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Research article

Bayesian and Non-Bayesian Estimation Methods for Simulating the Parameter of the Akshaya Distribution

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Abstract:
For modelling lifetime data from biological research and engineering, the "Akshaya distribution" is a model one-parameter continuous distribution that was proposed by [15]. The non-Bayesian and Bayesian estimation methods for the Akshaya's parameter are also presented in this study. The weighted least square estimation (WLSE), least square estimation (LSE), Cramer-von-Mises estimation (CVME), and maximum likelihood estimation (MLE), five traditional estimation approaches, are used to find the model parameter. The parameter of the suggested distribution was also determined using the squared error loss function and Bayesian estimating (BE) under independent gamma priors. Finally, a simulation study is used to expound on the applicability and value of the proposed distribution.

Keywords: Akshaya distribution; Bayesian procedure; maximum likelihood estimation; Anderson-Darling estimation; Cramer-von-Mises estimation least square estimation; Weighted least square estimation.
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1. Introduction

The statistical analysis and modelling of lifetime data are significant in many practical sciences, including those related to insurance, finance, biomedical, and engineering. As a result, many lifetime distributions have just recently been introduced. The features, estimation methods, and geological and medical applications of a new extended two-parameter distribution were proposed by Al-Kutubi et al. in their paper, [1]. Furthermore, Brooks and Steven [2] provided an in-depth overview of Markov chain Monte Carlo (MCMC) algorithms and looked at a number of implementation issues associated with MCMC methods.

In their 1953 life article, Epstein et al. [4] looked at statistical problems that occur when observations are made available in an organised way. Ghitany et al. [7] presented the statistical properties of a novel, two-parameter distribution that integrated the Weibull and generalised gamma distributions. Additionally, Ghitany et al. showed that the Lindley distribution is a better model than the exponential distribution in their study [8]. The generalised Lindley distribution was first introduced in 2011 by Nadarajah et al., who showed that it outperforms gamma, log-normal, Weibull, and exponential distributions when taking bathtub hazard rate into account [12]. In 2021, Mahmood et al. [19] published an enlarged Cosine generalised family of distributions for dependability modelling: characteristics and applications with simulation analysis, and Muse et al. [20] suggested a new flexible form of the log-logistic distribution. Citations [5], [17], and [6] in 2022 explored a family of produced distributions with applications.

By applying adaptive type-RN1 progressive hybrid censored competing risks, Okasha and Mustafa [13] used the E-Bayesian approach to estimate the Weibull distribution. For further detail on competing risk models, see Abushal et al. [26], Ramadan et al. [27], Sarhan et al. [22], and Sarhan et al. [21]. A new one-parameter Akash distribution that mixes exponential θ and gamma $(3, \theta)$ distributions was proposed by Rama Shanker in his paper cited in Shanker (2015). He suggested the Akshaya distribution, which had just one parameter and outperformed the Lindley and conventional exponential distributions, for modelling lifetime data. In their study, Shanker et al. [15] demonstrated the connections between the exponential distributions of Akash, Shanker, Lindley, and others as well as comparative analyses of these distributions. Other papers covered approaches to parameter estimation.

Shanker is credited with developing the ”Akshaya distribution,” [15] a revolutionary one-parameter continuous distribution that has been presented for modelling lifetime data from biological research and engineering. Among other mathematical and statistical characteristics, its shape, moments, hazard rate function, mean residual life function, stochastic ordering, mean deviations, and Bonferroni and Lorenz curves have been described. The conditions under which the Akshaya distribution is over-dispersed, equi-dispersed, and under-dispersed are explored together with some other one parameter lifetime distributions. The methods of moments and maximum likelihood estimation for estimating the parameter of the proposed distribution have been researched. Additionally, Ramadan et al. [23] discussed the generalised power Akshaya distribution and its uses. Given by is the probability density function (pdf) of the Akshaya distribution.

