	0.2285	0.9806
LWBXII	0.0498	0.4506	0.0975	0.5974	0.7380	0.5278
LGBXII	0.0378	0.7874	0.0696	0.7546	0.6516	0.6004

Conflict of interest

The authors state that they have no financial or other conflicts of interest to disclose with connection to this research.

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Supplementary (if necessary)