$$f(x;\theta) = \frac{\theta^4}{\theta^3 + 3\theta^2 + 6\theta + 6}(1 + x)^3 e^{-\theta x}, x, \theta > 0, \tag{1.1}$$

the formula for the cumulative distribution function (CDF) is

$$F(x;\theta) = 1 - \{1 + \frac{\theta^3 x^3 + 3\theta^2(\theta + 1)x^2 + 3\theta(\theta^2 + 2\theta + 2)x}{\theta^3 + 3\theta^2 + 6\theta + 6}\}e^{-\theta x}, x, \theta > 0, \tag{1.2}$$

and is followed by the hazard rate function

$$h(x;\theta) = \frac{\theta^4(1 + x)^3}{\theta^3 x^3 + 3\theta^2(\theta + 1)x^2 + 3\theta(\theta^2 + 2\theta + 2)x + (\theta^3 + 3\theta^2 + 6\theta + 6)}, x, \theta > 0. \tag{1.3}$$

Equation (1.3)’s rate function is an increasing function of x and θ. However, from a methodological perspective, the Akshaya distribution is not appropriate in many circumstances.

In this study, Bayesian and non-Bayesian methods for parameter estimation for the Akshaya distribution are introduced. In section 2, non-Bayesian inference techniques are discussed. The Bayesian estimation process for the unknowable parameter was investigated in Section 3. Section 4 presents a simulation exercise to illustrate the distribution's adaptability. The conclusion is presented in Section 5.

2. Non-Bayesian Inference Methods

In this part, the parameter of the Akshaya distribution is estimated using the weighted least square estimation, least square estimation, Cramer-von-Mises estimation, and maximum likelihood estimation.

2.1. Maximum likelihood estimation method

Let $(x_1, x_2, ..., x_n)$ be a random sample from Akshaya distribution, then the likelihood estimation function, L can be given as follows

$$\begin{aligned} L &= \prod_{i=1}^{n} f(x;\theta) \\ &= \frac{\theta^{4n}}{(\theta^3 + 3\theta^2 + 6\theta + 6)^n} e^{-\theta \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 + x_i)^3, \end{aligned} \tag{2.1}$$

and the natural log likelihood function is given by

$$ln(L) = n\{4ln(\theta) - ln(\theta^3 + 3\theta^2 + 6\theta + 6)\} - \theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \{3ln(1 + x_i)\}. \tag{2.2}$$

The first derivatives of the natural log likelihood function with respect to θ is given by

$$\frac{\partial}{\partial\theta} ln(L) = \frac{4n}{\theta} - \frac{3\theta^2 + 6\theta + 6}{\theta^3 + 3\theta^2 + 6\theta + 6} - \sum_{i=1}^{n} x_i, \tag{2.3}$$

The maximum likelihood estimate, $\hat{\theta}$ of θ is the solution of the equation $\frac{\partial}{\partial\theta} ln(L) = 0$ and so it can be obtained by solving the following fourth degree polynomial equation $\theta^4 \bar{x} + (3\bar{x} - 1)\theta^3 + 6(\bar{x} - 1)\theta^2 + 6(\bar{x} - 3)\theta - 24 = 0$

By setting $U(\theta) = 0$, the probability estimates of the model parameter can be calculated. Since the abovementioned equation is non-linear, the Newton-Raphson approach in R programming language is used to estimate the model parameter.

2.2. Anderson and Darling (AD) method of estimation

The function with respect to the model parameter θ is minimised to provide the Anderson and Darling estimates, which are represented as

$$AD(\theta) = -n - \frac{1}{n} \sum_{k=1}^{n} (2k - 1)(lnF(x_k) + ln\bar{F}(x_{n+1-k})),$$

where $\bar{F}(x) = 1 - F(x)$.

2.3. Cramer-von-Mises (CVM) method of estimation

Another significant estimating method by Macdonald [29] is the Cramer-von Mises. By minimising the function $C(\theta)$ with respect to the unknown parameters θ, it is possible to calculate the parameter estimate in the Cramer-von Mises estimation technique as follows:

$$C(\theta) = \frac{1}{12} + \sum_{k=1}^{n} \left[F(x_k) - \frac{2k-1}{2n} \right]^2$$

$$= \frac{1}{12} + \sum_{k=1}^{n} \left[1 - \left\{ 1 + \frac{\theta^3 x^3 + 3\theta^2(\theta+1)x^2 + 3\theta(\theta^2+2\theta+2)x}{\theta^3 + 3\theta^2 + 6\theta + 6} \right\} e^{-\theta x} - \frac{2k-1}{2n} \right]^2 .$$

2.4. Least square estimation (LSE) and weighted least square estimation (WLSE)

The least square and weighted least square methods of estimation are proposed by Swain et al. [28] to estimate the parameters of Beta distribution. In LSE method, the estimates of the parameters of the proposed model can be determined by minimizing the least square function $LS(\theta)$ with respect to unknown parameter, where

$$LS(\theta) = \sum_{k=1}^{n} \left[F(x_k) - \frac{k}{n+1} \right]^2$$

$$= \sum_{k=1}^{n} \left[1 - \left\{ 1 + \frac{\theta^3 x^3 + 3\theta^2(\theta+1)x^2 + 3\theta(\theta^2+2\theta+2)x}{\theta^3 + 3\theta^2 + 6\theta + 6} \right\} e^{-\theta x} - \frac{k}{n+1} \right]^2 .$$

Similar to this, the weighted least square function $WLS(\theta)$ is minimised to determine the WLSE of the unknown parameters:

$$WLS(\theta) = \sum_{k=1}^{n} \frac{(n+1)^2(n+2)}{k(n-k+1)} \left[F(x_k) - \frac{k}{n+1} \right]^2$$

$$= \sum_{k=1}^{n} \frac{(n+1)^2(n+2)}{k(n-k+1)} \left[1 - \left\{ 1 + \frac{\theta^3 x^3 + 3\theta^2(\theta+1)x^2 + 3\theta(\theta^2+2\theta+2)x}{\theta^3 + 3\theta^2 + 6\theta + 6} \right\} e^{-\theta x} - \frac{k}{n+1} \right]^2 .$$

3. Bayesian Estimation Method

In this section, the parameters θ, which are expected to follow the gamma prior distribution with parameters a and b, are estimated using the Bayesian estimation (BE) method. The form of the gamma prior density function is

$$g(u; a, b) = \frac{b^a}{\Gamma(a)} u^{a-1} e^{-ub}, \quad u, a, b > 0. \tag{3.1}$$

thus, θ's prior density is given by

$$g(\theta) = \prod_{i=1}^{n} g(\theta) \propto (\theta)^{a-1} e^{-\theta b}. \tag{3.2}$$

According to the Bayesian method, the posterior distribution function is given by

$$g(\theta|\underline{x}) = \frac{g(\theta)L(\underline{x})}{\int g(\theta)L(\underline{x})} \propto g(\theta)L(\underline{x}). \tag{3.3}$$

Substituting from Equations (3.2) and (2.1) into Equation (3.3) we get

$$g(\theta|\underline{x}) \propto (\theta)^{a-1} \frac{\theta^{4n}}{(\theta^3 + 3\theta^2 + 6\theta + 6)^n} e^{-(\theta(\sum_{i=1}^{n} x_i + b)} \prod_{i=1}^{n} (1 + x_i)^3. \tag{3.4}$$

Without determining the normalised constant, the posterior distribution is quantitatively summarised using the Markov Chain Monte Carlo (MCMC) method which introduced by Brooks et al. [2].

4. Simulation Study

This section uses numerically the inverse of the cumulative distribution function to produce random data for Akshaya. When the size is $n = 25, 50$, and 100, the R software is used to create various distribution samples. The experiment is run a total of 5000 times with initial values of $\theta = 0.5$, $\theta = 0.7$ and $\theta = 1.2$. In this study, four quantities are looked at.

(a) Mean of the estimated values (ME) of $\hat{\nu}$, $\hat{\nu} = \hat{\theta}$ which equals $\frac{1}{5000} \sum_{i=1}^{5000} \hat{\nu}_i$.
(b) Average bias of the MLE (AB) of $\hat{\nu}$ which equals $\frac{1}{5000} \sum_{i=1}^{5000} (\hat{\nu}_i - \nu)$.
(c) The mean squared error (MSE) of the MLE of $\hat{\nu}$ which equals $\frac{1}{5000} \sum_{i=1}^{5000} (\hat{\nu}_i - \nu)^2$.

Table 1. Five classical methods and Bayesian method for estimating θ with simulated data for sample size $n = 25$ and $\theta = 0.5$.

Name	MLE	LSE	WLSE	CVME	ADE	BE
initial	0.500000	0.500000	0.500000	0.500000	0.500000	0.500000
Mean	0.505167	0.503387	0.503232	0.466945	0.503102	0.505854
MSE	0.002593	0.002967	0.002817	0.006679	0.002718	0.002591
RMSE	0.050919	0.054469	0.053077	0.081727	0.052133	0.050900
Bias	0.005167	0.003387	0.003230	0.033055	0.003103	0.005854

Tables 1– 9 show that:

- As the sample size (n) increases, the absolute value of the average bias $|AB|$ for the parameters θ decreases.

Table 2. Five classical methods for estimating θ with simulated data for sample size $n = 50$ and $\theta = 0.5$.

Name	MLE	LSE	WLSE	CVME	ADE	BE
initial	0.500000	0.500000	0.500000	0.500000	0.500000	0.500000
Mean	0.502145	0.501473	0.501383	0.462905	0.501273	0.504385
MSE	0.001231	0.001445	0.001349	0.003953	0.001323	0.001201
RMSE	0.035091	0.038011	0.036739	0.062872	0.036374	0.034659
Bias	0.002145	0.001473	0.001383	0.037095	0.001273	0.004385

Table 3. Five classical methods for estimating θ with simulated data for sample size $n = 100$ and $\theta = 0.5$.

Name	MLE	LSE	WLSE	CVME	ADE	BE
initial	0.500000	0.500000	0.500000	0.500000	0.500000	0.500000
Mean	0.501054	0.500511	0.500533	0.460568	0.500469	0.505549
MSE	0.000633	0.000725	0.000683	0.002727	0.0006755	0.000577
RMSE	0.025157	0.026932	0.026127	0.052217	0.025991	0.024021
Bias	0.001054	0.000511	0.000533	0.039431	0.000469	0.005549

Table 4. Five classical methods and Bayesian method for estimating θ with simulated data for sample size $n = 25$ and $\theta = 2$.

Name	MLE	LSE	WLSE	CVME	ADE	BE
initial	0.700000	0.700000	0.700000	0.700000	0.700000	0.700000
Mean	0.707027	0.704290	0.703999	0.654369	0.703985	0.706351
MSE	0.005329	0.006135	0.005841	0.013931	0.005623	0.004702
RMSE	0.073006	0.078326	0.076429	0.118033	0.074984	0.068574
Bias	0.007027	0.004290	0.003999	0.045630	0.003985	0.006351

Table 5. Five classical methods for estimating θ with simulated data for sample size $n = 50$ and $\theta = 0.7$.

Name	MLE	LSE	WLSE	CVME	ADE	BE
initial	0.700000	0.700000	0.700000	0.700000	0.700000	0.700000
Mean	0.704152	0.702613	0.702748	0.647168	0.702668	0.707342
MSE	0.002617	0.002967	0.002823	0.007762	0.002772	0.002078
RMSE	0.051157	0.054471	0.053131	0.088103	0.0526521	0.045589
Bias	0.0041512	0.002613	0.002748	0.052832	0.002668	0.007342

Table 6. Five classical methods for estimating θ with simulated data for sample size $n = 100$ and $\theta = 0.7$.

Name	MLE	LSE	WLSE	CVME	ADE	BE
initial	0.700000	0.700000	0.700000	0.700000	0.700000	0.700000
Mean	0.703201	0.705619	0.701046	0.720688	0.716091	0.706171
MSE	0.001047	0.001906	0.000821	0.0070342	0.001972	0.001219
RMSE	0.00431	0.006436	0.004327	0.062910	0.047021	0.034924
Bias	0.002441	0.002058	0.001278	0.009212	0.0037021	0.006171

Table 7. Five classical methods for estimating θ with simulated data for sample size $n = 25$ and $\theta = 1.2$.

Name	MLE	LSE	WLSE	CVME	ADE	BE
initial	1.200000	1.200000	1.200000	1.200000	1.200000	1.200000
Mean	1.207612	1.204980	1.204901	1.219931	1.101126	1.503112
MSE	0.007933	0.009312	0.008732	0.008574	0.025123	0.109760
RMSE	0.089051	0.096491	0.093453	0.092572	0.158492	0.331293
Bias	0.0076014	0.004987	0.004904	0.019932	0.0988771	0.303121

Table 8. Five classical methods for estimating θ with simulated data for sample size $n = 50$ and $\theta = 1.2$.

Name	MLE	LSE	WLSE	CVME	ADE	BE
initial	1.200000	1.200000	1.200000	1.200000	1.200000	1.200000
Mean	1.214921	1.208991	1.208653	1.234544	1.111634	1.473873
MSE	0.016991	0.019201	0.0181541	0.018852	0.040082	0.110623
RMSE	0.130382	0.138572	0.134742	0.137305	0.200211	0.332612
Bias	0.0149223	0.008993	0.008654	0.034542	0.088371	0.273860

Table 9. Five classical methods for estimating θ with simulated data for sample size $n = 100$ and $\theta = 1.2$.

Name	MLE	LSE	WLSE	CVME	ADE	BE
initial	1.200000	1.200000	1.200000	1.200000	1.200000	1.200000
Mean	1.201261	1.218002	1.207801	1.230190	1.205421	1.451300
MSE	0.013201	0.016023	0.012503	0.013782	0.032014	0.101092
RMSE	0.107213	0.120051	0.120364	0.130163	0.208973	0.203114
Bias	0.014681	0.004573	0.004594	0.032041	0.051020	0.150240

- As the sample size (n) increases, the mean squared error (MSE) for the parameters θ decreases.

Additionally, the credible interval and 95% confidence interval for the Bayesian model for the parameter theta with actual values of $\theta = 0.5$, $\theta = 0.7$ and $n = 100$ can be stated as follows:
$(\hat{\theta_L}, \hat{\theta_U}) = (0.64207, 0.78254)$, and $(\hat{\theta_L}, \hat{\theta_U}) = (0.6400833, 0.7747177)$.
$(\hat{\theta_L}, \hat{\theta_U}) = (0.475630.57925)$, and $(\hat{\theta_L}, \hat{\theta_U}) = (0.4641415, 0.5558569)$. respectively

All hyper-parameters of the Bayesian technique are set to 0.001, which denotes an uninformative prior. In order to use MCMC, the proposal was transformed into a multivariate t distribution with four degrees of freedom, a variance-covariance matrix equal to the inverse Fisher matrix, and a mode equal to the vector of the maximum likelihood estimates. There were also 5000 draws totaling M. As a diagnostic test for the MCMC, we displayed the autocorrelations and traces for each parameter, as shown in Figures refAutocorrelation plot of theta, 3 and 4. The trace plots show a reasonable mixture of the sampled draws, and the autocorrelation plot reveals that the Lag rapidly decreases, indicating that the draws become roughly independent with time. Figure 1 shows the plot of the posterior function for the parameter θ.

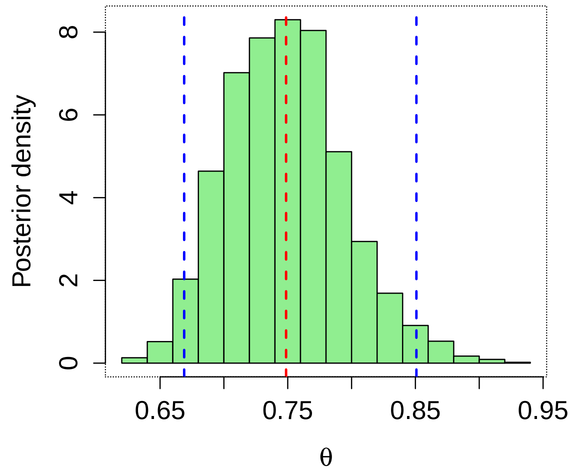

Figure 1. Marginal posterior density plots for the simulated data.

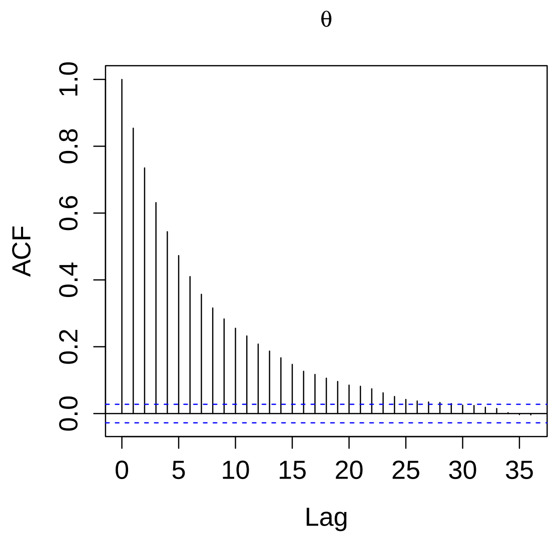

Figure 2. Autocorrelation plot for the simulated data.

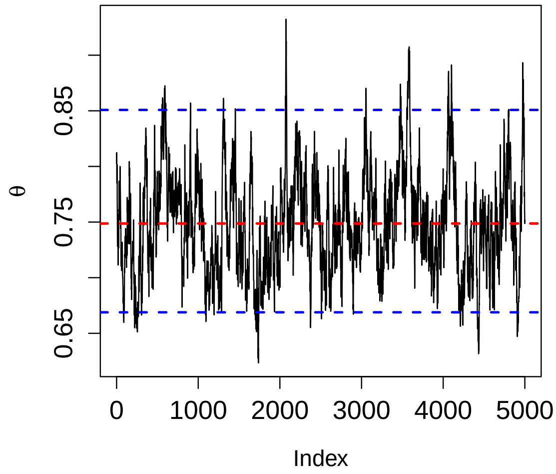

Figure 3. Trace plots for the simulated data.

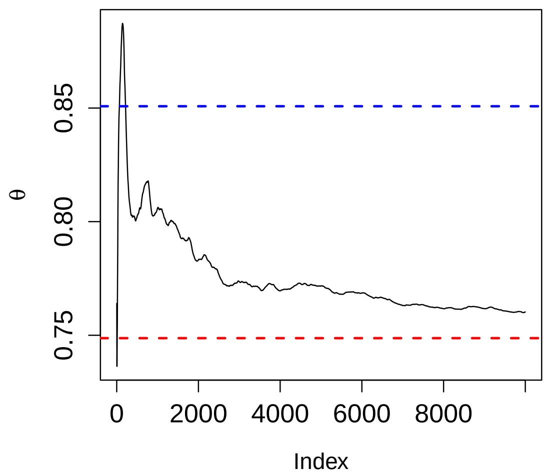

Figure 4. Cumsum plot for the simulated data.

5. Conclusions

The five non-Bayesian approaches of MLE, AD, CVM, LSE, and WLSE are used to estimate the parameters of the Akshaya model. Additionally, under the Square Error Loss Function, the parameter is determined using Bayesian estimation. The effectiveness of the model is evaluated using several estimation techniques. The mean of the estimated values is then displayed through the use of a simulation study. The MLE, AD, CVM LSE, WLSE, and BE estimators of the model parameter's average bias and mean square error are discussed. Additionally, the parameter's credible interval and confidence interval are determined.

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Conflict of interest

The authors declare no conflict of interest.

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